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Central Bank Purchases of Government Bonds

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Abstract

We develop a microfounded model, where agents have the possibility to trade money for government bonds in an over-the-counter market. It allows us to address important open questions about the effects of central bank purchases of government bonds, these being: under what conditions these purchases can be welfare-improving, what incentive problems they mitigate, and how large these effects are. Our main finding is that this policy measure can be welfare-improving, by correcting a pecuniary externality. Concretely, the value of money is increased as central bank's purchases of government bonds induce agents to increase their demand for money, which is welfare-improving.

Keywords: Monetary theory, over-the-counter markets, quantitative easing, money demand, pecuniary externality.

JEL Classification: E31, E40, E50, G12.

1 Introduction

Quantitativ Easing (QE, hereafter) denotes a central bank policy of purchasing financial assets such as government bonds, agency debt, or mortgage-backed securities. Empirical evidence suggests that QE is successful in reducing yields on these assets, while there is no sound conclusion about its effects on the allocation and welfare.[1]

In this paper, we focus on the effects of central bank purchases of government bonds. For this purpose, we construct a microfounded monetary model, where trading in financial markets is essential. The model allows us to understand which incentive problems QE mitigates, how to find the optimal degree of QE, under what conditions it is successful and what effects it has on quantities and prices.

In our model, agents face idiosyncratic liquidity shocks, and they hold a portfolio composed of money and government bonds. Money can be directly used to purchase goods and thus serves as

[1] See, for instance, Gagnon et al. (2011), D'Amigo and King (2012), Krishnamurthy and Vissing-Jorgensen (2012), or Bauer and Neely (2013), among others.

a medium of exchange. In contrast, government bonds cannot be used as a medium of exchange, but are a superior store of value.[2] The idiosyncratic liquidity shocks generate an ex-post inefficient allocation of the medium of exchange: Some agents will hold money, but have no current need for it, while other agents will hold insufficient amounts of money in order to satisfy their liquidity needs. A secondary financial market allows agents to trade money for bonds and so improves the allocation of the medium of exchange. The secondary financial market is an over-the-counter market, that embeds the recent advances in search theory.[3] QE is modelled as a central bank purchase of government bonds funded by the issuance of money; i.e., we abstract from the issuance of interest-bearing reserves and assume that the central bank directly controls the bond-to-money ratio. There are no aggregate shocks, and we derive our results in the monetary steady state equilibrium. Furthermore, we focus on the optimal degree of QE in an economy, where the efficient allocation is not attainable; i.e., on an economy with inflation rates above the Friedman rule.

Our main finding is that QE mitigates a pecuniary externality and so improves the allocation and welfare. This externality arises because in our incomplete market model the resulting equilibria might not be constrained efficient. In such an environment, government interventions can be welfare-improving.[4] In our model, the secondary financial market reduces the incentive to self-insure against liquidity shocks, and agents attempt to benefit from money held by other market participants. As a result, the aggregate demand for money is too low, and QE can mitigate this externality. The reasoning behind this result is as follows. QE reduces the bond-to-money ratio and as a result bonds become scarce and are priced above their fundamental value. This induces agents to increase their demand for money, which marginally increases the value of money and so the insurance for all market participants. This finding is very novel and not present in previous studies about the effects of QE. A further contribution of our work is that we show that QE is only successful for low inflation rates.

We calibrate the model to U.S. data in order to estimate the impact of QE on quantities and prices. For our baseline calibration, we find that the optimal degree of QE succeeds in reducing the yield of bonds by 0.52 percent. The social benefits associated with such a policy measure account for 0.014 percent of steady state consumption. Furthermore, we find that QE is only welfare-improving for inflation rates below 8 percent. The higher the bargaining power of bond vendors, the higher the benefits of QE and the higher the critical inflation rate up to which this policy measure is welfare-improving. In contrast, the lower the search frictions, the lower the benefits and the lower the critical inflation rate up to which QE is beneficial.

Based on empirical evidence, we argue that for the U.S. a calibration with equally distributed bargaining power among agents and low search frictions seems more realistic than our baseline

[2] It is socially beneficial that government bonds cannot be used as a medium of exchange. Otherwise, bonds would be perfect substitutes for money and thus be redundant. See Kocherlakota (2003), Andolfatto (2011), Berentsen and Waller (2011), or Berentsen et al. (2014), for a more detailed discussion.

[3] There is a rapidly growing literature which builds on the seminal contribution of Duffie et al. (2005). See, for instance, Duffie et al. (2008), Lagos and Rocheteau (2009), Lagos, Rocheteau and Weill (2011), Rocheteau and Wright (2013), Geromichalos and Herrenbrueck (2014), Geromichalos et al. (2014), or Berentsen et al. (2015b).

[4] See, for instance, Greenwald and Stiglitz (1986) or Berentsen et al. (2015b) for a more detailed discussion.

calibration. In such a case, the benefits associated with a central bank policy of purchasing government bonds are likely to be small and in the area of 0.002 percent of steady state consumption. Furthermore, such a policy measure is only beneficial for inflation rates below 4.3 percent and reduces the yield of bonds by 0.40 percent. Hence, even though we can verify that QE proves competent in substantially reducing the yield of bonds, our findings indicate that the overall social benefits are likely to be anemic.

In our model, an equilibrium can be of three types, which we denote by type-I, type-II, and type-III. In a type-I and a type-II equilibrium, bonds are priced at their fundamental value, and QE does not effect welfare. The reason is that trading is unconstrained in the type-I equilibrium, while in the type-II equilibrium only the cash constraint on bond buyers is binding. In contrast, in a type-III equilibrium, bonds are priced above their fundamental value and exhibit a liquidity premium. The reason is that in this equilibrium the bond constraint on bond vendors is binding. Therefore, in a type-III equilibrium QE has a direct impact on the allocation and welfare. A summary of the bond prices and the trading constraints in the respective equilibria is shown in Table 1:

Table 1: Bond prices and trading constraints

Equilibrium	Bond Price	Bond Constraint	Cash Constraint
Type-I	Fundamental	Non-Binding	Non-Binding
Type-II	Fundamental	Non-Binding	Binding
Type-III	Liquidity Premium	Binding	Non-Binding

In Figure 1, we show the regions of existence with respect to the inflation rate γ and the bond-to-money ratio $\mathcal{B}$. Furthermore, the figure also shows the optimal choice of $\mathcal{B}$, which coincides with the optimal degree of QE.

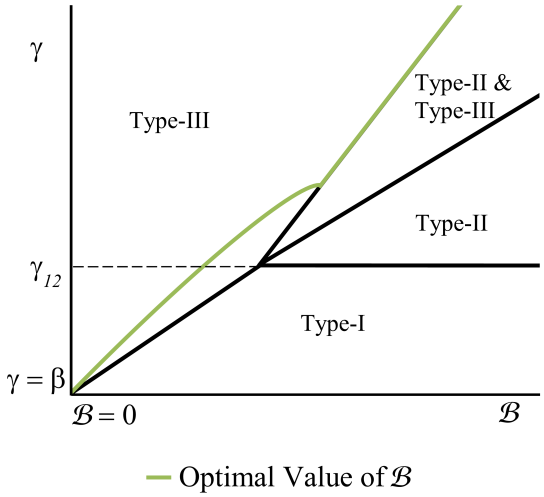

Figure 1: Regions of existence and optimal $\mathcal{B}$

The Friedman-rule is shown at the bottom-left corner of the above figure, where the inflation rate equals the rate of time preference, β. At the Friedman-rule, financial intermediation is redundant, and the efficient allocation is attainable. For inflation rates $\beta < \gamma < \gamma_{12}$, either the type-I or the type-III equilibrium exists. Hereafter, we will show that it is always welfare-improving to reduce the bond-to-money ratio in the type-I equilibrium, such that the bond constraint becomes binding, and the economy moves to a region in the type-III equilibrium, where welfare is maximized (highlighted by the green line).[5] For inflation rates above γ_{12}, either the type-II or the type-III equilibrium exists. The triangle in the top-right section shows that there may exist a region, where the type-II and the type-III equilibrium coexist. We show, that for $\gamma > \gamma_{12}$ and γ sufficiently close to γ_{12}, welfare is also maximized in the type-III equilibrium; i.e., QE is welfare-improving. However, if $\gamma > \gamma_{12}$ and γ is considerably higher than γ_{12}, then welfare is maximized in the type-II equilibrium, and the optimal policy is to keep the bond-to-money ratio at least at the border to the type-II equilibrium. In other words, if inflation is too high, a central bank policy of purchasing government bonds does not improve welfare, but is more likely to reduce it.

2 Literature

Williamson (2012) analyses the effects of central bank purchases of government bonds in a model building on Lagos and Wright (2005). In the baseline model, he assumes that in non-monitored meetings producers only accept money in exchange for goods, while in monitored-meetings they also accept claims to government bonds. As banks operate after consumers learn in which type of meeting they will participate in the subsequent goods market, consumers in the first type of meeting withdraw cash from the bank, while consumers in the second type of meeting receive the right to trade claims to bonds. When a consumer meets a producer in a monitored meeting, then he makes a take-it-or-leave it offer; i.e., he has all the bargaining power in the match. Williamson finds that a central bank purchase of government bonds only has an effect on quantities and prices in an equilibrium where bonds are scarce. That is, in such an equilibrium this policy measure further reduces the supply of bonds and thus results in an increase of the liquidity premium on bonds; i.e., in a decrease of the real interest rate. As a result, it reduces the availability of claims to bonds in monitored meetings. The author then extends the model to include privately issued debt beside government debt. In the extended version, he finds that due to the reduction in the stock of government bonds, a central bank purchase of government bonds induces an expansion of private debt; i.e., a higher issuance of corporate bonds. Thereafter, he includes additional financial market frictions in order to analyse issues in financial crises. He argues that if an exogenous shock results in a complete shut-down of the credit market, such that the economy ends up in an equilibrium where bonds are scarce, then a central bank purchase of government bonds may result in a further reduction in the real interest rate and worsen the scarcity problem, rather than improve it.

[5] Note that we assume that fiscal policy is passive and that, therefore, any change in the bond-to-money ratio $\mathcal{B}$ has no effect on the inflation rate γ. This will be clarified in Section 5.

Williamson (2014a) constructs a model of money, credit and banking, where he analyzes central bank purchases of long-maturity government bonds, denoted as QE.[6] Like in Williamson (2012), trade in the goods market is performed by money and claims to bonds. Williamson (2014a) focuses his analysis on an equilibrium where government bonds are scarce and thus are priced above their fundamental value. He finds that a scarcity of government bonds, as well as inferior properties of long-maturity bonds when used as collateral compared to short-maturity bonds, are preconditions for the existence of a term premium. Furthermore, he finds that QE reduces the yield of long-maturity bonds, which results in a flattening of the yield curve. Additionally, QE reduces inflation and so increases the real yield of bonds, even though the nominal yield falls. This increases the value of collateralizable wealth and so relaxes the bank's collateral constraints, which is welfare-improving. Note, that the above result is derived for a central bank applying a floor system, while a channel system limits the central banks' ability to perform QE.

Williamson (2014b) builds on Williamson (2012 and 2014a) and analyses the effects of central bank purchases of private assets in an economy where agents have incentives to fake the quality of collateral. Consumers own houses which are used as collateral to obtain a mortgage. Like in Williamson (2012 and 2014a), trade in the goods market is performed by money and claims to financial assets; i.e., claims to mortgages, government debt, and reserves. Consumers (banks) have the ability to fake the quality of houses (mortgages) at a cost. Also Williamson (2014b) focuses his analysis on an equilibrium where financial assets are scarce, and thus government bonds are priced above their fundamental value. If the cost of faking is low, then mortgages exhibit an interest rate spread and haircuts to convince lenders that the collateral is not faked. In such an environment, a central bank purchase of mortgages might not be feasible as the central bank might not be able to verify the quality of the private assets. Hence, only when the cost of faking is sufficiently high, such that neither consumers nor banks have an incentive to fake the quality of the collateral before the program starts, it might then be welfare-improving for a central bank to purchase private assets from banks. He finds that in order to be successful, the central bank needs to purchase the entire amount of outstanding mortgages. Doing so and issuing government bonds at same time, relaxes the bank's collateral constraints, which is welfare-improving.

Gertler and Karadi (2013) develop a macroeconomic model to analyze the effects of central bank purchases of long-maturity government bonds or private loans, denoted as large-scale asset purchases. Their model has some similarities to Williamson (2012, 2014a, and 2014b); i.e., they also assume that banks need to collateralize their deposits and that bonds with different maturities and from different issuers (government or private) have different properties for being used as collateral. However, Gertler and Karadi (2013) do not model the intermediation of banks explicitly, but rather assume that households can only hold short-term government bonds and bank deposits, which are in turn collateralized by private debt and long-term government bonds. They argue that central banks, as opposed to private intermediaries, obtain funds elastically; i.e., they can fund the purchase of long-term securities by issuing short-term debt. This provides

[6] The basic framework builds on Lagos and Wright (2005) and Rocheteau and Wright (2005), and has a similar structure to that of Williamson (2012).

central banks with a channel for large-scale asset purchases to be effective in reducing borrowing costs, if the aggregate balance sheet constraint of banks is binding; i.e., if financial assets are scarce. They find, further, that large-scale asset purchases help to flatten the yield curve, which works especially well when short-term rates are not expected to rise anytime soon.

Herrenbrueck (2014) develops a model where agents can trade financial assets (government bonds and physical capital) in over-the-counter markets in the spirit of Duffie et al. (2005). The over-the-counter markets feature search frictions in the way that agents need to find a broker to trade with. Once matched, agents make a take-it-or-leave-it offer to the broker; i.e., they have all the bargaining power in the match. The author finds that QE, defined as a central bank purchase of government bonds, results in an increase in the price of financial assets and so transfers wealth to agents who value it more; i.e., to agents who want to sell financial assets for money in order to consume. In turn, this reallocation of purchasing power stimulates consumption and investment. The drawback of this policy measure is that it reduces the available stock of bonds and that it depresses the private flow of funds, which results in a less efficient allocation of money. Whether QE is beneficial depends on which of the effects dominates.

In contrast to the above studies, we abstract from private assets, the central bank issuance of interest-bearing reserves, or brokers, and clearly focus on the economic mechanisms behind central bank purchases of government bonds funded by the issuance of money. Furthermore, we explicitly model bargaining frictions in the secondary bond market, which we believe are necessary ingredients for any over-the-counter market. We find that the main welfare-improving aspect of QE is that it corrects a pecuniary externality, and not that it transfers wealth to agents who value it more. In other words, it is not the resulting liquidity premium on bonds that improves welfare, but the incentive to increase the demand for money. Besides this, we show the full picture and do not solely focus on an equilibrium where bonds are scarce. This allows us to state up to which inflation rate QE is successful, thus providing an argument which is missing in the above studies.

Our paper is therefore also related to the literature that focuses on correcting pecuniary externalities, such as Berentsen et al. (2014 and 2015b). Berentsen et al. (2014) find that adding search frictions to a competitive secondary financial market can be welfare-improving for high inflation rates. The reason is that adding search frictions increases the demand for money, which is welfare-improving, but it also increases consumption variability, since only some agents have access to the secondary financial market. In a similar framework to ours, Berentsen et al. (2015b) find that the demand for money is too low in an equilibrium where trading is unconstrained and that imposing a financial transaction tax can correct this externality. However, imposing a financial transaction tax requires the central bank to operate the secondary financial market in order to perfectly enforce tax payment. In contrast, QE only requires the central bank to issue money in order to purchase government bonds, which is much easier to implement. In view of its tactical simplicity, QE has recently experienced much more popularity among policy makers than implementing a financial transaction tax. Thus, it is crucial to understand the economic mechanisms behind such a policy measure.

3 Environment

A $[0,1]$-continuum of agents live forever in discrete time. In each period, there are three markets that open sequentially.[7] The first market is a secondary bond market, where agents trade money for nominal bonds. The second market is a goods market, where agents produce or consume market-2 goods. The third market is a centralized market, where all agents consume and produce market-3 goods, and financial contracts are redeemed. This market is called the primary bond market. All goods are perfectly divisible and non-storable.

At the beginning of each period, agents receive an idiosyncratic i.i.d. preference shock that determines whether they are producers or consumers in the goods market. With probability n, an agent can produce but not consume, and with probability $1-n$, he can consume but not produce. In the goods market, trading is frictionless; i.e., agents trade against the market and take prices as given. Consumers get utility $u(q)$ from q consumption, where $u'(q), -u''(q) > 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. Producers incur a utility cost $c(q) = q$ from producing q units of market-2 goods.

In the primary bond market, trading is also perfectly frictionless and competitive. The market-3 good is produced and consumed by all agents using a linear production technology; i.e., h units of time produce h units of goods.[8] Agents get utility $U(x)$ from x consumption, where $U'(x)$, $-U''(x) > 0$, $U'(0) = \infty$, and $U'(\infty) = 0$. Agents discount between, but not within, periods. The term $\beta = (1+r)^{-1}$ denotes the discount factor between two consecutive periods, where $r > 0$ represents the real interest rate. A central bank operates in the primary bond market and issues two perfectly divisible and storable assets: money and one-period bonds. Both assets are intrinsically useless. Bonds are issued at a discount, and pay off one unit of money in the next-period primary bond market. Bonds are intangible objects; i.e., no physical object exists. In the goods market, agents cannot commit, and there is a lack of record-keeping. These two frictions imply that producers ask for immediate compensation from consumers. As bonds are intangible objects, only money can serve as a medium of exchange in the goods market.[9] The per-capital stock of money is denoted by M_t, and the per-capita stock of newly issued bonds is denoted by B_t at the end of period t. The issuance price of bonds in the primary bond market is denoted by ρ_t. Thus, the change of the stock of money in period t is given by

$$M_t - M_{t-1} = \tau_t M_{t-1} + B_{t-1} - \rho_t B_t, \tag{1}$$

which is determined by three components: the lump-sum money injections, $\tau_t M_{t-1}$, the money

[7] Our framework is similar to Berentsen et al. (2015b), which builds on Lagos and Wright (2005). However, the contribution of Berentsen et al. (2015b) is different. In particular, the authors investigate the social benefits of a financial transaction tax on bond transactions in an equilibrium where trading is unconstrained.

[8] The assumption of quasi-linear preferences in the primary bond market results in a degenerate end-of-period distribution of money holdings, which makes the model tractable (see Lagos and Wright, 2005).

[9] The necessary assumptions that make money essential are discussed in more detail in Kocherlakota (1998), Wallace (2001), Lagos and Wright (2005) and Shi (2006). A more detailed discussion about why bonds cannot be used as a medium of exchange can be found in Kocherlakota (2003), Andolfatto (2011), Berentsen and Waller (2011), or Berentsen et al. (2014 and 2015b).

created to redeem previously issued bonds, B_{t-1}, and the money withdrawal from selling B_t units of bonds at price ρ_t. We assume that there is a strictly positive initial stock of money M_0 and bonds B_0, where $B_0/M_0 = \mathcal{B}$ denotes the bond-to-money ratio. For $\tau_t < 0$, the central bank must be able to extract money via lump-sum taxes from the economy.

At the beginning of each period, and after the realization of the idiosyncratic preference shock, agents can trade money for bonds in the secondary bond market. Consumers and producers meet at random in bilateral meetings according to a reduced-form matching function $\mathcal{M}(n, 1-n)$. By assumption, the matching function has constant returns to scale, and is continuous and increasing with respect to each of its arguments. The probability that a consumer meets a producer is denoted by $\delta = \mathcal{M}(n, 1-n)(1-n)^{-1}$, and the probability that a producer meets a consumer is denoted by $\delta^p = \mathcal{M}(n, 1-n)\, n^{-1}$. Once in a meeting, agents bargain over the quantity of money and bonds to be exchanged. Agents who are able to participate in this market are called *active* and those who are not are called *passive*.

3.1 Efficient Allocation

As a benchmark exercise, we present the allocation chosen by a social planner who dictates consumption and production. The planner treats all agents symmetrically and his optimization problem is

$$\mathcal{W} = \max_{h,x,q} \left[(1-n)\, u(q) - n q_p\right] + U(x) - h, \tag{2}$$

subject to the feasibility constraint $h \geq x$ and the market clearing condition $n q_p \geq (1-n)\, q$. The efficient allocation satisfies $U'(x^*) = 1$, $u'(q^*) = 1$, and $h^* = x^*$.

4 Agent's Decisions

For notational simplicity, we omit time subscript t going forward. Next-period variables are indexed by $+1$, and previous-period variables by -1. In what follows, we study the agents' decisions in a representative period t and work backwards from the last market (primary bond market) to the first market (secondary bond market).

4.1 Primary Bond Market

In the primary bond market, agents can acquire any amount of money and newly issued bonds at price ρ. Agents want to hold money, because they will use it in the next-period goods market, if they turn out to be consumers. In contrast, bonds cannot be used as a medium of exchange; i.e., they are illiquid, but they can be traded for money in the next-period secondary bond market, which opens before the goods market. Furthermore, agents can produce and consume the market-3 goods using a linear production technology; they receive money for maturing bonds; can trade money for market-3 goods; and they receive the lump-sum money transfer T from the central bank. An agent entering the primary bond market with m units of money and b units of bonds

has the value function $V_3(m, b)$. He solves the following decision problem:

$$V_3(m,b) = \max_{x,h,m_{+1},b_{+1}} \left[U(x) - h + \beta V_1(m_{+1}, b_{+1})\right], \tag{3}$$

subject to

$$x + \phi m_{+1} + \phi\rho b_{+1} = h + \phi m + \phi b + \phi T, \tag{4}$$

where ϕ is the price of money in terms of market-3 goods, and h denotes hours worked. The first-order conditions with respect to m_{+1}, b_{+1} and x are $U'(x) = 1$, and

$$\frac{\beta\partial V_1}{\partial m_{+1}} = \rho^{-1}\frac{\beta\partial V_1}{\partial b_{+1}} = \phi. \tag{5}$$

According to (5), the marginal cost of taking one additional unit of bonds into the next period, $\rho\phi$, is lower than that for money, ϕ, for any $\rho < 1$. The reason is that bonds are only beneficial to agents who will be active consumers in the next period. Therefore, bonds exhibit a lower marginal benefit than money for $\rho < 1$, which is denoted by $\beta\partial V_1/\partial b_{+1}$. Due to the quasi-linear preferences, the choice of m_{+1} and b_{+1} is independent of m and b. As a result, each agent exits the primary bond market with the same amount of money and bonds. The envelope conditions are

$$\frac{\partial V_3}{\partial m} = \frac{\partial V_3}{\partial b} = \phi. \tag{6}$$

The above equation states that at the beginning of the primary bond market, the marginal value of money and bonds is equal to ϕ. This is because bonds are redeemed at their face value in this market.

4.2 Goods Market

We assume competitive pricing in the goods market; i.e., all agents take prices as given and trade against the market. Consider the consumer's decision problem, where p denotes the price of one unit of the market-2 good q:

$$V_2^c(m,b) = \max_q \left[\begin{array}{c} u(q) + V_3(m - pq, b) \\ s.t.\ m \geq pq. \end{array}\right]. \tag{7}$$

The constraint states that a consumer cannot spend more money than the amount he brings into this market. If the constraint is non-binding, we have $\frac{\partial q}{\partial m} = 0$ and $u'(q) = 1$. If the constraint is binding, we have $\frac{\partial q}{\partial m} = \phi$ and $u'(q) > 1$. In this case, the buyer's envelope conditions are

$$\frac{\partial V_2^c}{\partial m} = \phi u'(q) \qquad \text{and} \qquad \frac{\partial V_2^c}{\partial b} = \phi, \tag{8}$$

where we have used the envelope conditions in the primary bond market.

The producer's value function in the goods market is

$$V_2^p(m,b) = \max_{q_p} \left[-c\left(q_p\right) + V_3\left(m + pq_p, b\right)\right], \tag{9}$$

where q_p satisfies the market clearing condition

$$(1-n)[\delta\hat{q} + (1-\delta)q] = nq_p, \tag{10}$$

and where the symbol "$\hat{}$" denotes the quantities traded by active agents. According to (10), the produced quantity of producers, nq_p, equals the consumed quantity of active consumers, $(1-n)\delta\hat{q}$, plus the consumed quantity of passive consumers, $(1-n)(1-\delta)q$. The market clearing condition states that the consumed quantities of active and passive consumers are different, since active consumers could adjust their portfolio in the secondary bond market and thus can consume more in the goods market.

It is easy to see that $p\phi = c'\left(q_p\right) = 1$ holds in any monetary equilibrium. The reason is the following. For $p\phi < c'\left(q_p\right) = 1$, there is no trade, because it is suboptimal for producers to trade in the good market. For $p\phi > c'\left(q_p\right) = 1$, each single producer has an incentive to sell more goods by (9). Hence, selling any finite amount of goods is suboptimal; i.e., the optimal strategy by producers is not supported by the market clearing condition (10). Therefore, the equilibrium price must be $p\phi = c'\left(q_p\right) = 1$.

Taking the total derivative of (9) with respect to m and b and using (6) yields the envelope conditions of the producer:

$$\frac{\partial V_2^p}{\partial m} = \frac{\partial V_2^p}{\partial b} = \phi. \tag{11}$$

Because producers cannot use money or bonds in this market, their marginal benefit equals the price of money in the primary bond market.

4.3 Secondary Bond Market

In the secondary bond market, the terms of trade are determined by the proportional bargaining solution proposed by Kalai (1977), which is increasingly popular in monetary economics due to its monotonicity properties.[10] Consumers and producers are matched pairwise and bargain over the terms of trade. Let (m_j, b_j), $(\hat{m}_j, \hat{b}_j)$ denote the portfolios of an active agent before and after trading in the secondary bond market, respectively. By the market clearing condition, we have

$$\hat{m}_c - m_c = -(\hat{m}_p - m_p) \qquad \text{and} \qquad \hat{b}_p - b_p = -(\hat{b}_c - b_c).$$

[10] Kalai bargaining is discussed in more detail in Aruoba et al. (2007), or Rocheteau and Wright (2005). For its application to financial markets, see for instance Geromichalos and Herrenbrueck (2014), Geromichalos et al. (2014) or Berentsen et al. (2015). A detailed discussion of the monotonicity properties is provided by Chun and Thomson (1988).

Let $d_m \equiv \hat{m}_c - m_c$ and $d_b \equiv \hat{b}_p - b_p$ be the trading amounts of money and bonds in the secondary bond market. Hence, we have the budget constraints for producers and consumers,

$$\phi m_p \geq \phi d_m \qquad \text{and} \qquad \phi b_c \geq \phi d_b, \tag{12}$$

which state that producers cannot offer more money than they have, and consumers cannot offer more bonds than they have.

The Kalai constraint states that the trade surplus is split among producers and consumers according to their bargaining power. It is given by

$$(1-\eta)\left[u(\hat{q}) - u(q) - \phi d_b\right] = \eta\phi(d_b - d_m), \tag{13}$$

where η denotes the bargaining power of a consumer, and $1-\eta$ is the bargaining power of a producer. The trade surplus in the secondary bond market is $u(\hat{q}) - u(q) - \phi d_m$, where $u(\hat{q}) - u(q) - \phi d_b$ is the consumer's surplus and $d_b - d_m$ is the producer's surplus. An active agent's decision problem is

$$K(m_c, m_p, b_c, b_p) \equiv \max_{d_m, d_b} \left[u(\hat{q}) - u(q) - \phi d_m\right] \quad s.t. \quad (12) \text{ and } (13). \tag{14}$$

Note that if $K(m_c, m_p, b_c, b_p)$ is differentiable with respect to $x = m_c, m_p, b_c, b_p$, then

$$\frac{\partial K}{\partial x} = u'(\hat{q})\frac{\partial \hat{q}}{\partial x} - u'(q)\frac{\partial q}{\partial x} - \phi\frac{\partial d_m}{\partial x}. \tag{15}$$

If the budget constraints for producers and consumers (12) are non-binding; i.e., if $\phi m_p > \phi d_m$ and $\phi b_c > \phi d_b$, then the first-order condition of the maximization problem in (14) with respect to d_m is

$$u'(\hat{q})\frac{\partial \hat{q}}{\partial d_m} - \phi = \phi\left[u'(\hat{q}) - 1\right] = 0, \tag{16}$$

which means that active consumers can consume the efficient quantity, such that $u'(\hat{q}) = 1$. Finally, we can derive the value function of a consumer and a producer before entering the secondary bond market as

$$\begin{aligned} V_1^c(m_c, b_c) &= \delta\eta K(m_c, m, b_c, b) + V_2^c(m_c, b_c), \\ V_1^p(m_p, b_p) &= \delta_p(1-\eta)K(m, m_p, b, b_p) + V_2^p(m_p, b_p), \end{aligned} \tag{17}$$

when the trading partner has a portfolio (m, b).

5 Monetary Equilibrium

We focus on symmetric, stationary monetary equilibria, where all agents follow identical strategies and where real variables are constant over time. The gross growth rate of bonds is denoted by $\zeta \equiv B/B_{-1}$, and the gross growth rate of the money supply is denoted by $\gamma \equiv M/M_{-1}$. In

a stationary monetary equilibrium, the real stock of money and bonds must be constant; i.e., $\phi M = \phi_{+1} M_{+1}$ and $\phi B = \phi_{+1} B_{+1}$, which implies $\gamma = \zeta = \phi/\phi_{+1}$. Hence, we can rewrite the central bank's budget constraint (1) as

$$\gamma - 1 - \tau = \mathcal{B}\left(1 - \rho\gamma\right). \tag{18}$$

Hereafter, we assume that any change in the bond-to-money ratio $\mathcal{B}$ is offset through an adjustment in the lump-sum tax τ, such that (18) holds; i.e., any change in $\mathcal{B}$ has no effect on the inflation rate γ. Thus, we follow the literature that builds on Lagos and Wright (2005) and assume that the fiscal policy is purely passive.

In what follows, we present three stationary monetary equilibria. In the first equilibrium, labeled type-I, the producer's cash constraint and the consumer's bond constraint (12) are non-binding in the secondary bond market. In the second equilibrium, labeled type-II, the producer's cash constraint is binding and the consumer's bond constraint is non-binding. In the third equilibrium, labeled type-III, the producer's cash constraint is non-binding and the consumer's bond constraint is binding. All proofs are relegated to the Appendix.

5.1 Type-I Equilibrium

A type-I equilibrium is characterized by

$$\begin{aligned} u'(\hat{q}) &= 1, & (19)\\ \phi m_p &> \phi d_m, & (20)\\ \phi b_c &> \phi d_b. & (21) \end{aligned}$$

Equations (20) and (21) simply mean that the constraints of money and bond holdings are non-binding in the secondary bond market.

Proposition 1 *A type-I equilibrium is a list* $\{\hat{q}, q, q_p, \rho\}$ *satisfying (10) and*

$$\begin{aligned} 1 &= u'(\hat{q}), & (22)\\ \frac{\gamma}{\beta} &= (1-n)\delta\eta\left[u'(\hat{q}) - u'(q)\right] + (1-n)u'(q) + n, & (23)\\ \rho &= \frac{\beta}{\gamma}. & (24) \end{aligned}$$

Equation (22) is obtained from the first-order condition in the secondary bond market (16) and states that active agents consume the efficient quantity; i.e., such that $u'(\hat{q}) = 1$. Equation (23) is derived from the marginal value of money in the secondary bond market. The left-hand side of (23) represents the marginal cost of acquiring one additional unit of money in the primary bond market, and the right-hand side of the equation denotes the marginal benefit. With probability $1-n$ the agent will be a consumer in the goods market, in which case he has the marginal utility $u'(q)$. With probability $(1-n)\delta$ he will be an active consumer and additionally obtain a fraction

η of the surplus $[u'(\hat{q}) - u'(q)]$. Finally, with probability n the agent will be a producer in the goods market, in which case he receives a marginal utility of 1. Equation (24) is derived from the marginal value of bonds in the secondary bond market and states that bonds are priced at their fundamental value, β/γ, in the primary bond market.

5.2 Type-II Equilibrium

A type-II equilibrium is characterized by

$$\begin{aligned} u'(\hat{q}) &> 1, && (25)\\ \phi m_p &= \phi d_m, && (26)\\ \phi b_c &> \phi d_b. && (27) \end{aligned}$$

Equation (25) means that an active consumer does not consume the optimal amount of goods in the goods market, because the constraint on the producer's money holdings is binding in the secondary bond market (26). As in the type-I equilibrium, the meaning of (27) is that the constraint on the consumer's bond holdings is non-binding in the secondary bond market.

Proposition 2 *A type-II equilibrium is a list* $\{\hat{q}, q, q_p, \rho\}$ *satisfying (10) and*

$$\begin{aligned} \hat{q} &= 2q, && (28)\\ \frac{\gamma}{\beta} &= (1-n)\delta\eta\left[u'(\hat{q}) - u'(q)\right] + (1-n)u'(q) + n + n\delta^p(1-\eta)\left[u'(\hat{q}) - 1\right], && (29)\\ \rho &= \frac{\beta}{\gamma}. && (30) \end{aligned}$$

Equation (28) is a direct consequence of the active consumers' binding cash-constraint in the goods market and the fact that active producers transfer all their money to their trading partner in the secondary bond market. Equation (29) is similar to (23), except for the last term on the right-hand side of the equation. Because the cash-constraint of an active producer is binding in the secondary bond market, he can earn a strictly positive surplus on his money holdings $[u'(\hat{q}) - 1]$, according to his bargaining power $1 - \eta$. Like in the type-I equilibrium, bonds are priced at their fundamental value, (30), because the bond constraint of an active consumer is non-binding.

5.3 Type-III Equilibrium

A type-III equilibrium is characterized by

$$\begin{aligned} u'(\hat{q}) &> 1, && (31)\\ \phi m_p &> \phi d_m, && (32)\\ \phi b_c &= \phi d_b. && (33) \end{aligned}$$

Also in the type-III equilibrium, an active consumer does not consume the efficient quantity. Equation (32) means that the producer's cash-constraint is non-binding in the secondary bond market, while (33) means that the consumer's bond constraint is binding in the secondary bond market.

Proposition 3 *A type-III equilibrium is a list* $\{\hat{q}, q, q_p, \rho\}$ *satisfying (10) and*

$$\mathcal{B} = \frac{(1-\eta)[u(\hat{q})-u(q)]+\eta(\hat{q}-q)}{q}, \tag{34}$$

$$\frac{\gamma}{\beta} = (1-n)\left[\delta u'(\hat{q})\frac{\eta+(1-\eta)u'(q)}{\eta+(1-\eta)u'(\hat{q})}+(1-\delta)u'(q)\right]+n, \tag{35}$$

$$\rho = \frac{\beta}{\gamma}\left[1+(1-n)\delta\eta\frac{u'(\hat{q})-1}{\eta+(1-\eta)u'(\hat{q})}\right]. \tag{36}$$

Equation (34) is derived from the Kalai condition (13). Equation (35) is derived from the marginal value of money in the secondary bond market. With probability $(1-n)\delta$, an agent will be an active consumer, in which case he obtains a share of the surplus equal to $u'(\hat{q})\left[\eta+(1-\eta)u'(q)\right]\left[\eta+(1-\eta)u'(\hat{q})\right]^{-1}$. With probability $(1-n)(1-\delta)$, he will be a passive consumer and obtain the marginal utility $u'(q)$. With probability n, he will be a producer in the goods market, in which case he receives a marginal utility of 1, because his cash-constraint is non-binding in the secondary bond market. Equation (36) is derived from the marginal value of bonds in the secondary bond market and states that bonds exhibit a liquidity premium equal to $(1-n)\delta\eta\left[u'(\hat{q})-1\right]\left[\eta+(1-\eta)u'(\hat{q})\right]^{-1}$. The reason for this result is the binding bond constraint of consumers in the secondary bond market.[11]

6 Regions of Existence

Hereafter, we derive the regions of existence of each type of equilibrium with respect to the inflation rate γ and the bond-to-money ratio $\mathcal{B} = \frac{B}{M} = \frac{b}{m}$. We focus on an economy where the efficient allocation is not attainable; i.e, $\gamma > \beta$ and $\delta < 1$.[12] Note that $\phi b = \frac{b}{m}\phi m = \mathcal{B}\phi m = \mathcal{B}q$ holds in any type of equilibrium. The following three Propositions guarantee existence of equilibria, even though they might not be unique, which is clarified in the subsequent four lemmas.

[11] See, for instance, Geromichalos et al. (2014) for a more detailed analysis under what conditions a liquidity premium exists in the primary bond market.

[12] It is easy to show that under competitive pricing, the type-I equilibrium coincides with the type-I equilibrium under Kalai bargaining for $\eta = 1$ (see, for instance, Berentsen et al., 2014 or 2015b). With $\delta = 1$ and $\eta = 1$, the type-I equilibrium only exists at the Friedman rule ($\gamma_{12} = \beta$); i.e., the efficient allocation, $u'(q^*) = 1$, is only attainable at $\gamma = \beta$. In contrast, under Kalai bargaining, it holds that $q = \hat{q} = q^*$ for $\delta = 1$ and $\eta < 1$ for $\beta < \gamma < Min(\gamma_{12}, \gamma_{13})$, with $\gamma_{12} = \beta\left\{(1-n)\eta+(1-n)(1-\eta)u'(0.5q^*)+n\right\}$, and with $\gamma_{13} = \beta\left\{(1-n)\eta+(1-n)(1-\eta)u'(q)+n\right\}$, where $\mathcal{B}q = (1-\eta)[u(q^*)-u(q)]+\eta(q^*-q)$ by Proposition 4. Hence, for $\delta = 1$ and $\eta < 1$, the efficient allocation is attainable for inflation rates above the Friedman rule, which is shown by Geromichalos and Herrenbrück (2014).

Proposition 4 *There exists a constant* γ_{12} *and a function* $\mathcal{B}_{13}(\gamma)$ *such that the type-I equilibrium is supported if, and only if,* $\gamma < \gamma_{12}$ *and* $\mathcal{B} > \mathcal{B}_{13}(\gamma)$.

Proposition 5 *There exists a function* $\mathcal{B}_{23}(\gamma)$ *such that the type-II equilibrium is supported if, and only if,* $\gamma > \gamma_{12}$ *and* $\mathcal{B} > \mathcal{B}_{23}(\gamma)$.

Proposition 6 *There exists a function* $\mathcal{B}_{32}(\gamma)$ *such that the type-III equilibrium is supported if, and only if, (i)* $\gamma \leq \gamma_{12}$ *and* $\mathcal{B} < \mathcal{B}_{13}(\gamma)$*; or (ii)* $\gamma > \gamma_{12}$ *and* $\mathcal{B} < \mathcal{B}_{32}(\gamma)$.

The following lemma characterizes the properties of $\mathcal{B}_{13}(\gamma)$, $\mathcal{B}_{23}(\gamma)$, and $\mathcal{B}_{32}(\gamma)$.

Lemma 7 $\mathcal{B}_{13}(\gamma)$, $\mathcal{B}_{23}(\gamma)$, *and* $\mathcal{B}_{32}(\gamma)$ *satisfy the following properties;*

(i) $\mathcal{B}_{13}(\gamma)$ *is increasing in* $\gamma < \gamma_{12}$,

(ii) $\mathcal{B}_{23}(\gamma)$ *is increasing in* $\gamma > \gamma_{12}$,

(iii) $\mathcal{B}_{32}(\gamma)$ *is increasing in* $\gamma > \gamma_{12}$,

(iv) $\lim_{\gamma \to \gamma_{12}} \mathcal{B}_{13}(\gamma) = \lim_{\gamma \to \gamma_{12}} \mathcal{B}_{23}(\gamma) = \lim_{\gamma \to \gamma_{12}} \mathcal{B}_{32}(\gamma)$.

The functions and critical values described in Propositions 4-6 are visualized in Figure 2 for ease of understanding.

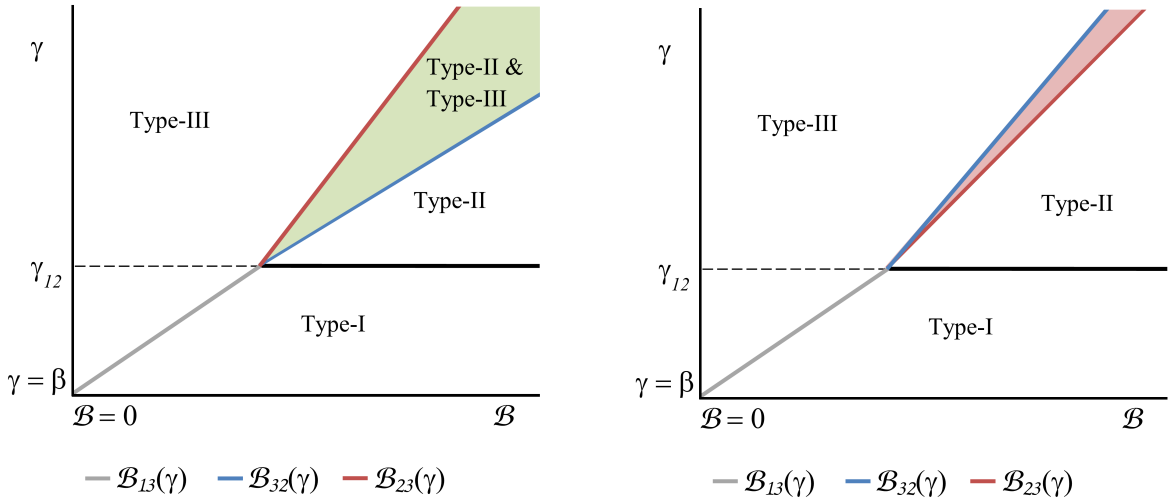

FIGURE 2: REGIONS OF EXISTENCE

The chart on the left-hand side of Figure 2 shows that there exists a region with $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$ for $\gamma > \gamma_{12}$. In such a case, the region between $\mathcal{B}_{23}(\gamma)$ and $\mathcal{B}_{32}(\gamma)$ supports the type-II and the

type-III equilibrium.[13] The chart on the right-hand side of Figure 2 shows that there exists a region with $\mathcal{B}_{32}(\gamma) < \mathcal{B}_{23}(\gamma)$ for large $\gamma > \gamma_{12}$. In such a case, the region between $\mathcal{B}_{32}(\gamma)$ and $\mathcal{B}_{23}(\gamma)$ does not support trading in the secondary financial market, so that active and passive agents consume the same quantity.[14]

The following two lemmas show the existence of an overlapping region supporting the type-II and the type-III equilibrium, which is highlighted in green in the chart on the left-hand side of Figure 2.

Lemma 8 *If $u'(2q) - \eta u'(q) + \eta > 0$ for all $q < \frac{\hat{q}}{2}$, then $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$ for all $\gamma > \gamma_{12}$.*

Lemma 9 *If there exists a constant $a < \frac{q^*}{2}$ with $u'(q^*) = 1$, such that $u'(2q) - \eta u'(q) + \eta > 0$ for all $a < q < \frac{q^*}{2}$, then $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$ for small $\gamma > \gamma_{12}$.*

The following lemma shows the existence of a region which does not support trading in the secondary financial market for large $\gamma > \gamma_{12}$. This region is highlighted in red in the chart on the right-hand side of Figure 2.

Lemma 10 *If there exists a constant a, such that $u'(2q) - \eta u'(q) + \eta < 0$ for all $q < a$, then $\mathcal{B}_{32}(\gamma) < \mathcal{B}_{23}(\gamma)$ for large $\gamma > \gamma_{12}$.*

7 Optimal Bond-to-Money Ratio

In this section, we present the main result of our paper; i.e., how to find the optimal bond-to-money ratio, which coincides with the optimal degree of QE. For this purpose, we first need to derive the welfare function:

$$(1-\beta)\mathcal{W} \equiv U(x^*) - x^* + (1-n)\left[\delta u(\hat{q}) + (1-\delta)u(q)\right] - nq_p, \tag{37}$$

where $U(x^*) - x^*$ denotes the agent's utility in the primary bond market, $(1-n)\left[\delta u(\hat{q}) + (1-\delta)u(q)\right]$ denotes the agent's expected utility in the goods market if he turns out to be a consumer, and $-nq_p$ denotes the expected utility of an agent if he becomes a producer in the goods market. Differentiating (37) with respect to $\mathcal{B}$ yields

$$(1-\beta)\frac{\partial \mathcal{W}}{\partial \mathcal{B}} = (1-n)\left[\delta u'(\hat{q})\frac{\partial \hat{q}}{\partial \mathcal{B}} + (1-\delta)u'(q)\frac{\partial q}{\partial \mathcal{B}}\right] - n\frac{\partial q_p}{\partial \mathcal{B}}. \tag{38}$$

[13] When $\mathcal{B}_{23}(\gamma) \leq \mathcal{B}_{32}(\gamma)$ for $\gamma > \gamma_{12}$, the region between $\mathcal{B}_{23}(\gamma)$ and $\mathcal{B}_{32}(\gamma)$ also supports an equilibrium in which the bond constraint of active consumers and the money constraint of active producers are binding simultaneously. This equilibrium is different from the type-II and the type-III equilibrium. It is a list $\{\hat{q}, q, q_p, \rho\}$ satisfying (10) and $\hat{q} = 2q$, $\mathcal{B}q = (1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q)$, $\frac{\gamma}{\beta} \leq (1-n)\delta\eta\left[u'(\hat{q}) - u'(q)\right] + (1-n)u'(q) + n + n\delta^p(1-\eta)\left[u'(\hat{q}) - 1\right]$, $\frac{\gamma}{\beta} \geq (1-n)\left[\delta u'(\hat{q})\frac{\eta+(1-\eta)u'(q)}{\eta+(1-\eta)u'(\hat{q})} + (1-\delta)u'(q)\right] + n$, $\rho \geq \frac{\beta}{\gamma}$, $\rho \leq \frac{\beta}{\gamma}\left[1 + (1-n)\delta\eta\frac{u'(\hat{q})-1}{\eta+(1-\eta)u'(\hat{q})}\right]$. The proof is available by request.

[14] More precisely, this equilibrium is a list $\{\hat{q}, q, q_p, \rho\}$ satisfying (10) and $\hat{q} = q$, $\frac{\gamma}{\beta} = (1-n)u'(q) + n$, $\rho = \frac{\beta}{\gamma}$.

In the type-I and type-II equilibrium, we always have $\frac{\partial \hat{q}}{\partial \mathcal{B}} = \frac{\partial q}{\partial \mathcal{B}} = \frac{\partial q_p}{\partial \mathcal{B}} = 0$, and hence $(1-\beta)\frac{\partial \mathcal{W}}{\partial \mathcal{B}} = 0$. In contrast, in the type-III equilibrium, welfare critically depends on the bond-to-money ratio. Proposition 4 formulates a condition under which it is optimal to increase or decrease the bond-to-money ratio.

Proposition 11 *Let $(\gamma, \mathcal{B})$ support the type-III equilibrium. (i) If $\Theta(\gamma, \mathcal{B}) > 0$, then welfare is increasing in $\mathcal{B}$; (ii) if $\Theta(\gamma, \mathcal{B}) < 0$, then welfare is decreasing in $\mathcal{B}$, where*

$$\Theta(\gamma, \mathcal{B}) \equiv \delta \left[u'(\hat{q}(\gamma, \mathcal{B})) - 1\right] \frac{\partial \hat{q}}{\partial \mathcal{B}} + (1-\delta) \left[u'(q(\gamma, \mathcal{B})) - 1\right] \frac{\partial q}{\partial \mathcal{B}},$$

$$\begin{aligned} \frac{\partial \hat{q}}{\partial \mathcal{B}} &= \frac{q(\gamma, \mathcal{B})}{\eta + (1-\eta)u'(\hat{q}(\gamma, \mathcal{B}))} + \frac{\mathcal{B} + \eta + (1-\eta)u'(q(\gamma, \mathcal{B}))}{\eta + (1-\eta)u'(\hat{q}(\gamma, \mathcal{B}))} \frac{\partial q}{\partial \mathcal{B}} > 0, \\ \frac{\partial q}{\partial \mathcal{B}} &= \frac{A(\hat{q}(\gamma, \mathcal{B}), q(\gamma, \mathcal{B}))}{B(\hat{q}(\gamma, \mathcal{B}), q(\gamma, \mathcal{B}), \mathcal{B})} < 0, \end{aligned}$$

with

$$\begin{aligned} A(\hat{q}, q) &\equiv -(1-n)\delta u''(\hat{q}) q \left[\eta + (1-\eta)u'(q)\right] \eta, \\ B(\hat{q}, q, \mathcal{B}) &\equiv (1-n)\delta u''(\hat{q}) \left[\mathcal{B} + \eta + (1-\eta)u'(q)\right] \left[\eta + (1-\eta)u'(q)\right] \eta \\ &\quad + (1-n)u''(q) \left[\eta + (1-\eta)u'(\hat{q})\right]^2 \left[\eta(1-\delta) + (1-\eta)u'(\hat{q})\right]. \end{aligned}$$

In Proposition 11, we show that the contribution of $\frac{\partial \hat{q}}{\partial \mathcal{B}} > 0$, while the contribution of $\frac{\partial q}{\partial \mathcal{B}} < 0$. Hence, the contribution of $\frac{\partial q_p}{\partial \mathcal{B}}$ depends on which of the two effects dominates. The reason for $\frac{\partial q}{\partial \mathcal{B}} < 0$ is as follows. In the secondary bond market, agents have the possibility to trade money for bonds after the realization of their idiosyncratic preference shock. The possibility to do so decreases the demand for money and hence its value. Decreasing the bond-to-money ratio helps to mitigate this externality in the type-III equilibrium and induces agents to increase their demand for money.

The following theorem states that if an economy is in the type-I equilibrium, then it is always optimal to decrease the bond-to-money ratio such that the bond constraint of active consumers becomes binding in the secondary bond market and the type-III equilibrium exists.

Theorem 12 *Let $(\gamma, \mathcal{B})$ support the type-I equilibrium. Then, welfare will be improved by decreasing $\mathcal{B}$.*

The reasoning behind the result stated in Theorem 12 is as follows. In the type-I equilibrium we have $\gamma < \gamma_{12}$ and $\mathcal{B} > \mathcal{B}_{13}(\gamma)$, thus decreasing the bond-to-money ratio will not affect welfare. However, at $\mathcal{B} = \mathcal{B}_{13}(\gamma)$, welfare will be further improved by decreasing $\mathcal{B}$. For ease of understanding, Figure 3 stylistically shows the evolution of welfare as a function of $\mathcal{B}$ for $\gamma < \gamma_{12}$.

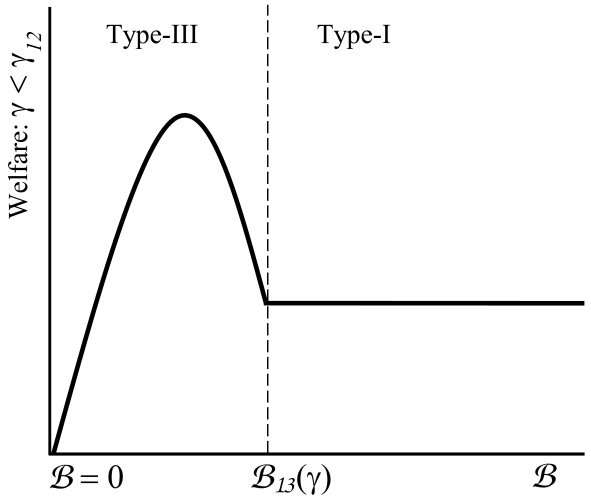

FIGURE 3: WELFARE AS A FUNCTION OF $\mathcal{B}$ FOR $\gamma < \gamma_{12}$

The finding presented in Theorem 12 is similar to the result in Berentsen et al. (2015b), who show that the demand for money is too low in the type-I equilibrium and that a financial transaction tax can correct this pecuniary externality.

8 Numerical Example

We illustrate the findings presented in the previous section with a numerical example. We set a model period to one year and choose the following functional forms for preferences and technology: $u(q) = q^{1-\alpha}/(1-\alpha)$, $c(q) = q$, and $U(x) = A\log(x)$. From Lemma 8 it follows that for $u(q) = q^{1-\alpha}/(1-\alpha)$, we need $\alpha \in (0,1)$ and $\eta \leq 2^{-\alpha}$ to have an overlapping region supporting the type-II and the type-III equilibrium for all $\gamma > \gamma_{12}$.[15] Otherwise, by Lemma 10, there exists a region which does not support trading in the secondary financial market for $\gamma > \gamma_{12}$. For the matching technology in the secondary bond market, we follow related studies and choose $\mathcal{M}(n, 1-n) = n(1-n)$, which implies $\delta = n$ and $\delta^p = 1-n$.

The following parameters need to be identified: (i) preference parameters: (β, A, α); (ii) the technology parameter: (n); (iii) bargaining power: (η); (iv) and policy parameters $(\mathcal{B}, \gamma)$.

The parameters are identified by using half a century of U.S. data from the first quarter of 1960 to the fourth quarter of 2010.[16] The parameters β, γ, $\mathcal{B}$, η, and n can be set equal to their target values. That is, we set $\beta = 0.97$ to replicate the real discount factor in the data, measured as the difference between the AAA corporate bond rate and the inflation rate, $\gamma = 1.04$. The

[15] Note, that for $u(q) = q^{1-\alpha}/(1-\alpha)$, $\alpha \in (0,1)$, and $\eta \leq 2^{-\alpha}$, we have $u'(2q) - \eta u'(q) + \eta = 2^{-\alpha}q^{-\alpha} - \eta q^{-\alpha} + \eta > 0$. Hence, $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$ for all $\gamma > \gamma_{12}$.

[16] All data sources are provided in the Appendix.

bond-to-money ratio is set to $\mathcal{B} = 3.03$, which is equal to the total amount of outstanding debt divided by M1 adjusted for retail sweeps.[17] Finally, we set $n = 0.5$ in order to maximize the number of matches and assume $\eta = 0.5$ for the baseline calibration. The remaining unknowns are A and α. We calibrate them simultaneously by matching the average level of money demand, denoted as $\mathcal{MD} = 0.17$, and its elasticity with respect to the AAA interest rate, denoted as $\xi = -0.43$.[18]

The model's money demand is

$$\mathcal{MD} = \frac{\phi M_{-1}}{Y} = \frac{\phi m}{A + (1-n)\left[\delta\phi\hat{m} + (1-\delta)\,\phi m\right]},$$

where $\phi m = \phi M_{-1} = q$ and $\phi\hat{m} = \hat{q}$.[19] The model's money demand depends on the interest rate in the primary bond market $i \equiv 1/\rho - 1$, and on α and A via $u(q)$ and $U(x)$. We estimate the model's elasticity of money demand with respect to i in the same way as its empirical counterpart; i.e., by ordinary least squares and a log-log specification. Table 2 presents the calibrated parameter values, where the model replicates both money demand targets exactly.

Table 2: Calibrated parameter values[a]

β	γ	$\mathcal{B}$	η	n	A	α
0.97	1.04	3.03	0.5	0.5	3.11	0.36

[a] Table 1 displays the calibrated parameter values. The parameters β, γ, B, η, and n are set equal to their target values, while A and α are obtained simultaneously by matching the average money demand and its elasticity with respect to the AAA corporate bond rate.

The calibrated parameter values allow us to obtain the critical values γ_{12} and $\mathcal{B}_{13}(\gamma)$, and to study the effect of a reduction in $\mathcal{B}$ on consumption and asset prices. In particular, we calculate the optimal value of the bond-to-money ratio, which solves $\Theta(\gamma, \mathcal{B}^*) = 0$ from Proposition 11. Having obtained $\mathcal{B}^*$ allows us to calculate the percentage of total consumption that agents would be willing to give up in order to be in a steady state with $\mathcal{B} = \mathcal{B}^*$ instead of $\mathcal{B} = 3.03$, which is denoted as $1 - \Delta$. Furthermore, we calculate the impact of this policy measure on the yield of bonds in the primary bond market, denoted as $\Delta_i = 1/\rho(\mathcal{B}^*) - 1/\rho(\mathcal{B})$. Finally, we also provide the share of goods market consumption on total consumption, labeled as s_{GM}. Table 3 gives a brief overview of our results.

[17] We follow the findings of Berentsen et al. (2015a) and use M1 adjusted for retail sweeps instead of M1, because it represents the stock of transaction media more accurately.

[18] Money demand is measured as the ratio of M1 adjusted for retail sweeps divided by the nominal gross domestic product. The elasticity of money demand with respect to the AAA interest rate is estimated by ordinary least squares and a log-log specification.

[19] The model's money demand is defined as the ratio of the real stock of money, $\phi M_{-1} = \phi m = q$, divided by the real output of the economy, Y. The latter term is obtained by the sum of the real output of the primary bond market, which equals A for $U(x) = A\log(x)$, and the real output of the goods market, which equals $(1-n)\left[\delta\phi\hat{m} + (1-\delta)\,\phi m\right]$.

TABLE 3: CALIBRATION RESULTS[a]

γ_{12}	$\mathcal{B}_{13}(\gamma)$	$\mathcal{B}^*$	$1-\Delta$	Δ_i	s_{GM}
1.07	0.73	0.50	0.00014	-0.0052	0.11

[a] Table 2 displays the calibration results. The term γ_{12} denotes the critical inflation rate that separates the type-I from the type-II equilibrium and the term $B_{13}(\gamma)$ denotes the critical bond-to-money ratio that separates the type-I from the type-III equilibrium at the calibrated value of $\gamma = 1.04$. The table also shows the optimal bond-to-money ratio, B^*, which is calculated such that $\Theta(\gamma, B^*) = 0$. It also shows the percentage of total consumption that agents would be willing to give up in order to be in a steady state with $B = B^*$, instead of $B = 3.03$, denoted as $1-\Delta$, and the effect on the yield of bonds of such a policy measure, denoted as Δ_i. Finally, it also shows the size of the goods market, s_{GM}.

Table 3 shows that $\gamma_{12} = 1.07 > 1.04$; i.e., that the U.S. economy is in the type-I equilibrium at the calibrated value of γ. Furthermore, we have $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$, since $\eta = 0.5 < 2^{-\alpha} = 0.78$, and thus there exists an overlapping region supporting the type-II and the type-III equilibrium for all $\gamma > \gamma_{12}$. We find that $\mathcal{B}^* = 0.50$, which is significantly lower than its empirical counterpart of $\mathcal{B} = 3.03$. The benefits associated with a policy of reducing $\mathcal{B}$ to $\mathcal{B}^*$ are relatively low and in the region of 0.014 percent of total consumption. However, reducing $\mathcal{B}$ to $\mathcal{B}^*$ has a major impact on the yield of bonds in the primary bond market; i.e., it reduces $i \equiv 1/\rho - 1$ by 0.52 percent. Finally, the share of goods market consumption on total consumption is, with 11 percent, in line with related studies.[20]

We now investigate how sensitive our results are with respect to the search and bargaining frictions in the secondary bond market. In order to do this, we provide two robustness-checks. Firstly, we keep $n = 0.5$ and recalibrate the model for $\eta = (0.1, 0.3, 0.7, 0.9)$. Secondly, we keep $\eta = 0.5$ and recalibrate the model for $n = (0.1, 0.3, 0.7, 0.9)$. In Figure 4, we show the development of $\mathcal{B}^*$ as a function of γ for the two robustness exercises as compared to the baseline calibration (solid black line).

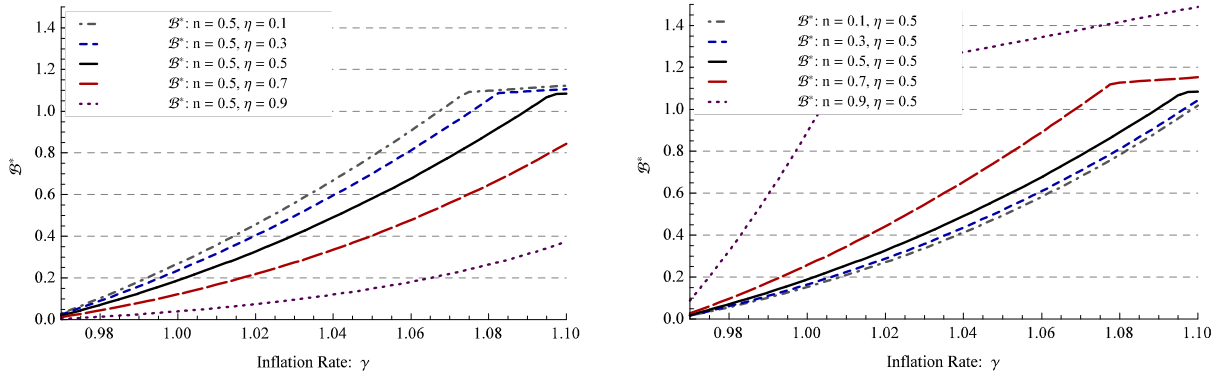

FIGURE 4: $\mathcal{B}^*$ AS A FUNCTION OF γ

[20] See, for instance, Lagos and Wright (2005), Aruoba et al. (2011), or Berentsen et al. (2014 and 2015b).

The chart on the left-hand side of Figure 4 shows that the lower the bargaining power of consumers in the secondary bond market, the higher the optimal bond-to-money ratio. That is, for $\eta = 0.1$, we find $\mathcal{B}^* = 0.68$ at the calibrated value of γ, while for $\eta = 0.9$, we obtain a value of $\mathcal{B}^* = 0.12$.

The chart on the right-hand side of the above figure shows that if search frictions are low in the secondary bond market, i.e., if n is high, then the optimal value of $\mathcal{B}^*$ is also high. That is, for $n = 0.1$, we find $\mathcal{B}^* = 0.42$ at the calibrated value of γ, while for $n = 0.9$, we obtain $\mathcal{B}^* = \mathcal{B}_{23}(\gamma) = 1.28$. Hence, for $n = 0.9$, the economy is in a region of the type-II equilibrium, where it is not welfare-improving to reduce $\mathcal{B}$. Furthermore, higher values of n result in lower values of γ_{12}. Note, however, that the kink in the above figures does not represent γ_{12}. Since welfare is continuous in each type of equilibrium and since welfare is maximized (in terms of $\mathcal{B}$) in the interior of the region of the type-III equilibrium for $\gamma = \gamma_{12}$, then it must also be maximized in the interior of the region of the type-III equilibrium, when $\gamma > \gamma_{12}$ and γ is sufficiently close to γ_{12} by above continuities. Hence, only when γ is considerably higher than γ_{12}, it is not optimal to reduce $\mathcal{B}$ in the type-II equilibrium, such that $\mathcal{B}^* = \mathcal{B}_{23}(\gamma)$.

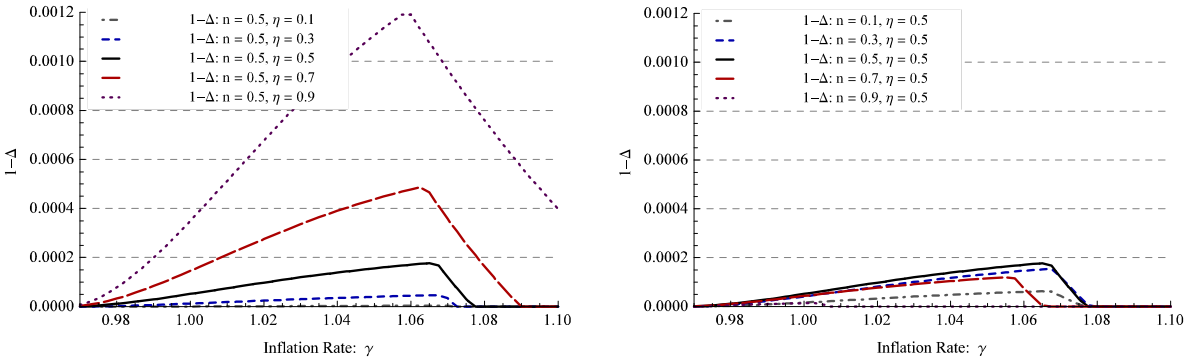

FIGURE 5: $1 - \Delta$ AS A FUNCTION OF γ

Figure 5 shows the development of $1 - \Delta$ as a function of γ. The left-hand side of Figure 5 shows that higher values of η result in higher benefits of reducing $\mathcal{B}$ to $\mathcal{B}^*$ in terms of total consumption. In particular, for $\eta = 0.1$, we find $1 - \Delta \approx 0.000$ percent at the calibrated value of γ, while for $\eta = 0.9$, we obtain a value of $1 - \Delta = 0.099$ percent. Comparing the left-hand side of Figure 5 to its counterpart in Figure 4 shows clearly that it can be optimal to reduce $\mathcal{B}$ in the type-II equilibrium if γ is sufficiently close to γ_{12}. In particular, the kink in Figure 5 represents γ_{12}, and the development of $1 - \Delta$ shows that the benefits of reducing $\mathcal{B}$ to $\mathcal{B}^*$ are the highest at $\gamma = \gamma_{12}$ and decreasing thereafter.

The right-hand side of Figure 5 shows that for higher values of n, the benefits of reducing $\mathcal{B}$ to $\mathcal{B}^*$ in terms of total consumption increase up to $n = 0.5$ and decrease thereafter. In particular, for $n = 0.1$, we find $1 - \Delta \approx 0.005$ percent at the calibrated value of γ, while for $n = 0.7$ we obtain a value of $1 - \Delta = 0.012$ percent, which is lower than the value obtained for $n = 0.5$ of $1 - \Delta = 0.014$ percent. The reason behind this result is that most agents are made worse off

for high values of n, while only a few passive agents can increase their consumption due to the higher value of money as a consequence of the reduction in $\mathcal{B}$. For $n = 0.9$, we find that it is optimal to reduce $\mathcal{B}$ only for $\gamma < 1.01$, where $\gamma_{12} = 1.00$.

In Figure 6, we show the effect of reducing $\mathcal{B}$ to $\mathcal{B}^*$ on the yield of bonds in the primary bond market, Δ_i. Again, we find that the effect of reducing $\mathcal{B}$ to $\mathcal{B}^*$ is most pronounced for high values of η or intermediate values of n.

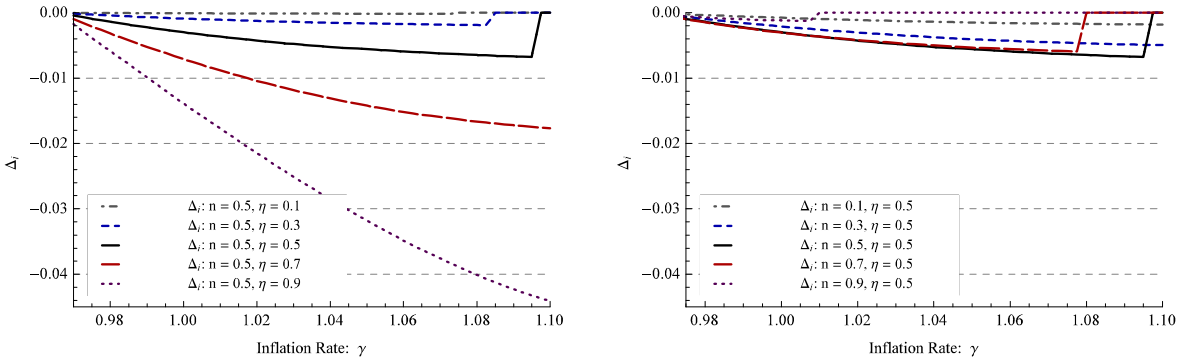

FIGURE 6: Δ_ρ AS A FUNCTION OF γ

The above analysis shows that a policy measure of reducing $\mathcal{B}$ to $\mathcal{B}^*$ improves the allocation and welfare if the bargaining power of consumers in the secondary bond market is sufficiently high. Furthermore, if search frictions are low, reducing $\mathcal{B}$ is only beneficial for low inflation rates and the gains in terms of total consumption are low, because most agents are made worse off and only a few benefit from the higher value of money.

8.1 Empirical Evidence

In the market for U.S. Treasuries, search frictions are low and mainly caused by delays due to the search for suitable counterparties.[21] Therefore, our baseline value of $n = 0.5$ is likely to be too low. A possibility to obtain a more realistic estimate of n is to replicate the effect on i caused by recent implementations of QE. D'Amico and King (2013) estimate that the central bank purchases of U.S. government bonds, which took place in 2009, were successful in reducing yields by around 0.3 percent. Thus, we use $\Delta_i = -0.003$ as a target to calibrate n. Furthermore, insights from experimental economics indicate that $\eta = 0.5$ is a reasonable assumption in bilateral anonymous meetings.[22] Therefore, we stick to the value of $\eta = 0.5$ used in the baseline calibration.

[21] See, for instance, Duffie et al. (2005) for a more detailed discussion about search frictions in financial markets or Krishnamurthy (2002) for the empirical price effects of search frictions in the market for U.S. Treasuries.

[22] Forsythe et al. (1994) conduct two experiments of ultimatum (take-it-or-leave-it) and dictator (take-it) games in the U.S. in 1988. They find that if subjects get paid, they tend to act "fairly" and share the pie evenly in the ultimatum games, which contradicts the subgame perfect Nash equilibrium (which implies an offer of 0). Furthermore, the authors find that their results are independent of the size of the pie. Roth et al. (1991) conduct ultimatum games in Israel, Japan, the U.S., and Yugoslavia in 1989 and 1990. They find that the proposals made

Due to the new target, we need to calibrate one more parameter simultaneously. That is, we calibrate A, α, and n simultaneously by matching the average level of money demand, $\mathcal{MD} = 0.17$, its elasticity with respect to the AAA interest rate, $\xi = -0.43$, and the effect of QE on the yield of bonds, $\Delta_i = -0.003$. Table 4 summarizes the calibration results.

TABLE 3: EMPIRICAL EVIDENCE - CALIBRATION RESULTS[a]

n	γ_{12}	$\mathcal{B}_{13}(\gamma)$	$\mathcal{B}^*$	$1-\Delta$	Δ_i	s_{GM}
0.76	1.04	1.18	0.95	0.00002	-0.0040	0.07

[a] Table 4 is Table 3's counterpart for a calibration that also targets Δ_i in order to estimate n. For a description of the reported variables, we refer the reader to Table 3.

Following the new calibration procedure, we are still able to hit the money demand targets perfectly, but the closest we can get to the target of Δ_i is $\Delta_i = -0.0040$. Table 3 shows that the estimate of n increases from $n = 0.5$ to $n = 0.76$. This results in an optimal bond-to-money ratio of $\mathcal{B}^* = 0.95$, which is nearly twice as high as in the baseline calibration, where we obtained a value of $\mathcal{B}^* = 0.50$. We find that $\gamma_{12} = 1.04$ and that QE is only welfare-improving for inflation rates below 4.3 percent. Furthermore, the benefits associated with such a policy are low; i.e., agents are only willing to give up $1 - \Delta = 0.002$ percent of total consumption in order to be in a steady state with $\mathcal{B} = \mathcal{B}^*$ instead of $\mathcal{B} = 3.03$.

9 Conclusion

We develop a general equilibrium model, where agents can trade money for government bonds in a secondary financial market, which features search and bargaining frictions. The possibility to do so reduces the incentive to self-insure against liquidity shocks, and as a consequence agents rely on the liquidity provision by other market participants. In such an environment, QE can be welfare-improving for low inflation rates. The reason is that such a policy measure reduces the bond-to-money ratio, which results in a price of bonds above the fundamental value. Hence, bonds become scarce and less attractive, which induces agents to increase their demand for money. In turn, this marginally increases the value of money and so improves the allocation and welfare.

We calibrate the model to U.S. data and find that for a realistic parameterization with low search frictions and equally distributed bargaining power among producers and consumers, QE proves competent in reducing the yield of bonds by around 0.40 percent. However, the benefits of such a policy measure in terms of total consumption are likely to be small and in the area of 0.002 percent. Furthermore, QE is only successful for inflation rates below 4.3 percent.

by bargainers are the highest in the U.S. and Yugoslavia with a modal proposal of 50 percent of the pie and the lowest in Japan and Israel with a modal proposal of 40 percent of the pie. Furthermore, the authors find that the size of the pie does not affect their results in a meaningful way. Slonim and Roth (1998) conduct an ultimatum game in the Slovak Republic in 1994. They find that subjects offer between 41.5 percent and 44 percent and that a bigger size of the pie results in lower offers when players have the possibility to gain experience.

10 Appendix - For Online Publication

10.1 Proofs

Proof of Proposition 1. *Derivation of (22).* Since the short selling constraints (12) are not binding in a type-I equilibrium, it is obvious from (16) that (22) holds.

Derivation of (23). The marginal value of money of an agent at the beginning of a period can be written as follows:

$$\frac{\partial V_1}{\partial m} = (1-n)\frac{\partial V_1^p}{\partial m} + n\frac{\partial V_1^c}{\partial m}.$$

Using (17), we can rewrite the above equation as

$$\frac{\partial V_1}{\partial m} = (1-n)\left[\delta\eta\frac{\partial K}{\partial m_c} + \frac{\partial V_2^c}{\partial m_c}\right] + n\left[\delta_p(1-\eta)\frac{\partial K}{\partial m_p} + \frac{\partial V_2^p}{\partial m_p}\right],$$

where

$$\frac{\partial K}{\partial m_c} := \frac{\partial K(m_c, m_p, b_c, b_p)}{\partial m_c}\mid_{(m_c,m_p,b_c,b_p)=(m,m,b,b)},$$
$$\frac{\partial K}{\partial m_p} := \frac{\partial K(m_c, m_p, b_c, b_p)}{\partial m_p}\mid_{(m_c,m_p,b_c,b_p)=(m,m,b,b)}.$$

We can use the envelope condition of a consumer in the goods market, (8), to replace $\frac{\partial V_2^c}{\partial m_c}$, and (11) to replace $\frac{\partial V_2^p}{\partial m_p}$. Furthermore, in a symmetric equilibrium, we have $\frac{\partial K}{\partial m_c} = \phi - \phi u'(q)$ and $\frac{\partial K}{\partial m_p} = 0$, because $\frac{\partial \hat{q}}{\partial m_c} = 0$, $\frac{\partial q}{\partial m_c} = \frac{1}{p} = \phi$, $\frac{\partial d_m}{\partial m_c} = -1$, and $\frac{\partial \hat{q}}{\partial m_p} = \frac{\partial q}{\partial m_p} = \frac{\partial d_m}{\partial m_p} = 0$ with (15). Using these expressions to replace $\frac{\partial K}{\partial m_c}$ and $\frac{\partial K}{\partial m_p}$ and (5) updated one period, we get (23).

Derivation of (24). Following the same procedure as in the derivation of (23), we can rewrite the marginal value of bonds at the beginning of a period as

$$\frac{\partial V_1}{\partial b} = (1-n)\left[\delta\eta\frac{\partial K}{\partial b_c} + \frac{\partial V_2^c}{\partial b_c}\right] + n\left[\delta_p(1-\eta)\frac{\partial K}{\partial b_p} + \frac{\partial V_2^p}{\partial b_p}\right].$$

Using (8) to replace $\frac{\partial V_2^c}{\partial b_c}$, (11) to replace $\frac{\partial V_2^p}{\partial b_p}$, $\frac{\partial K}{\partial b_p} = \frac{\partial K}{\partial b_c} = 0$, and (5) updated one period, we get (24). ■

Proof of Proposition 2. *Derivation of (28).* Since the cash-constraint of active and passive consumers in the goods market is binding, we have $m_c + d_m = p\hat{q}$ for an active consumer and $m_c = pq$ for a passive consumer. Furthermore, because active producers are cash-constrained in the secondary bond market, we have $m_p = d_m = M$. Using $m_p = m_c = M$ and rearranging terms, we obtain (*28*).

Derivation of (29). The marginal value of money of an agent at the beginning of a period

can be written as

$$\frac{\partial V_1}{\partial m} = (1-n)\left[\delta\eta\frac{\partial K}{\partial m_c} + \frac{\partial V_2^c}{\partial m_c}\right] + n\left[\delta_p(1-\eta)\frac{\partial K}{\partial m_p} + \frac{\partial V_2^p}{\partial m_p}\right],$$

where we used (17). We can use the envelope condition of a consumer in the goods market, (8), to replace $\frac{\partial V_2^c}{\partial m_c}$ and (11) to replace $\frac{\partial V_2^p}{\partial m_p}$. In a symmetric equilibrium, we have $\frac{\partial K}{\partial m_c} = \phi u'(\hat{q}) - \phi u'(q)$ and $\frac{\partial K}{\partial m_p} = \phi u'(\hat{q}) - \phi$, because $\frac{\partial \hat{q}}{\partial m_c} = \frac{\partial q}{\partial m_c} = \frac{1}{p} = \phi$, $\frac{\partial d_m}{\partial m_c} = 0$, and $\frac{\partial \hat{q}}{\partial m_p} = \phi$, $\frac{\partial q}{\partial m_p} = 0$ and $\frac{\partial d_m}{\partial m_p} = 1$ with (15). Using these expressions to replace $\frac{\partial K}{\partial m_c}$ and $\frac{\partial K}{\partial m_p}$ and (5) updated one period, we get (23).[23]

Derivation of (30). The derivation of (30) is equal to the derivation of (24) and is not repeated here. ■

Proof of Proposition 3. *Derivation of (34).* Since the bond constraint of active consumers is binding in the secondary bond market, we have $d_b = b$. Since the cash-constraint of active and passive consumers is binding in the goods market, we have $m_c + d_m = p\hat{q}$ and $m_c = pq$. Using these expressions and the first-order condition of producers in the goods market, $p\phi = 1$, in (13), we obtain (34).

Derivation of (35). The marginal value of money of an agent at the beginning of a period can be written as

$$\frac{\partial V_1}{\partial m} = (1-n)\left[\delta\eta\frac{\partial K}{\partial m_c} + \frac{\partial V_2^c}{\partial m_c}\right] + n\left[\delta_p(1-\eta)\frac{\partial K}{\partial m_p} + \frac{\partial V_2^p}{\partial m_p}\right],$$

where we used (17). We can use the envelope condition of a consumer in the goods market, (8), to replace $\frac{\partial V_2^c}{\partial m_c}$ and (11) to replace $\frac{\partial V_2^p}{\partial m_p}$. Furthermore, we can use $\frac{\partial K}{\partial m_p} = 0$, because $\frac{\partial \hat{q}}{\partial m_p} = \frac{\partial q}{\partial m_p} = \frac{\partial d_m}{\partial m_p} = 0$ with (15),[24] to obtain

$$\frac{\partial V_1}{\partial m} = (1-n)\left[\delta\eta\frac{\partial K}{\partial m_c} + \phi u'(q)\right] + n\phi. \tag{39}$$

Next, we need to derive $\frac{\partial K}{\partial m_c}$. By (15), we have

$$\frac{\partial K}{\partial m_c} = u'(\hat{q})\frac{\partial \hat{q}}{\partial m_c} - u'(q)\frac{\partial q}{\partial m_c} - \phi\frac{\partial d_m}{\partial m_c}. \tag{40}$$

[23] Since $\hat{q} = \phi(m_c + d_m) = \phi(m_c + m_p)$ and $q = \phi m_p$, $\frac{\partial \hat{q}}{\partial m_c} = \frac{\partial \hat{q}}{\partial m_p} = \frac{\partial q}{\partial m_c} = \phi$. Since $d_m = m_p$ is binding, equation (25) holds. Thus, if a buyer has a little more (or less) m_c, agents will still trade $d_m = m_p$. Hence $\frac{\partial d_m}{\partial m_c} = 0$. If a producer has a little more (or less) m_p, then he will still trade the whole m_p. Hence $\frac{\partial d_m}{\partial m_p} = 1$. $\frac{\partial q}{\partial m_p} = 0$ is obvious.

[24] It is obvious that $\frac{\partial \hat{q}}{\partial m_p} = \frac{\partial q}{\partial m_p} = \frac{\partial d_m}{\partial m_p} = 0$, since $\hat{q}$, q, and d_m do not depend on the producer's money holdings, because the producer's cash constraint is non-binding in the type-III equilibrium.

Next, we have to find $\frac{\partial \hat{q}}{\partial m_c}$ and $\frac{\partial d_m}{\partial m_c}$. By (13), we have

$$(1-\eta)\left[u'(\hat{q})\frac{\partial \hat{q}}{\partial m_c} - u'(q)\frac{\partial q}{\partial m_c}\right] = -\eta\phi\frac{\partial d_m}{\partial m_c} \tag{41}$$

and since $\hat{q} = \phi(m_c + d_m)$, $q = \phi m_c$, we also have

$$\frac{\partial \hat{q}}{\partial m_c} = \phi + \phi\frac{\partial d_m}{\partial m_c} \qquad \text{and} \qquad \frac{\partial q}{\partial m_c} = \phi. \tag{42}$$

By solving the system of equations (41) and (42), we can derive

$$\frac{\partial \hat{q}}{\partial m_c} = \phi\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})}, \tag{43}$$

$$\frac{\partial d_m}{\partial m_c} = \frac{1-\eta}{\phi\eta}\left[u'(\hat{q})\frac{\partial \hat{q}}{\partial m_c} - u'(q)\frac{\partial q}{\partial m_c}\right]. \tag{44}$$

Using (43) and (44) in (40), we have

$$\begin{aligned}\frac{\partial K}{\partial m_c} &= u'(\hat{q})\frac{\partial \hat{q}}{\partial m_c} - u'(q)\frac{\partial q}{\partial m_c} + \frac{1-\eta}{\eta}\left[u'(\hat{q})\frac{\partial \hat{q}}{\partial m_c} - u'(q)\frac{\partial q}{\partial m_c}\right] \\ &= \frac{1}{\eta}\left[u'(\hat{q})\frac{\partial \hat{q}}{\partial m_c} - u'(q)\frac{\partial q}{\partial m_c}\right] = \frac{1}{\eta}\left[\phi u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} - \phi u'(q)\right],\end{aligned}$$

where we have used $\frac{\partial q}{\partial m_c} = \phi$. Using the above equation in (39) and (5) updated one period, we obtain (35).

Derivation of (36). The marginal value of bonds at the beginning of a period can be written as

$$\frac{\partial V_1}{\partial b} = (1-n)\left[\delta\eta\frac{\partial K}{\partial b_c} + \frac{\partial V_2^c}{\partial b_c}\right] + n\left[\delta_p(1-\eta)\frac{\partial K}{\partial b_p} + \frac{\partial V_2^p}{\partial b_p}\right]. \tag{45}$$

By (34), we can find

$$\frac{\partial \hat{q}}{\partial b} = \frac{\phi}{(1-\eta)u'(\hat{q}) + \eta} \qquad \text{and} \qquad \frac{\partial d_m}{\partial b} = \frac{1}{(1-\eta)u'(\hat{q}) + \eta},$$

since $\mathcal{B} = \frac{\phi b}{\phi m} = \frac{\phi b}{q}$. Using the above equations to replace $\frac{\partial \hat{q}}{\partial b_c}$ and $\frac{\partial d_m}{\partial b_c}$, $\frac{\partial q}{\partial b_c} = 0$, and $\frac{\partial \hat{q}}{\partial b_p} = \frac{\partial q}{\partial b_p} = \frac{\partial d_m}{\partial b_p} = \frac{\partial q}{\partial b_c} = 0$, we obtain

$$\begin{aligned}\frac{\partial K}{\partial b_c} &= \phi\frac{u'(\hat{q}) - 1}{(1-\eta)u'(\hat{q}) + \eta}, \\ \frac{\partial K}{\partial b_p} &= 0.\end{aligned}$$

Using the two above equations in (45), (8) to replace $\frac{\partial V_2^c}{\partial b_c}$, (11) to replace $\frac{\partial V_2^p}{\partial b_p}$, and (5) updated one period, we obtain (36). ■

Proof of Proposition 4. In the type-I equilibrium, agents must hold enough money and bonds to support the optimal amount of consumption for active consumers, $u'(\hat{q}) = 1$. First, consider the following system of equations

$$\begin{aligned} u'(\hat{q}) &= 1, \\ \hat{q} &= 2q, \\ \frac{\gamma}{\beta} &= (1-n)\delta\eta + (1-n)(1-\delta\eta)u'(q) + n, \end{aligned}$$

with three variables $\hat{q}$, q, and γ. Let $\hat{q}_{12}$, q_{12}, and γ_{12} be the solution of the system of equations. The existence of the solution is immediate. By construction, γ_{12} is the threshold of the inflation rate at which the cash constraint of active producers is just binding, $\phi m_p = \phi d_m$. From the proof of proposition 2, we know that $\hat{q} = 2q$. Hence $\gamma \leq \gamma_{12}$ must hold in the type-I equilibrium.

Furthermore, the bond-to-money ratio must be high enough in order to support $u'(\hat{q}) = 1$. Hence, for a given $\gamma \leq \gamma_{12}$, consider the following system of equations

$$\begin{aligned} u'(\hat{q}) &= 1, \\ \mathcal{B}q &= (1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q), \\ \frac{\gamma}{\beta} &= (1-n)\delta\eta + (1-n)(1-\delta\eta)u'(q) + n, \end{aligned}$$

with three variables $\hat{q}$, q, and $\mathcal{B}$. Let $\hat{q}_{13}(\gamma)$, $q_{13}(\gamma)$, and $\mathcal{B}_{13}(\gamma)$ be the solution of the system of equations. The existence of the solution is immediate, since u' is continuous and $u'(\infty) = 0$. By construction, $\mathcal{B}_{13}(\gamma)$ is the threshold of the bond-to-money ratio at which the bond constraint of active consumers is just binding. Hence, $\mathcal{B} \geq \mathcal{B}_{13}(\gamma)$ must hold in the type-I equilibrium. ■

Proof of Proposition 5. First, consider the following system of equations,

$$\begin{aligned} u'(\hat{q}) &= 1, \\ \hat{q} &= 2q, \\ \frac{\gamma}{\beta} &= (1-n)\delta u'(\hat{q}) + (1-n)(1-\delta\eta)u'(q) - (1-n)\delta(1-\eta) + n, \end{aligned}$$

with three variables $\hat{q}$, q, and γ. Let $\hat{q}_{21}$, q_{21}, and γ_{21} be the solution of the system of equations. With $u'(\hat{q}) = 1$, we have

$$\frac{\gamma}{\beta} = (1-n)(1-\delta\eta)u'(q) + (1-n)\delta\eta + n.$$

Hence the solution exists, and $\gamma_{21} = \gamma_{12}$. By the construction of $\gamma_{21} = \gamma_{12}$, it follows that γ_{12} is the threshold of the inflation rate at which active consumers can consume the optimal amount of goods, $u'(\hat{q}) = 1$. Hence $\gamma \geq \gamma_{12}$ must hold in the type-II equilibrium.

In the type-II equilibrium, the bond-to-money ratio must be high enough such that the bond constraint is not binding in the secondary bond market. Hence, given that $\gamma \geq \gamma_{12}$, we consider the following system of equations

$$\begin{aligned}\hat{q} &= 2q,\\ \mathcal{B}q &= (1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q),\\ \frac{\gamma}{\beta} &= (1-n)\delta u'(\hat{q}) + (1-n)(1-\delta\eta)u'(q) - (1-n)\delta(1-\eta) + n,\end{aligned}$$

with three variables $\hat{q}$, q, and $\mathcal{B}$. Let $\hat{q}_{23}(\gamma)$, $q_{23}(\gamma)$, and $\mathcal{B}_{23}(\gamma)$ be the solution of the system of equations. The existence of the solution is immediate, since u' is continuous and $u'(0) = \infty$. By construction, $\mathcal{B}_{23}(\gamma)$ is the threshold of the bond-to-money ratio at which the bond constraint is just binding conditional on the given inflation rate γ. Hence $\mathcal{B} > \mathcal{B}_{13}(\gamma)$ must hold in the type-II equilibrium. ■

Proof of Proposition 6. *Case $\gamma \leq \gamma_{12}$:* Given $\gamma \leq \gamma_{12}$, consider the following system of equations,

$$\begin{aligned}u'(\hat{q}) &= 1,\\ \mathcal{B}q &= (1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q),\\ \frac{\gamma}{\beta} &= (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} + (1-n)(1-\delta)u'(q) + n,\end{aligned}$$

with three variables $\hat{q}$, q, and $\mathcal{B}$. Let $\hat{q}_{31}(\gamma)$, $q_{31}(\gamma)$, and $\mathcal{B}_{31}(\gamma)$ be the solution of the system of equations. With $u'(\hat{q}) = 1$, we have

$$\frac{\gamma}{\beta} = (1-n)(1-\delta\eta)u'(q) + (1-n)\delta\eta + n.$$

Hence the solution exists, and $\mathcal{B}_{31}(\gamma) = \mathcal{B}_{13}(\gamma)$. By the construction of $\mathcal{B}_{31}(\gamma) = \mathcal{B}_{13}(\gamma)$, it follows that $\mathcal{B}_{13}(\gamma)$ is the threshold of the bond-to-money ratio at which active consumers can consume the optimal amount of goods $u'(\hat{q}) = 1$. Hence, $\mathcal{B} \leq \mathcal{B}_{13}(\gamma)$ must hold in the type-III equilibrium conditional on $\gamma \leq \gamma_{12}$.

Case $\gamma > \gamma_{12}$: Given $\gamma > \gamma_{12}$, consider the system of equations,

$$\begin{aligned}\hat{q} &= 2q,\\ \mathcal{B}q &= (1-\eta)\left[u(\hat{q}) - u(q)\right] + \eta(\hat{q} - q),\\ \frac{\gamma}{\beta} &= (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} + (1-n)(1-\delta)u'(q) + n\end{aligned}$$

with three variables $\hat{q}$, q, and $\mathcal{B}$. Let $\hat{q}_{32}(\gamma)$, $q_{32}(\gamma)$, and $\mathcal{B}_{32}(\gamma)$ be the solution of the system of equations. The existence of the solution is immediate, since u' is continuous, $\lim_{q\to 0}\frac{(1-n)\delta u'(2q)}{\eta+(1-\eta)u'(2q)} = \frac{(1-n)\delta}{(1-\eta)} \in \mathbb{R}$, and $u'(0) = \infty$. By construction, $\mathcal{B}_{32}(\gamma)$ is the threshold of the bond-to-money ratio

at which the cash constraint in the secondary bond market is also binding. Hence, $\mathcal{B} < \mathcal{B}_{32}(\gamma)$ must hold in the type-III equilibrium conditional on $\gamma > \gamma_{12}$. ■

Proof of Lemma 7. *(i):* Note that $\hat{q}_{13}(\gamma)$, $q_{13}(\gamma)$, and $\mathcal{B}_{13}(\gamma)$ are the solution of the system of equations

$$\begin{aligned} u'(\hat{q}) &= 1, \\ \mathcal{B}q &= (1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q), \\ \frac{\gamma}{\beta} &= (1-n)\delta\eta + (1-n)(1-\delta\eta)u'(q) + n. \end{aligned}$$

From the last equation, it is easy to see that q_{13} is decreasing in γ. From the first and the second equation, we have

$$\frac{\partial \mathcal{B}}{\partial q} = \frac{(1-\eta)(-u'(q) - \eta)}{q} - \frac{(1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q)}{q^2} < 0.$$

Therefore, as γ increases, q_{13} decreases and $\mathcal{B}_{13}(\gamma)$ increases.

(ii): Note that $\hat{q}_{23}(\gamma)$, $q_{23}(\gamma)$, and $\mathcal{B}_{23}(\gamma)$ are the solution of the system of equations

$$\begin{aligned} \hat{q} &= 2q, \\ \mathcal{B}q &= (1-\eta)[u(\hat{q}) - u(q)] + \eta(\hat{q} - q), \\ \frac{\gamma}{\beta} &= (1-n)\delta u'(\hat{q}) + (1-n)(1-\delta\eta)u'(q) - (1-n)\delta(1-\eta) + n. \end{aligned}$$

From the first and the last equation, it is easy to see that q_{23} is decreasing in γ. From the first and the second equation, we have

$$\frac{q^2}{1-\eta}\frac{\partial \mathcal{B}}{\partial q} = [2qu'(2q) - qu'(q)] - [u(2q) - u(q)] = [2qu'(2q) - u(2q)] - [qu'(q) - u(q)].$$

Note that $xu'(x) - u(x)$ is decreasing in x, since its derivative is

$$u'(x) + u''(x) - u'(x) < 0.$$

Hence $\frac{\partial \mathcal{B}}{\partial q} < 0$. Therefore, as γ increases, q_{23} decreases and $\mathcal{B}_{23}(\gamma)$ increases.

(iii): Note that $\hat{q}_{32}(\gamma)$, $q_{32}(\gamma)$, and $\mathcal{B}_{32}(\gamma)$ are the solution of the system of equations

$$\begin{aligned} \hat{q} &= 2q, \\ \mathcal{B}q &= (1-\eta)\left[u(\hat{q}) - u(q)\right] + \eta(\hat{q} - q), \\ \frac{\gamma}{\beta} &= (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} + (1-n)(1-\delta)u'(q) + n. \end{aligned}$$

From the first and the last equation, we have

$$\frac{\gamma}{\beta} = (1-n)\delta u'(2q)\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(2q)} + (1-n)(1-\delta)u'(q) + n.$$

If we differentiate the right-hand side of the above equation, then we obtain

$$\begin{aligned}
&2(1-n)\delta u''(2q)\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(2q)} + (1-n)(1-\delta)u''(q) \\
&+ (1-n)\delta u'(2q)\frac{(1-\eta)u''(q)}{\eta + (1-\eta)u'(2q)} - (1-n)\delta u'(2q)\frac{2[\eta + (1-\eta)u'(q)](1-\eta)u''(2q)}{[\eta + (1-\eta)u'(2q)]^2}.
\end{aligned}$$

Multiplying the above formula by $[\eta + (1-\eta)u'(2q)]^2$, we obtain

$$\begin{aligned}
&2(1-n)\delta u'''(q)][\eta + (1-\eta)u'(2q)] \\
&+ (1-n)(1-\delta)u'''(2q)]^2 \\
&+ (1-n)\delta u'(2q)(1-\eta)u'''(2q)] \\
&- 2(1-n)\delta u'(2q)[\eta + (1-\eta)u'(q)](1-\eta)u''(2q) \\
&= 2(1-n)\delta u'''(q)]\eta \\
&+ (1-n)(1-\delta)u'''(2q)]^2 \\
&+ (1-n)\delta u'(2q)(1-\eta)u'''(2q)] < 0.
\end{aligned}$$

The last inequality follows immediately from $u'' < 0$. Hence q_{32} is decreasing in γ. From the first and the second equation, we have

$$\frac{\partial \mathcal{B}}{\partial q} < 0,$$

as is shown in *(ii)*. Therefore, as γ increases, q_{32} decreases and $\mathcal{B}_{32}(\gamma)$ increases.

(iv): The proof is immediate by noticing that the following three formulas are identical when $u'(\hat{q}) = 1$:

$$\begin{aligned}
&(1-n)\delta\eta + (1-n)(1-\delta\eta)u'(q) + n \\
&= (1-n)\delta u'(\hat{q}) + (1-n)(1-\delta\eta)u'(q) - (1-n)\delta(1-\eta) + n \\
&= (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} + (1-n)(1-\delta)u'(q) + n.
\end{aligned}$$

■

Proof of Lemma 8. Given $\gamma > \gamma_{12}$, we have

$$\begin{aligned}\frac{\gamma}{\beta} &= (1-n)\delta u'(2q_{23}(\gamma)) + (1-n)(1-\delta\eta)u'(q_{23}(\gamma)) - (1-n)\delta(1-\eta) + n \\ &= (1-n)\delta u'(2q_{32}(\gamma))\frac{\eta + (1-\eta)u'(q_{32}(\gamma))}{\eta + (1-\eta)u'(2q_{32}(\gamma))} + (1-n)(1-\delta)u'(q_{32}(\gamma)) + n.\end{aligned}$$

Let

$$\begin{aligned}f_2(q) &\equiv (1-n)\delta u'(2q) + (1-n)(1-\delta\eta)u'(q) - (1-n)\delta(1-\eta) + n, \\ f_3(q) &\equiv (1-n)\delta u'(2q)\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(2q)} + (1-n)(1-\delta)u'(q) + n,\end{aligned}$$

then

$$f_2(q_{23}(\gamma)) = f_3(q_{32}(\gamma)) = \frac{\gamma}{\beta}.$$

Note that

$$f_2(q) - f_3(q) = \frac{(1-n)\delta(1-\eta)}{\eta + (1-\eta)u'(2q)}\left[u'(2q) - 1\right]\left[u'(2q) - \eta u'(q) + \eta\right].$$

Since by assumption $u'(2q) - \eta u'(q) + \eta > 0$, we have $f_2(q) > f_3(q)$. Moreover, because

$$\frac{\partial f_2}{\partial q} = 2(1-n)\delta u''(2q) + (1-n)(1-\delta\eta)u''(q) < 0,$$

we must have

$$q_{23}(\gamma) > q_{32}(\gamma).$$

Otherwise, we would have a contradiction, $\frac{\gamma}{\beta} = f_2(q_{23}(\gamma)) > f_3(q_{32}(\gamma)) = \frac{\gamma}{\beta}$.

By (13), the bond-to-money ratio is

$$\mathcal{B} = (1-\eta)\frac{u(2q) - u(q)}{q} + \eta.$$

Hence, if we differentiate $\mathcal{B}$ with respect to q, we obtain

$$\frac{\partial \mathcal{B}}{\partial q} = \frac{2u'(2q) - u'(q)}{q} - \frac{u(2q) - u(q)}{q^2} < 0,$$

as we showed in the proof of Lemma 7. Therefore $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$, because $q_{23}(\gamma) > q_{32}(\gamma)$. ■

Proof of Lemma 9. As $\gamma > \gamma_{12}$ approaches γ_{12}, $q_{32}(\gamma) < \frac{q^*}{2}$ approaches $\frac{q^*}{2}$. Hence, for a given a, we have $q_{32}(\gamma) > a$ for a sufficiently small γ. From the previous proof, we have

$$f_2(q) - f_3(q) = \frac{(1-n)\delta(1-\eta)}{\eta + (1-\eta)u'(2q)}\left[u'(2q) - 1\right]\left[u'(2q) - \eta u'(q) + \eta\right].$$

For a small γ, so for $q_{32}(\gamma) > a$, we have $u'(2q_{32}(\gamma)) - \eta u'(q_{32}(\gamma)) + \eta > 0$. Therefore, it holds that $f_2(q_{32}(\gamma)) > f_3(q_{32}(\gamma))$. Because also $\frac{\partial f_2}{\partial q} < 0$, and

$$\frac{\gamma}{\beta} = f_2(q_{23}(\gamma)) = f_3(q_{32}(\gamma)) = \frac{\gamma}{\beta},$$

we must have

$$q_{23}(\gamma) > q_{32}(\gamma).$$

Otherwise, we would have a contradiction, $\frac{\gamma}{\beta} = f_2(q_{23}(\gamma)) > f_3(q_{32}(\gamma)) = \frac{\gamma}{\beta}$. At last, we have

$$\frac{\partial \mathcal{B}}{\partial q} = \frac{2u'(2q) - u'(q)}{q} - \frac{u(2q) - u(q)}{q^2} < 0.$$

Since $q_{23}(\gamma) > q_{32}(\gamma) > a$, we must have $\mathcal{B}_{32}(\gamma) > \mathcal{B}_{23}(\gamma)$. ■

Proof of Lemma 10. As γ approaches ∞, $q_{32}(\gamma)$ approaches zero. Hence for a given a, $q_{32}(\gamma) < a$ for a sufficiently large γ. From the previous proof, we have

$$f_2(q) - f_3(q) = \frac{(1-n)\delta(1-\eta)}{\eta + (1-\eta)u'(2q)} \left[u'(2q) - 1\right] \left[u'(2q) - \eta u'(q) + \eta\right].$$

For large γ; i.e., for $q_{32}(\gamma) < a$, we have $u'(2q_{32}(\gamma)) - \eta u'(q_{32}(\gamma)) + \eta < 0$. Therefore, it holds that $f_2(q_{32}(\gamma)) < f_3(q_{32}(\gamma))$. Because also $\frac{\partial f_2}{\partial q} < 0$, and

$$\frac{\gamma}{\beta} = f_2(q_{23}(\gamma)) = f_3(q_{32}(\gamma)) = \frac{\gamma}{\beta},$$

we must have

$$q_{23}(\gamma) < q_{32}(\gamma).$$

Otherwise, we would have a contradiction, $\frac{\gamma}{\beta} = f_2(q_{23}(\gamma)) < f_3(q_{32}(\gamma)) = \frac{\gamma}{\beta}$. At last, we have

$$\frac{\partial \mathcal{B}}{\partial q} = \frac{2u'(2q) - u'(q)}{q} - \frac{u(2q) - u(q)}{q^2} < 0.$$

Since $q_{23}(\gamma) < q_{32}(\gamma) < a$, we must have $\mathcal{B}_{32}(\gamma) < \mathcal{B}_{23}(\gamma)$. ■

Proof of Proposition 11. Consider any $(\gamma, \mathcal{B})$ that supports the type-III equilibrium. Note that $\hat{q} = \hat{q}(\gamma, \mathcal{B})$, $q = q(\gamma, \mathcal{B})$, and $q_p = q_p(\gamma, \mathcal{B})$ are completely determined by $(\gamma, \mathcal{B})$ in the type-III equilibrium, because $\hat{q}$, q and q_p are solutions of the system of equations,

$$nq_p = (1-n)\left[\delta\hat{q} + (1-\delta)q\right] \tag{46}$$

$$\mathcal{B}q = (1-\eta)\left[u(\hat{q}) - u(q)\right] + \eta(\hat{q} - q), \tag{47}$$

$$\frac{\gamma}{\beta} = (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} + (1-n)(1-\delta)u'(q) + n. \tag{48}$$

By differentiating equation (46) with respect to $\mathcal{B}$, we obtain

$$n\frac{\partial q_p}{\partial \mathcal{B}} = (1-n)\left[\delta\frac{\partial \hat{q}}{\partial \mathcal{B}} + (1-\delta)\frac{\partial q}{\partial \mathcal{B}}\right].$$

Hence, we can simplify (38) as

$$(1-\beta)\frac{\partial \mathcal{W}}{\partial \mathcal{B}} = (1-n)\left\{\delta\left[u'(\hat{q})-1\right]\frac{\partial \hat{q}}{\partial \mathcal{B}} + (1-\delta)\left[u'(q)-1\right]\frac{\partial q}{\partial \mathcal{B}}\right\}.$$

By differentiating equation (47) with respect to $\mathcal{B}$, we have

$$q + \mathcal{B}\frac{\partial q}{\partial \mathcal{B}} = (1-\eta)\left[u'(\hat{q})\frac{\partial \hat{q}}{\partial \mathcal{B}} - u'(q)\frac{\partial q}{\partial \mathcal{B}}\right] + \eta\left(\frac{\partial \hat{q}}{\partial \mathcal{B}} - \frac{\partial q}{\partial \mathcal{B}}\right);$$

i.e., by rearranging the above equation we obtain

$$\frac{\partial \hat{q}}{\partial \mathcal{B}} = \frac{q}{\eta + (1-\eta)u'(\hat{q})} + \frac{\mathcal{B} + \eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})}\frac{\partial q}{\partial \mathcal{B}}. \tag{49}$$

By differentiating equation (48) with respect to $\mathcal{B}$, we have

$$\begin{aligned} 0 = {} & (1-n)\delta u''(\hat{q})\frac{\partial \hat{q}}{\partial \mathcal{B}}\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} + (1-n)\delta u'(\hat{q})\frac{(1-\eta)u''(q)}{\eta + (1-\eta)u'(\hat{q})}\frac{\partial q}{\partial \mathcal{B}} \\ & - (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{[\eta + (1-\eta)u'(\hat{q})]^2}(1-\eta)u''(\hat{q})\frac{\partial \hat{q}}{\partial \mathcal{B}} + (1-n)(1-\delta)u''(q)\frac{\partial q}{\partial \mathcal{B}}. \end{aligned} \tag{50}$$

With the equations (49) and (50), we have

$$\begin{aligned} 0 = {} & (1-n)\delta u''(\hat{q})\left[\frac{q}{\eta + (1-\eta)u'(\hat{q})} + \frac{\mathcal{B} + \eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})}\frac{\partial q}{\partial \mathcal{B}}\right]\frac{\eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})} \\ & + (1-n)\delta u'(\hat{q})\frac{(1-\eta)u''(q)}{\eta + (1-\eta)u'(\hat{q})}\frac{\partial q}{\partial \mathcal{B}} \\ & - (1-n)\delta u'(\hat{q})\frac{\eta + (1-\eta)u'(q)}{[\eta + (1-\eta)u'(\hat{q})]^2}(1-\eta)u''(\hat{q})\left[\frac{q}{\eta + (1-\eta)u'(\hat{q})} + \frac{\mathcal{B} + \eta + (1-\eta)u'(q)}{\eta + (1-\eta)u'(\hat{q})}\frac{\partial q}{\partial \mathcal{B}}\right] \\ & + (1-n)(1-\delta)u''(q)\frac{\partial q}{\partial \mathcal{B}}. \end{aligned}$$

By factoring $\frac{\partial q}{\partial \mathcal{B}}$, we have

$$
\begin{aligned}
&-(1-n)\delta u''(\hat{q})\frac{q}{\eta+(1-\eta)u'(\hat{q})}\frac{\eta+(1-\eta)u'(q)}{\eta+(1-\eta)u'(\hat{q})}\\
&+(1-n)\delta u'(\hat{q})\frac{\eta+(1-\eta)u'(q)}{\left[\eta+(1-\eta)u'(\hat{q})\right]^2}(1-\eta)u''(\hat{q})\frac{q}{\eta+(1-\eta)u'(\hat{q})}\\
&=\left\{\begin{array}{c}
(1-n)\delta u''(\hat{q})\frac{\mathcal{B}+\eta+(1-\eta)u'(q)}{\eta+(1-\eta)u'(\hat{q})}\frac{\eta+(1-\eta)u'(q)}{\eta+(1-\eta)u'(\hat{q})}\\
+(1-n)\delta u'(\hat{q})\frac{(1-\eta)u''(q)}{\eta+(1-\eta)u'(\hat{q})}\\
-(1-n)\delta u'(\hat{q})\frac{\eta+(1-\eta)u'(q)}{\left[\eta+(1-\eta)u'(\hat{q})\right]^2}(1-\eta)u''(\hat{q})\frac{\mathcal{B}+\eta+(1-\eta)u'(q)}{\eta+(1-\eta)u'(\hat{q})}\\
+(1-n)(1-\delta)u''(q)
\end{array}\right\}\frac{\partial q}{\partial \mathcal{B}}.
\end{aligned}
$$

If we multiply the above equation by $\left[\eta+(1-\eta)u'(\hat{q})\right]^3$, we obtain

$$
\begin{aligned}
&-(1-n)\delta u''(\hat{q})q\left[\eta+(1-\eta)u'(q)\right]\left[\eta+(1-\eta)u'(\hat{q})\right]\\
&+(1-n)\delta u'(\hat{q})\left[\eta+(1-\eta)u'(q)\right](1-\eta)u''(\hat{q})q\\
&=\left\{\begin{array}{c}
(1-n)\delta u''(\hat{q})\left[\mathcal{B}+\eta+(1-\eta)u'(q)\right]\left[\eta+(1-\eta)u'(q)\right]\left[\eta+(1-\eta)u'(\hat{q})\right]\\
+(1-n)\delta u'(\hat{q})(1-\eta)u''(q)\left[\eta+(1-\eta)u'(\hat{q})\right]^2\\
-(1-n)\delta u'(\hat{q})\left[\eta+(1-\eta)u'(q)\right](1-\eta)u''(\hat{q})\left[\mathcal{B}+\eta+(1-\eta)u'(q)\right]\\
+(1-n)(1-\delta)u''(q)\left[\eta+(1-\eta)u'(\hat{q})\right]^3
\end{array}\right\}\frac{\partial q}{\partial \mathcal{B}}.
\end{aligned}
$$

After simplifying the above equation, we have

$$
\begin{aligned}
&-(1-n)\delta u''(\hat{q})q\left[\eta+(1-\eta)u'(q)\right]\eta\\
&=\left\{\begin{array}{c}
(1-n)\delta u''(\hat{q})\left[\mathcal{B}+\eta+(1-\eta)u'(q)\right]\left[\eta+(1-\eta)u'(q)\right]\eta\\
+(1-n)u''(q)\left[\eta+(1-\eta)u'(\hat{q})\right]^2\left[\eta(1-\delta)+(1-\eta)u'(\hat{q})\right]
\end{array}\right\}\frac{\partial q}{\partial \mathcal{B}}.
\end{aligned}
$$

Hence, we can denote

$$\frac{\partial q}{\partial \mathcal{B}}=\frac{A(\hat{q},q)}{B(\hat{q},q,\mathcal{B})},$$

where

$$
\begin{aligned}
A(\hat{q},q)&\equiv-(1-n)\delta u''(\hat{q})q\left[\eta+(1-\eta)u'(q)\right]\eta,\\
B(\hat{q},q,\mathcal{B})&\equiv(1-n)\delta u''(\hat{q})\left[\mathcal{B}+\eta+(1-\eta)u'(q)\right]\left[\eta+(1-\eta)u'(q)\right]\eta\\
&\quad+(1-n)u''(q)\left[\eta+(1-\eta)u'(\hat{q})\right]^2\left[\eta(1-\delta)+(1-\eta)u'(\hat{q})\right].
\end{aligned}
$$

Note that since $u''<0$, we have

$$A(\hat{q},q)>0,\qquad\text{and}\qquad B(\hat{q},q,\mathcal{B})<0.$$

Hence, $\frac{\partial q}{\partial \mathcal{B}} < 0$. Moreover,

$$\begin{aligned}\left[\eta + (1-\eta)u'(\hat{q})\right] B(\hat{q}, q, \mathcal{B}) \frac{\partial \hat{q}}{\partial \mathcal{B}} &= qB(\hat{q}, q, \mathcal{B}) + \left[\mathcal{B} + \eta + (1-\eta)u'(q)\right] A(\hat{q}, q) \\ &= q(1-n)u''(q)\left[\eta + (1-\eta)u'(\hat{q})\right]^2 \left[\eta(1-\delta) + (1-\eta)u'(\hat{q})\right] < 0.\end{aligned}$$

Therefore, $\frac{\partial \hat{q}}{\partial \mathcal{B}} > 0$. At last, let

$$\begin{aligned}\Theta(\gamma, \mathcal{B}) \equiv{}& \delta\left[u'(\hat{q}(\gamma, \mathcal{B})) - 1\right] \frac{\partial \hat{q}}{\partial \mathcal{B}} + (1-\delta)\left[u'(q(\gamma, \mathcal{B})) - 1\right] \frac{\partial q}{\partial \mathcal{B}} \\ ={}& \delta\left[u'(\hat{q}(\gamma, \mathcal{B})) - 1\right] \frac{q(\gamma, \mathcal{B})}{\eta + (1-\eta)u'(\hat{q}(\gamma, \mathcal{B}))} \\ &+ \left\{\delta\left[u'(\hat{q}(\gamma, \mathcal{B})) - 1\right] \frac{\mathcal{B} + \eta + (1-\eta)u'(q(\gamma, \mathcal{B}))}{\eta + (1-\eta)u'(\hat{q}(\gamma, \mathcal{B}))} + (1-\delta)\left[u'(q(\gamma, \mathcal{B})) - 1\right]\right\} \\ &\times \frac{A(\hat{q}(\gamma, \mathcal{B}), q(\gamma, \mathcal{B}))}{B(\hat{q}(\gamma, \mathcal{B}), q(\gamma, \mathcal{B}), \mathcal{B})},\end{aligned}$$

then $\frac{\partial \mathcal{W}}{\partial \mathcal{B}} = \frac{1-n}{1-\beta}\Theta(\gamma, \mathcal{B})$. Thus, if $\Theta(\gamma, \mathcal{B}) > 0$, then welfare will be improved by increasing $\mathcal{B}$, and if $\Theta(\gamma, \mathcal{B}) < 0$, then welfare will be improved by decreasing $\mathcal{B}$. ■

Proof of Theorem 12. By Proposition 4, $\gamma < \gamma_{12}$, and $\mathcal{B} > \mathcal{B}_{13}(\gamma)$. Decreasing the bond-to-money ratio $\mathcal{B}$ up to $\mathcal{B}_{13}(\gamma)$ will not change welfare, since it supports the type-I equilibrium. But at $\mathcal{B} = \mathcal{B}_{13}(\gamma)$,

$$\Theta(\gamma, \mathcal{B}_{13}(\gamma)) = (1-\delta)\left[u'(q(\gamma, \mathcal{B}_{13}(\gamma))) - 1\right] \frac{A(\hat{q}(\gamma, \mathcal{B}_{13}(\gamma)), q(\gamma, \mathcal{B}_{13}(\gamma)))}{B(\hat{q}(\gamma, \mathcal{B}_{13}(\gamma)), q(\gamma, \mathcal{B}_{13}(\gamma)), \mathcal{B}_{13}(\gamma))} < 0.$$

The inequality comes from $A(\hat{q}, q) > 0$, $B(\hat{q}, q, \mathcal{B}) < 0$, and $u'(q) - 1 > 0$. Welfare will be improved by decreasing $\mathcal{B}$ further at $\mathcal{B}_{13}(\gamma)$ by Proposition 11. ■

10.2 Data Source

All the data that we used for the calibration is downloadable from the Federal Reserve Bank of St. Louis FRED® database. Table A.1 gives a brief overview of the respective identifiers.

Table A.1: Data source

Description	Identifier	Period
AAA Moody's corporate bond	AAA	60:Q1-10:Q4
Consumer price index	CPIAUCSL	60:Q1-10:Q4
U.S. total public debt	GFDEBTN	66:Q1-10:Q4
M1 money stock	M1SL	60:Q1-66:Q4
M1 sweep-adjusted money stock	M1ADJ	67:Q1-10:Q4
Nominal GDP	GDP	60:Q1-10:Q4

Because the time series of the total public debt is only available from 1966:Q1, we construct the quarterly data in the period from 1960:Q1 to 1965:Q4 with the data provided by http://www.treasurydirect.gov/govt/reports/pd/mspd/mspd.htm. To obtain the quarterly data, we apply the same aggregation method as the Federal Reserve Bank of St. Louis FRED® database, which is defined as the average of the monthly data. Since the time series of the sweep-adjusted M1 is only available from 1967:Q1, we use the time series of M1 (identifier M1SL) in the period from 1960:Q1 to 1966:Q4.

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