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Variable Retirement and the Effects of Social Insurance on Savings, Wealth, and Welfare

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Abstract:

We construct a Blanchard-style overlapping generations model consisting of long-lived individuals who have uninsurable idiosyncratic risk resulting from uncertain retirement periods and medical costs in retirement. Without social insurance, such individuals must save for these eventualities. We examine the impact of pay-as-you-go social insurance policies (public pensions and medicare coverage) on individual and aggregate consumption, saving, and wealth levels as well as wealth distribution. We also derive expressions for optimal (Pareto improving) social insurance policies.

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1. Introduction

Social insurance programs constitute the largest and most rapidly growing domestic spending programs in most industrialized economies. In the United States, for example, the two main social insurance programs are Social Security and Medicare, both of which provide benefits to retirees financed by payroll taxes on working generations, a financing scheme commonly called pay-as-you-go (PAYG). In recent years, these programs have generated increasing interest among policymakers, mainly due to doubts about their long-run solvency, given present financing arrangements, but also to concerns about the impact that such programs may have on private saving and capital formation. Ever since the pioneering work of Samuelson (1958), economists have used some form of overlapping generations model to analyze the impacts of these intergenerational tax-transfer programs. One difficulty with the Samuelson-style overlapping generations model has been its tractability. In most cases, it is difficult to express aggregate savings in a transparent form that reveals the impact of social insurance policies in economies having realistic demographic characteristics, such as long and uncertain lifetimes for households. Another has been the relative neglect of the risk-sharing aspects of programs such as Social Security and Medicare, in favor of their inter temporal and intergenerational consumption shifting aspects.[1]

In this paper we develop a tractable overlapping generations model based on the Blanchard (1985) finite horizon model. This provides a convenient framework to analyze the impact of social insurance policies on saving and consumption rates, the level of wealth and its distribution, both by workers and retirees, as well as the corresponding economy-wide aggregates. The model is also used to analyze welfare effects and to characterize optimal social insurance policies. Unlike the standard version of the Blanchard model, where life-cycle saving motives are introduced by assuming that each worker's earnings decline exponentially over time, we assume stochastic retirement (loss of human capital earnings) and the presence of uncertain medical expenses upon retirement. Medical expenses and retirement consumption can be funded either by a PAYG social insurance program, or by private savings by workers. All risks are idiosyncratic, so that an

[1] The risk sharing aspects of social security have been analyzed by Bohn (1999), Rust (1999), Shiller (1998) and Storesletten et al (2004), among others.

important feature of the analysis is its integration of the risk-sharing aspects of social insurance policies with their life-cycle consumption-shifting properties.

As in Blanchard (1985) and Yaari (1965), we assume that private insurance companies (an annuity industry) supply actuarially fair contracts that agents use to insure against the risk of uncertain lifetimes (mortality risk). While this assumption is important for rendering such models tractable, it limits the scope for analyzing social insurance policies in a meaningful way because perfect annuities markets obviate the need for social security in the first place[2]. In our model, however, workers still face an uncertain period of future retirement in which they will have no earnings, and they are unable to insure against this future loss of earnings or the uncertain retirement medical expenses. Although such risks are idiosyncratic and can be fully pooled, we assume that unspecified insurance market failures prevent private insurance markets from mediating the risk of lost of future earnings and medical expenses.[3]

Using a Constant Absolute Risk Aversion (CARA) preference function, we characterize equilibrium savings and consumption behavior by workers and retirees in terms of: (i) social insurance policy parameters, such as social security benefit and Medicare co-payment rates; (ii) demographic factors, such as the dependency rate, life-span, and the fraction of an average adult life spent in retirement; (iii) preference parameters, such as the degree of risk aversion/inter-temporal substitution; and (iv) economic variables, such as earnings and the real interest rate. Our analysis leads to a number of findings that we summarize in a series of propositions. First, individual savings and consumption levels, as well as their respective growth rates, fall upon retirement, except when social insurance is "complete", in which case they remain unchanged. Second, social insurance reduces the level of aggregate savings, and can do so by an amount that exceeds the additional dollar value of their benefits (that is, crowding out of private saving by social insurance can exceed 100%). Third, we characterize the optimal social insurance policy and show that a reduction in social insurance from the optimum increases wealth concentration in the economy. However, after some

[2] The assumption of a "real annuities" market in the Blanchard model contradicts one of the key welfare arguments for having social security—the absence (until the hedging arrangements made possible by the recent introduction of inflation indexed treasury bonds) of privately-provided real annuity financial instruments.

[3] One could construct such an argument in our model by having two or more classes of agents with different retirement risks. If information about individual risk is private knowledge, adverse selection can lead to a missing market.

point, further reductions in social insurance can decrease wealth concentration. Fourth, the optimal social insurance level is complete only at the golden rule point; if the rate of return on capital exceeds the population growth rate, optimal social insurance is incomplete.

Our approach differs from other contributions in the social insurance literature in several key respects. Like Diamond and Mirrlees (1978, 1986), we allow for a variable and stochastic retirement period. But Diamond and Mirrlees consider a two period model in which agents exercise some control over the risk of retirement in the second period, creating a social insurance moral hazard, a condition we preclude. Like Auerbach and Kotlikoff (1987), we consider agents who have long and uncertain lifetimes. However, the 55-period Samuelson-type overlapping-generations model they employ, while useful for simulating the effects of changes in social insurance policy, is intractable for deriving an exact decision rule that can be used to derive the impact of policy and other parameters on aggregate savings. Gertler (1999) does incorporate stochastic retirement of the type used in this paper to analyze the effects of social insurance. He assumes a recursive form of utility function and numerically simulates the effects of policy rather than deriving exact savings expressions, as we do using the CARA utility function. Our paper also relates to other studies that explore features of the wealth distribution implied by savings motivated by life-cycle considerations (Huggett, 1996) and savings motivated by uninsured idiosyncratic risk (Aiyagari, 1994). Combined life-cycle and uninsured idiosyncratic risk motives stimulate saving in our analysis, and we explore the impact of social insurance on the resulting wealth distribution.[4]

The remainder of the paper proceeds as follows. Section 2 lays out the stochastic environment. In section 3, we derive the optimal consumption plans for members of retired and working cohorts, and consider the impact of social insurance policy – both public pension benefits and Medicare benefits – on individual consumption and saving. Sections 4 to 6 then derive expressions for aggregate saving, wealth and consumptions levels in the economy, as well as for the distribution of wealth. We determine the impact of changes in social insurance policy variables and demographic parameters on these variables. In section 7, we derive expressions for optimal and

[4] Huggett and Ventura (1999) analyze the distributional effects of social security reform on a variety of economic measures, including welfare and consumption.

Pareto improving social insurance policies. In section 8, we provide some brief conclusions and discuss some caveats. Insofar as possible technical details are relegated to an Appendix.

2. The Stochastic Environment

Our framework extends the well known Blanchard (1985) model to a situation in which each agent faces two sources of risk: (i) the probability of death, and (ii) the probability of losing his/her human capital earnings, either through retirement or through becoming permanently disabled.

As in Blanchard, the mortality hazard rate, denoted by p, is constant over time. The probability density function, describing time until death, for an individual born at time v, is given by the exponential distribution function, $pe^{-p(t-v)}$. Thus, the probability of dying in the time interval (v,t) is $\int_v^t pe^{-p(s-v)}ds = 1-e^{-p(t-v)}$, implying that the probability of being alive at time t is $e^{-p(t-v)}$.

As mentioned, households are also at risk of losing their earnings (labor income), an event that we will refer to as "retirement". Analogous to Blanchard's "perpetual youth" assumption, we assume that the retirement hazard rate, denoted π, is constant through life. The probability density function describing time until retirement for an individual born at time v is $\pi e^{-\pi(t-v)}$, so that the probability of retiring in the time interval (v,t) is $\int_v^t \pi e^{-\pi(s-v)}ds = 1-e^{-\pi(t-v)}$. Thus, conditional on survival, the probability of being employed at time t is $e^{-\pi(t-v)}$.

We assume that the two distributions generating p and π are independent, hence the probability of being alive and working at time t is $e^{-(p+\pi)(t-v)}$. An immediate consequence of these exponential distributions is that $1/p$ is the individual's expected lifetime, and $\frac{1}{p+\pi}$ is his/her expected working horizon, so that $\left(1/p - 1/(p+\pi)\right) = (1/p)\cdot\left(\pi/(p+\pi)\right)$ measures the expected length of retirement for someone who has not yet retired. Thus, if $\pi > 0$, every individual in the working phase expects a positive period of retirement where $\pi/(p+\pi)$ is the fraction of the individual's remaining life that he/she can expect to be retired.

The specification of these probability distributions by the exponential function is not meant to capture reality, but, as Blanchard notes, is key to rendering the aggregation – a crucial element of the model – tractable. By interpreting π as representing the probability of losing human capital

earnings, either through retirement or disability, the assumption that it is independent of age is more palatable. On balance, we find the specification of random termination of employment by an exponential function to be no more objectionable than it is for death. Overall, we subscribe to the view, effectively expressed by Blanchard, that the payoffs in terms of tractability and insights offered justify the obvious unrealism of the exponential function.

3. Individual Behavior

Working individuals anticipate a life-cycle comprising two phases; in the first the individual is a worker, and the second, the individual is retired.[5] We begin by determining the savings and consumption behavior of individual agents, proceeding sequentially. We consider the behavior of retirees, and then, given their expected behavior in retirement, determine their decisions as workers. We assume that individuals maximize their expected lifetime utility with no bequests. As in Blanchard, there is individual but no aggregate uncertainty with respect to population size, thus providing scope for insurance in the form of premium payments to the living from the estates of the dying, which raises the private rate of return on capital by the mortality rate. *p*.

3.1 Retirees

Consider an individual who retires at precisely time *t*, having accumulated financial assets (capital), $a(t)$, while employed. These assets pay an instantaneous rate of return, $r + p$, where *r* is the real rate of return on capital (assumed fixed) and *p* is the premium due to the presence of the insurance scheme. Upon the termination of his/her employment, at time *t*, the individual begins to receive constant social security benefits, *b*, at each point of time. We assume that the present value of medical expenses needed to maintain an individual's health during retirement is stochastic, and denoted by the random variable $\tilde{m}$ At the time they retire, individuals learn the state of their retirement health and thus learn their future medical expenses with certainty. We shall let $\eta \in [0,1]$ denote the individual's share of his/her retirement medical expenses.

Conditional upon retirement at time *t*, and the realization of known future certain medical

[5] Nonetheless, some fraction of workers in the economy will die before retiring.

expenses, *m*, an individual chooses a retirement consumption path to maximize

$$E\left\{U^R(t)\right\} = \int_t^{\infty} U\left(c^R(z,t)\cdot e^{-(p+\delta)(z-t)}dz \qquad \delta > 0 \right. \tag{1a}$$

subject to the accumulation equation

$$\dot{a}^R(z,t) = (r+p)\cdot\left(a^R(z,t) - \eta\cdot m\right) + b - c^R(z,t) \tag{1b}$$

where $a^R(z,t)$ denotes the assets and $c^R(z,t)$ denotes the consumption at time $z \geq t$ of an individual who retires at time *t*.[6] Note that uncertainty about future consumption for a retiree comes solely from the probability of death, and for the exponential distribution this is incorporated by augmenting the agent's time discount rate from δ to $\delta + p$ in (1a). Note further, that in writing the accumulation equation, we let $\dot{a}^R(z,t) \equiv \partial a^R(z,t)/\partial z$ and $(r+p)\cdot\eta\cdot m$ denotes the (constant) known flow of private retirement medical expenses.

Maximizing (1a) subject to (1b) with respect to $c^R(z,t)$ and $a^R(z,t)$, yields the conventional optimality conditions

$$U'(c^R) = \lambda^R \tag{2a}$$

$$r = \delta - \frac{\dot{\lambda}^R}{\lambda^R} \tag{2b}$$

where $\lambda^R(z,t)$ is the shadow value of wealth to a retiree. In addition, the transversality condition $\lim_{z\to\infty} \lambda^R(z,t)a^R(z,t)e^{-(p+\delta)(z-t)} = 0$ must hold. Equations (2a) and (2b) can be combined to yield[7]

$$\frac{-U''\left(c^R(z,t)\right)}{U'\left(c^R(z,t)\right)}\cdot\dot{c}^R(z,t) = r - \delta\,. \tag{3}$$

where $\dot{c}^R(z,t) \equiv \partial c^R(z,t)/\partial z$.

Henceforth, we shall restrict ourselves to the CARA utility function, $U = -\beta e^{-\gamma c}$, where γ is the constant coefficient of absolute risk aversion. In this case (3) simplifies to:

$$\dot{c}^R(z,t) = \frac{r-\delta}{\gamma} \tag{3'}$$

[6] The superscript *R* refers to "retirement"

[7] Equation (2a) equates the marginal utility of consumption to the shadow value of wealth, while (2b) equates the rate of return on capital to the rate of return on consumption. We assume $r \geq \delta$, thus ensuring non-negative savings.

the solution to which is

$$c^R(z,t) = c^R(t,t) + \frac{r-\delta}{\gamma}(z-t). \tag{4}$$

Thus, during the retirement phase of an individual's life-cycle, consumption grows at the linear rate $(r-\delta)/\gamma$ per unit of time. Substituting (4) into the accumulation equation, (1b), integrating over *z*, and invoking the transversality condition, we find that the optimal path for $a^R(z,t)$ is

$$a^R(z,t) = a^R(t,t) + \frac{r-\delta}{\gamma(r+p)}(z-t) \qquad z \geq t \tag{5}$$

where, at the instant of retirement ($z = t$), the retirement consumption must satisfy

$$c^R(t,t) = (r+p)\cdot\left(a^R(t,t) - \eta\cdot m\right) + b - \left(\frac{r-\delta}{\gamma(r+p)}\right) \tag{6}$$

and $a^R(t,t) \equiv a(t)$. Combining equations (4) to (6) yields

$$c^R(z,t) = (r+p)\cdot\left(a^R(z,t) - \eta\cdot m\right) + b - \left(\frac{r-\delta}{\gamma\cdot(r+p)}\right) \qquad \text{for all } z \geq t \tag{7}$$

which implies that the retirement benefit and medicare policies affect the level of retirement consumption, but not its (linear) growth rate.

The quantity $s^R \equiv (r-\delta)/[\gamma(r+p)] = \dot{a}^R(z,t)$ is the equilibrium savings rate (per unit of time) by each retiree. This is independent of both the date of retirement and current time, and is therefore the same for all retirees. It is also independent of retirement benefits, which are fully reflected in the level of consumption. As expected, retirement savings vary positively with expected lifetime $1/p$. Finally, the retirement savings rate is zero if and only if $r = \delta$, in which case consumption during retirement remains constant.

Substituting (4) into the retiree's utility function yields the maximized expected retirement utility, conditional on retiring at time *t* with financial wealth, $a(t) \equiv a^R(t,t)$, and realized (known future) medical costs *m*:

$$W^R\left(a(t)\,|\,t,m\right) = \frac{-\beta}{r+p}\cdot e^{-\gamma\left[(r+p)\cdot(a(t)-\eta\cdot m)+b-s^R\right]} \tag{8}$$

For notational convenience we denote the retiree's flow of medical costs by $\tilde{\mu} \equiv (r+p)\eta\tilde{m}$, and the

probability distribution determining $\tilde{\mu}$ by $P(\tilde{\mu})$. Thus, prior to the revelation of private medical costs, *m*, the expected maximized utility to an individual retiring at time *t* with financial assets $a(t)$ is

$$\underset{\tilde{\mu}}{E}\left\{W^R\left(a(t)\,|\,t,\tilde{\mu}\right)\right\} = \frac{-\beta}{r+p}\cdot e^{-\gamma\left[(r+p)\cdot a(t)+b-s^R\right]}\Psi(\tilde{\mu}) \qquad (8')$$

where the expected value, $\Psi(\mu)\equiv\int_{\tilde{\mu}} P(\tilde{\mu})\cdot e^{\gamma\cdot\tilde{\mu}}d\tilde{\mu}$, adjusts utility for the retiree's flow of medical co-payments. We assume that $\tilde{m}$ is distributed uniformly over the interval $\left[\bar{m}-M,\bar{m}+M\right]$, where $\bar{m}$ is the mean of $\tilde{m}$, in which case $\Psi(\mu)=e^{\gamma\cdot\bar{\mu}}\cdot D(\hat{\mu})$, where $\bar{\mu}\equiv\eta\cdot(r+p)\cdot\bar{m}>0$, is the retiree's average co-payment, $\hat{\mu}\equiv\eta\cdot(r+p)\cdot M$, and $D(\hat{\mu})\equiv\left((e^{\gamma\hat{\mu}}-e^{-\gamma\hat{\mu}})/(2\gamma\hat{\mu})\right)\geq 1$, with $D(0)=1$. We can thus express the expected maximum retirement utility, prior to the revelation of medical costs as

$$\underset{\tilde{\mu}}{E}\left\{W^R\left(a(t)\,|\,t,\tilde{\mu}\right)\right\} = \frac{-\beta}{r+p}\cdot e^{-\gamma\left[(r+p)\cdot a(t)+b-\bar{\mu}-s^R\right]}D(\hat{\mu}). \qquad (8'')$$

Expected retirement utility increases with benefits, *b*, but is reduced by both an increase in the average private retirement medical expenses $\bar{\mu}$ and an increase in $\hat{\mu}$, which corresponds to an increase in the variance of the retiree's private medical expenses. [8]

3.2 Workers

Consider now an individual entering the economy at time *v*. We refer to such individuals as having "vintage *v*" or "belonging to cohort *v*". They maximize their expected lifetime utility

$$E\left\{U(v)\,|\,v\right\} = \int_v^\infty -\beta\left[e^{-\gamma\cdot c^E(t,v)} + \pi\frac{1}{r+p}\cdot e^{-\gamma\left[(r+p)\cdot a^E(t,v)+b-\bar{\mu}-s^R\right]}D(\hat{\mu})\right]e^{-(p+\delta+\pi)(t-v)}dt \qquad (9a)$$

subject to the accumulation equation while employed[9]

$$\dot{a}^E(t,v) = (r+p)\cdot a^E(t,v)+h-\tau-c^E(t,v) \qquad (9b)$$

where now $\dot{a}^E(t,v)\equiv\partial a^E(t,v)/\partial t$, $\dot{c}^E(t,v)\equiv\partial c^E(t,v)/\partial t$ and we are assuming the CARA utility function throughout the individual's lifetime. The first term in the integral in (9a) refers to utility conditional on being employed at time *t*, while the second is the expected maximized utility after

[8] Note further that $\hat{\mu}=\sqrt{3\cdot V(\tilde{\mu})}$ where $V(\tilde{\mu})$ is the variance of the retiree's medical expenses.

[9] The superscript *E* refers to employment.

retirement, (8"), assuming that retirement occurs precisely at *t*. This component incorporates the accumulation of financial assets after retirement, as described by (1b). Both components are conditional on survival to time *t*. In the budget constraint, (9b), *h* denotes human capital earnings, which are assumed constant and exogenous, while τ is the lump-sum tax imposed (on workers only) to finance the social insurance programs.

Optimizing now with respect to $c^E(t,v)$ and $a^E(t,v)$ yields the first order conditions during the employment phase of an individual's life-cycle,

$$\beta\gamma e^{-\gamma c^E} = \lambda^E \tag{10a}$$

$$r+\pi\frac{\beta\gamma e^{-\gamma\left[(r+p)\cdot\left[a^E(t,v)-\bar{\mu}\right]+b-s^R\right]}D(\hat{\mu})}{\lambda^E} = \delta+\pi-\frac{\dot{\lambda}^E}{\lambda^E} \tag{10b}$$

where λ^E denotes the shadow value of capital to workers, together with the transversality condition. Equation (10a) is analogous to (2a), while (10b) equates the expected rate of return on capital to the expected rate of return on consumption. In addition to *r*, the former reflects the fact that investment augments the stock of financial assets at the time of retirement, upon which the subsequent consumption level during retirement depends. The return to consumption includes the retirement risk, as reflected by its hazard rate. Differentiating (10a) with respect to *t* and using (10b), yields

$$\dot{c}^E(t,v) = \frac{r-\delta}{\gamma}-\frac{\pi}{\gamma}\cdot\left[1-\frac{D(\hat{\mu})\cdot e^{-\gamma\left[(r+p)\cdot a^E(t,v)+b-\bar{\mu}-s^R\right]}}{e^{-\gamma\cdot c^E(t,v)}}\right]. \tag{11}$$

To derive the equilibrium savings and consumption rates during employment, we proceed as follows. First, we rewrite the budget constraint, (9b), as

$$c^E(t,v) = (r+p)\cdot a^E(t,v)+h-\tau-s^E(t,v) \tag{12}$$

where $s^E(t,v)\equiv\dot{a}^E(t,v)$ is the level of saving of an individual at time *t* for an individual of vintage *v* who remains employed at time *t*. Taking the time derivative of (12), yields

$$\dot{c}^E(t,v) = (r+p)\cdot s^E(t,v)-\dot{s}^E(t,v) \tag{13}$$

Combining equations (11), (12), and (13) implies that the savings rate at time *t* of an individual of vintage *v* follows the differential equation

$$\dot{s}^E(t,v) = (r+p)\left(s^E(t,v) - s^R\right) + \frac{\pi}{\gamma}\left(1 - D(\hat{\mu})e^{\gamma\left[h-\tau-b+\bar{\mu}+s^R-s^E(t,v)\right]}\right) \tag{14}$$

where we recall $s^R \equiv (r-\delta)/[\gamma(r+p)]$. Equation (14) is seen to be of the form

$$\dot{s}^E(t,v) = f\left(s^E(t,v)\right), \quad f' > 0$$

This relationship implies that $s^E(t,v)$ diverges to plus or minus infinity (with t) at an increasing rate, unless $\dot{s}^E(t,v) \equiv 0$. Using the relationship $s^E(t,v) \equiv \dot{a}^E(t,v)$ this can be shown to violate the transversality condition to the agent's optimization problem. Thus the only viable solution is for the savings rate of every worker to remain constant over time. Setting $\dot{s}^E(t,v) = 0$ in (14) the solution for the constant savings level of each worker can be expressed as

$$s^E = s^R + \frac{\pi}{\gamma(r+p)}\left(D(\hat{\mu})e^{\gamma\left[h-\tau-b+\bar{\mu}+s^R-s^E\right]} - 1\right) = s^R + \frac{\pi}{\gamma(r+p)} \cdot \left(\chi(.) - 1\right) \tag{15}$$

where $\chi(.) \equiv D(\hat{\mu})e^{\gamma\left[h-\tau-b+\bar{\mu}+s^R-s^E\right]}$.[10] The quantity $\chi(.)$ is a measure of the household's risk that is uninsured by government policy.[11] If the household is fully insured $\chi(.) = 1$ and $s^E = s^R$. Incomplete insurance of lost earnings ($h - \tau - b > 0$) or uncertain medical co-payments in retirement ($\bar{\mu} > 0, \hat{\mu} > 0$) implies that $\chi(.) > 1$ so $s^E > s^R$ and saving falls upon retirement.

Equation (15) plays a fundamental role in our analysis of the impact of social insurance policy on savings and consumption behavior. Given the stationarity of the exponential distribution, the consumption-savings plan chosen by the individual at time v remains optimal as long as the indivudal works. Hence, s^E as determined in (15) is the savings function for any worker, and is independent of both current time and the time the person entered the labor force. Equation (15) defines an implicit function for s^E, and solving, the constant savings rate of workers can be

[10] In Section 7 we show that the optimal insurance policy can be expressed in terms of κ and χ, enabling us to characterize the change in the working-period and retirement savings rate in terms of the optimal insurance policy.

[11] This model relates to the literature on precautionary saving, which makes use of the CARA utility function for the same tractability reasons we invoke in this paper; see Blanchard and Mankiw (1988). The precautionary savings motive relates to the uncertainty of future income *relative* to that of future expenses, and an increase in such uncertainty increases saving as long as the third derivative of the utility function is positive, as it is with the CARA form. In our model, uncertainty about retirement and death are described by exponential distributions, which are summarized by single parameters, p and π respectively. An increase in either parameter increases the waiting time and the variance, making it difficult to disentangle life-cycle saving effects from precautionary saving effects. In contrast, medical expenses are parameterized by two parameters, $\bar{\mu}$ and $\hat{\mu}$, reflecting the mean and variance of future medical expenses, respectively. An increase in current savings in response to an increase in $\hat{\mu}$ can be interpreted as being a pure precautionary savings effect.

expressed in the following form

$$s^E = s^R + \varphi\left(D(\hat{\mu}), h - \tau - b + \bar{\mu}, \frac{\pi}{r+p} \right). \tag{16}$$

This emphasizes that the change in savings rate at retirement depends upon the extent of income insurance, as measured by the difference between the net before-retirement earnings rate and the after-retirement benefit rate $(h-\tau-b)$ and the mean and variance of retirement medical co-payments.

We summarize these results in:

Proposition 1: (Working-period and retirement individual savings behavior)

(i) As $\pi \to 0$ so that the probability of retirement declines, $s^E \to s^R$, and the savings rate tends to remain unchanged at all times.

(ii) If $h-\tau = b$ (full insurance of the loss of earnings) and $\bar{\mu} = 0$ and $D(\hat{\mu}) = 1$ (no medical co-payments in retirement), $s^E = s^R$, and the savings rate is the same in the employment and retirement phases of the lifecyle.

(iii) More generally, if $h-\tau > b$ (incomplete insurance of earnings), $\bar{\mu} > 0$ and/or $D(\hat{\mu}) > 1$ (some medical co-payments in retirement), then $s^E > s^R$ and the individual's savings level drops upon retirement.

Differentiating (15), we can derive the effect of the level of earnings, *h*, and the social insurance policy variables $(b, \tau, \bar{\mu}, \hat{\mu})$ on the level of savings of workers:

$$\frac{\partial s^E}{\partial h} = -\frac{\partial s^E}{\partial \tau} = -\frac{\partial s^E}{\partial b} = \frac{\partial s^E}{\partial \bar{\mu}} = \frac{\gamma \cdot D(\hat{\mu})}{D'(\hat{\mu})} \cdot \frac{\partial s^E}{\partial \hat{\mu}} = \frac{\kappa(.)}{1+\kappa(.)} > 0 \tag{17a}$$

where for notational convenience we let $\kappa(.) \equiv \left(\pi/(r+p)\right)\chi(.)$. We can further show

$$\frac{\partial s^E}{\partial \pi} = \frac{s^E - s^R}{\pi(1+\kappa(.))} > 0 = \frac{\partial s^R}{\partial \pi};\ \frac{\partial s^E}{\partial p} = -\frac{(r-\delta)}{\gamma(r+p)^2} - \frac{(s^E - s^R)}{(1+\kappa(.))(r+p)} < -\frac{(r-\delta)}{\gamma(r+p)^2} = \frac{\partial s^R}{\partial p} < 0 \tag{17b}$$

and summarize these results in

Proposition 2: (Policy and demographic effects on individual savings behavior)

(i) The retirement individual savings rate is independent of retirement/insurance policy, whereas the working-period individual savings rate increases partially with *h*, $\bar{\mu}$, and $\hat{\mu}$, but decreases with τ and *b*.

(ii) An increase in retirement risk, π, raises the working-period savings rate, but leaves the retirement savings rate unaffected. An increase in mortality risk reduces both the working-period and retirement savings rates.

The intuition for these results is as follows. First, an increase in earnings induces more savings from workers, while higher social insurance payments and higher expected retirement benefits have the opposite effect. In addition, higher expected, and more uncertain, retirement medical expenses encourage more savings by workers. Second, given *p*, an increase in π reduces the expected working period and increases the expected retirement period, thus inducing higher savings by workers. Given π, an increase in *p* reduces both life expectancy and the expected retirement length, reducing the need to save while working. Since it also reduces life expectancy of retirees, the reduction in *p* also reduces the retirement savings rate, though by a lesser amount.

3.3 Individual Consumption and Consumption Growth

We now consider what happens to the individual's consumption level and its growth rate upon retirement. Suppose an individual of vintage v retires at time *T*. At that time, his/her consumption level, when employed, would be

$$c^E(T,v) = (r+p)\cdot a^E(T,v) + h - \tau - s^E. \tag{18a}$$

Immediately upon retirement, (6) implies that the consumption level becomes

$$c^R(T,T) = (r+p)\,a^R(T,T) - \bar{\mu} + b - s^R. \tag{18b}$$

Because wealth accumulation occurs continuously, $a^E(T,v) = a^R(T,T) \equiv a(T)$, and subtracting (18b) from (18a) we obtain

$$c^R(T,T) - c^E(T,v) = -\bar{\mu} - (h - \tau - b) - \left(s^R - s^E\right). \tag{19}$$

With full insurance of the loss of earnings ($h-\tau=b$) and no medical co-payments $(\bar{\mu}=0)$, $s^E=s^R$ and consumption is unchanged upon retirement. That is, $c^R(T,T)=c^E(T,v)$. However, medical co-payments and/or incomplete insurance of earnings cause income loss upon retirement. The responses summarized in (17) imply that the fall in savings upon retirement is less than the fall in income, thus $c^R(T,T)<c^E(T,v)$ and consumption drops at the retirement date *T*.

This finding relates to the so-called "retirement consumption puzzle". While a systematic fall in consumption on a *fully expected* retirement date contradicts optimizing behavior over an individual's life-cycle, a fall in consumption when retirement is *unexpected* does not contradict the life-cycle model. Stephens and Haider (2003) find that some part of the retirement consumption drop can be explained by unexpected retirement. However, they also find that consumption falls, even for individuals who retire when they expect to (leaving the retirement consumption puzzle intact). In our model, all retirement is unexpected and this is reflected in the drop in consumption at that instant.

Recalling (13), and (3) individual consumption growth rates before and after retirement are

$$\dot{c}^E(t,v)=(r+p)\cdot s^E(t,v)=(r+p)s^E \tag{13'}$$

$$\dot{c}^R(z,t)=\frac{r-\delta}{\gamma}=(r+p)s^R \tag{3'}$$

respectively, and subtracting yields

$$\dot{c}^R-\dot{c}^E=(r+p)(s^R-s^E) \tag{20}$$

so that the individual's consumption growth rate declines upon retirement. Greater social insurance reduces both the drop in consumption and the drop in the consumption growth rate upon retirement.

We can summarize consumption behavior upon retirement as follows.

Proposition 3: (Working-period and retirement individual consumption behavior)

(i) With stochastic retirement, in the absence of full social insurance, individual consumption levels and rates of growth drop upon retirement.

(ii) An increase in retirement benefit *b* leads to a partial increase in working-

period consumption, together with a full increase in retirement consumption.

(iii) An increase in worker tax τ leads to a partial $(1/(1+\kappa(.)))$ decrease in working-period consumption, but has no effect on retirement consumption (other than through the cumulative impact on wealth).

(iv) An increase in earnings h leads to a partial $(1/(1+\kappa(.)))$ increase in working-period consumption, and no effect on retirement consumption (other than through the cumulative impact on wealth).

(v) The change in the consumption growth rate at the time of retirement resulting from changes in $h, \tau, b, \bar{\mu}, \hat{\mu}$ is brought about entirely by its impact on the working-period growth rate.

4. Aggregate Savings and Social Insurance

We now consider the level of aggregate savings and how it is impacted by changes in social insurance policy. We shall subsequently discuss the analogous consequences for aggregate wealth, and consumption.[12]

Let *n* denote the constant rate of growth in the effective labor supply, which is taken to be exogenous. We normalize units so that the initial cohort size is $(n+p)$. The population at time *t* is obtained by aggregating over the surviving members of all cohorts, namely

$$N(t) = \int_{-\infty}^{t} (n+p) e^{n\cdot v} e^{-p\cdot(t-v)} dv = e^{nt}. \tag{21a}$$

Similarly, the labor force and the number of retirees at time *t* are respectively

$$E(t) = \int_{-\infty}^{t} (n+p) e^{n\cdot v} e^{-(p+\pi)\cdot(t-v)} dv = \frac{n+p}{n+p+\pi} \cdot e^{nt} \tag{21b}$$

and

[12] In general, when analyzing the impacts of changes in a social security policy parameters on aggregate variables, we conduct "comparative economies" and consider only long-run impacts. That is, we are consider how aggregate values differ for two economies that are identical in all respects except the policy parameters. Equivalently, we are examining the policy impacts in the "long-run". These impacts are approached asymptotically in this economy. When examining the impact on aggregate consumption, we do explicitly consider both the short- and long-run impacts.

$$R(t) = N(t) - E(t) = \frac{\pi}{n+p+\pi} \cdot e^{nt}. \tag{21c}$$

For notational convenience we let

$$\theta^E \equiv \frac{n+p}{n+p+\pi},\ \theta^R \equiv \frac{\pi}{n+p+\pi} = 1-\theta^E$$

denote the fractions of the population employed and retired, respectively. As an empirical fact we assume that the number of workers exceeds the number of retirees so that $n+p>\pi$. The ratio of retirees to workers, $\theta^R/\theta^E = \pi/(n+p)$, which we will denote by θ, is commonly called the *dependency rate* in the social insurance literature. In the U.S., the dependency rate is currently around $1/3$.[13]

With all workers and retirees having the same respective savings rates, (s^E and s^R), total savings by the two groups are

$$S^E(t) = s^E E(t) = s^E \theta^E e^{nt}$$

$$S^R(t) = s^R R(t) = s^R \theta^R e^{nt}$$

Aggregate private saving in the economy at time *t* is thus equal to $S(t) = s \cdot e^{nt}$ where *s*, the average per capita savings in the economy, is given by

$$s = \left[\theta^R s^R + \theta^E s^E\right] = \left[s^R + \theta^E \varphi\left[D(\hat{\mu}), h-\tau-b+\bar{\mu}, \frac{\pi}{r+p}\right]\right]. \tag{22}$$

Note that social insurance policies affect aggregate savings entirely through their impacts on the savings of workers. Thus (17a) implies

$$\frac{\partial s}{\partial x} = \theta^E \frac{\partial s^E}{\partial x},\ \ x = h, \tau, b, \bar{\mu}, \hat{\mu}.$$

We can also compute: $\partial s/\partial \pi, \partial s/\partial p$ and analyze the effect of demographic structure on the average savings rate. An increase in π, for example, has two offsetting effects. It raises the savings rate of workers, but it also raises the number of retired people, who have a lower savings rate. Using (15), the net effect is

[13] We may observe that the dependency rate depends only upon demographic factors and is independent of all macro policy instruments. The dependency rate $\theta = 1/3$ implies $n+p = 3\pi$.

$$\frac{\partial s}{\partial \pi} = \frac{(s^E - s^R)}{\pi(1+\theta)}\left(\frac{1}{1+\kappa(.)} - \frac{\theta}{1+\theta}\right). \tag{23}$$

The smaller the dependency rate θ and the greater the return on capital *r*, the more likely is the positive effect due to the increased savings of workers likely to dominate, and vice versa.[14]

We now consider the effect of a PAYG social insurance scheme on aggregate private savings and its distribution. For PAYG, the revenues raised by taxing workers must just finance the retirement and net medical benefits. That is,

$$E(t)\tau = R(t)\left[b + (r+p)\bar{m} - \bar{\mu}\right] \tag{24}$$

implying

$$\tau = \theta\left(b + (r+p)\bar{m} - \bar{\mu}\right) \tag{25}$$

where $(r+p)\bar{m} - \bar{\mu}$ is the government's share of the cost of a retiree's average flow of medical expenses, and $\bar{\mu} = \eta(r+p)\bar{m}$ where η is the medical expense co-payment rate.[15]

Suppose the government raises the rate of social security benefits per retiree, db. This requires an increase in the per capita (lump-sum) tax rate, $d\tau = \theta \cdot db$. The total effect on average *per capita* savings in the economy is thus:

$$\frac{ds}{db} = \frac{1}{1+\theta}\frac{ds^E}{db} = \frac{1}{1+\theta}\cdot\left(\frac{\partial s^E}{\partial b} + \frac{\partial s^E}{\partial \tau}\frac{d\tau}{db}\right) = -\frac{\kappa(.)}{1+\kappa(.)} > -1. \tag{26}$$

That is, an increase in the individual retirement benefit reduces per capita saving partially.

We can also derive an analogous relationship in terms of aggregate quantities. Denoting aggregate social security benefits by $B(t) = b \cdot \theta^R e^{nt}$, and aggregate social insurance taxes by $T(t) = \tau \cdot \theta^E e^{nt}$, we obtain

$$\frac{dS(t)}{dB(t)} = \frac{1+\theta}{\theta}\cdot\frac{ds}{db} = -\frac{1+\theta}{\theta}\cdot\frac{\kappa(.)}{1+\kappa(.)} = -\frac{(n+p+\pi)\chi(.)}{r+p+\pi\chi(.)}. \tag{27}$$

In general, a PAYG funded increase in social security benefits decreases aggregate private saving,

[14] As we shall show in Section 7, setting social security optimally implies $\kappa(.) = \theta$, in which case (23) is positive.

[15] It is assumed that all individual risk is independently distributed and the population is sufficiently large so that only average benefits and taxes affect the government budget.

and the decrease may be greater or less than dollar for dollar (i.e. -1). With "full social insurance", $\chi(.)=1$, and $dS(t)/dB(t) \geq -1$ if and only if $r \geq n$. In Section 7, we shall show that optimal social security requires $\kappa(.)=\theta$, in which case, $dS(t)/dB(t)=-1$. We shall also show that for optimal social insurance policies, $\chi^* = (r+p)/(n+p)$. If $\chi > \chi^*$ then $dS(t)/dB(t) \leq -1$.[16]

Proposition 4: (The effect of social security on aggregate private savings)

In general, a PAYG funded increase in social security benefits leads to a reduction in aggregate private savings which may be greater or smaller than the increment in benefits. When social insurance is set optimally, an incremental dollar of social security benefits leads to exactly a dollar fall in aggregate private savings. But if social insurance is incomplete, an incremental dollar of social security benefits will lead to a drop in saving of more than one dollar.

The effect of an increase in medical insurance benefits takes into account the effects of the reductions in both $\bar{\mu}$ and $\hat{\mu}$ on saving. We consider a change in the Medicare co-payment rate η which implies that $d\bar{\mu} = d\hat{\mu}$, in which case

$$\frac{dS(t)}{dM(t)} = \frac{dS(t)}{dB(t)} - \frac{1}{\theta} \cdot \frac{D'(\hat{\mu})}{\gamma D(\hat{\mu})} \cdot \frac{\kappa(.)}{1+\kappa(.)}. \tag{28}$$

Hence the fall in saving from an increase in Medicare benefits financed by PAYG must be greater than the fall in saving from an equal dollar increase in Social Security benefits. This reflects the precautionary saving motive attached to uncertain future medical expenses.

Proposition 4a: (The effect of Medicare on aggregate private savings)

In general, a PAYG funded increase in Medicare benefits causes a larger reduction in private savings than an equal increase in social security benefits. If Medicare fully insures deviations from average medical expenses across consumers so $\hat{\mu}=0$, changes in Medicare benefits have the same impact on aggregate private savings as an equal value change in Social Security benefits.

[16] Noting the relationships $\kappa = (\pi/(r+p))\chi$, $\theta = \pi/(n+p)$ immediately yields the expression for χ^*.

5. Aggregate Wealth

The aggregation needed to derive total wealth and its allocation are more complex, due to the fact that each individual's wealth depends on age, years worked, and in the case of retirees, years retired. Here we simply provide the key relationships, with the details provided in Appendix, A.1.

5.1 Wealth Levels

We assume that all individuals receive the same initial endowment of wealth, a_0, (possibly zero). The wealth of a worker of cohort *v* at time *t* is

$$a^E(t,v) = a_0 + s^E \cdot (t-v), \tag{29}$$

Aggregating over individuals of each cohort, we find that the total wealth of workers at time *t* is $A^E(t) = E(t) \cdot a^E$, where a^E, the average per-capita wealth of workers is a constant over time and given by

$$a^E = a_0 + \frac{s^E}{p+n+\pi} \tag{30}$$

Comparing (29) to (30) we see that $1/(n+p+\pi)$ is equivalent to the average number of years worked and can be interpreted as being the average working horizon of workers.

Turning to retirees, the wealth at time *t* of a retiree of vintage *v* who worked *y* units of time before retiring is

$$a^R(t,y,v) = a_0 + s^E \cdot y + s^R \cdot (t-y-v). \tag{31}$$

This reflects the fact that during *y* years of working, the individual saved at the rate s^E, while for the $(t-y-v)$ years of retirement he/she saved at s^R. Aggregating over all retirees by cohort and length of time worked, we show that the wealth of all retirees at time *t*, $A^R(t) = R(t) \cdot a^R$, where a^R, the average per-capita wealth of all retirees, is constant over time and given by

$$a^R = a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{p+n}. \tag{32}$$

Comparing (32) with (30), we see $a^R = a^E + s^R/(p+n) > a^E$, so that average retiree's wealth exceeds the average worker's wealth.

The wealth of workers and retirees combined is $A(t) = A^E(t) + A^R(t)$.[17] Thus, overall per capita wealth at time *t* is

$$\frac{A(t)}{N(t)} = a = \theta^E a^E + \theta^R a^R = a_0 + \frac{s^E}{p+n+\pi} + \frac{\theta^R s^R}{p+n} = a_0 + \frac{1}{p+n} s \,. \tag{33}$$

The average per capita wealth can be computed by simply adding the average savings rate times a "multiplier" $1/(p+n)$ to the initial endowment. In the absence of population growth the multipier is simply the life expectancy, but with growing population it is smaller because part of the accumulating wealth must be set aside for the growing population. From these relationships we can derive

Proposition 5: (The effect of social insurance on per capita private wealth)

A PAYG increase in social security reduces the per capita wealth of both workers and retirees by the same *absolute* amount. Since retirees have higher average wealth, this causes the *relative* share of wealth of retirees, $A^R(t)/A(t)$, to increase.

5.2 Distribution of Wealth

An important issue concerns the distributional consequences of social security, particularly on wealth. The effect of social security on wealth inequality among retirees has received some attention recently; see Gokhale (2001) and Gokhale and Kotlikoff (2002). Accordingly, we now derive expressions describing the distribution wealth in this economy. To do this we determine the fraction of the population at time *t* that holds wealth less than a particular value $\hat{a}$. The computational details are found in the Appendix, A.2.

We first consider the distribution of wealth among workers. In the Appendix we show that the number of workers with wealth less than or equal to $\hat{a}$ is $E(t, a \le \hat{a}) = E(t) \cdot \nu^E(a \le \hat{a})$ where

[17] We can also easily show $\dot{A}^E/A^E = \dot{A}^R/A^R = \dot{A}/A = n$. Note also that we can express aggregate savings $S(t) = (n+p) \cdot A(t) = \dot{A}(t) + p \cdot A(t)$. That is, aggregate accumulation of annuity wealth by living households at each instant is equal to net new annuity wealth, which is determined by the population growth rate, plus the purchase of annuity contracts by living households from households who die, consistent with Blanchard (1985).

$$\nu^E\left(a \le \hat{a}\right) = 1 - e^{-(n+p+\pi)\cdot\frac{\hat{a}}{s^E}} \tag{34}$$

is the fraction of worker population with wealth less than or equal to $\hat{a}$. Hence, the cumulative distribution of wealth among workers is exponential with mean $s^E/(n+p+\pi)$ and variance $\left(s^E/(n+p+\pi)\right)^2$. The conventional measure of concentration in a distribution is the Gini ratio. The Gini ratio for the exponential distribution is exactly ½, so that social insurance has no impact on the concentration of wealth among workers.

Similarly, let $R\left(t, a \le \hat{a}\right) = R(t)\cdot\nu^R\left(a \le \hat{a}\right)$ be the number of retirees with wealth less than or equal to $\hat{a}$ where *R(t)* is the number of retirees at time *t* and $\nu^R\left(a \le \hat{a}\right)$ is the fraction of the retiree population of retirees with wealth less than or equal to $\hat{a}$. In the Appendix, we show that this fraction is described by the mixed exponential distribution

$$\nu^R\left(a \le \hat{a}\right) = \xi^R \cdot \left(1 - e^{-(n+p+\pi)\cdot\frac{\hat{a}}{s^E}}\right) + \left(1-\xi^R\right)\cdot\left(1 - e^{-(n+p)\cdot\frac{\hat{a}}{s^R}}\right) \tag{35}$$

where $\xi^R \equiv (\theta^E s^E)/(\theta^E s^E - s^R)$.

In the Appendix, we show that the Gini ratio for the distribution of wealth among retirees is

$$G^R = \frac{1}{2}\left\{1 + \frac{\sigma}{(1+\sigma)^2}\right\}. \tag{36}$$

where $\sigma \equiv s^R/(\theta^E s^E)$. First, we find that, providing $s^R > 0$, $\sigma > 0$ so $G^R > 1/2$. Thus, wealth concentration is greater among retirees than among workers. Second, an increase in social insurance benefits decreases s^E and increases σ, while $\partial G^R/\partial\sigma > (<) 0$ as $1-\sigma > (<) 0$. When social insurance is complete, $s^E = s^R$ and $1-\sigma = 1-1/\theta^E < 0$. Thus, a decrease in social insurance benefits increases wealth concentration among retirees. However, the relationship between social insurance and wealth concentration among retirees is not monotonic. With incomplete social insurance, it is possible for $\theta^E s^E > s^R$, so $1-\sigma > 0$. In this case, a decrease in social insurance can decrease the level of wealth concentration among retirees. An explanation for this non-monotonicity is provided after we derive the wealth distribution for the whole population.

Combining (34) and (35) we can determine the distribution of wealth across the whole population from the relationship $N\left(t, a \le \hat{a}\right) = E\left(t, a \le \hat{a}\right) + R\left(t, a \le \hat{a}\right) = \nu^N\left(a \le \hat{a}\right)\cdot N\left(t\right)$, where

$$\nu^N\left(a \le \hat{a}\right) = \xi^N \cdot \left[1 - e^{-(n+p+\pi)\cdot\frac{\hat{a}}{s^E}}\right] + \left(1-\xi^N\right)\cdot\left[1 - e^{-(n+p)\cdot\frac{\hat{a}}{s^R}}\right] \tag{37}$$

is the fraction of the population that has wealth less than $\hat{a}$, and $\xi^N \equiv \left(\theta^E(s^E - s^R)\right)\Big/\left(\theta^E s^E - s^R\right)$. The population wealth distribution is also a mixed exponential distribution. It reduces to the pure exponential form in two cases, if: (i) $s^R = 0$, implying $\xi^N = 0$, or (ii) $s^E = s^R \neq 0$ implying $\xi^N = 1$.

The mean per capita wealth for this mixed distribution can be expressed as $\bar{a} = \left(s/(n+p)\right)$, where $s \equiv \theta^E s^E + \theta^R s^R$ is average per capita saving, while the variance, $V(a)$ can be written as $V\left(a\right) = \left(s/(n+p)\right)^2 - 2\cdot\theta^E s^R(s - s^R)/(n+p)^2$. Since an increase in social insurance benefits causes *s* to decline, this causes both the mean and variance of the wealth distribution to decline as well.

In the Appendix, we derive the Gini coefficient for the mixed exponential distribution of wealth across the whole population, namely:

$$G^N = \frac{1}{2}\left[1 + \frac{\theta^R s^R(s - s^R)}{s(s + s^R\theta^E)}\right] \tag{38}$$

from which it follows immediately that $G^N \geq 1/2$ since $s \geq s^R$ as $s^E \geq s^R$.[18,19] In the case where $s = s^E = s^R$, the distribution of wealth in the economy is a pure exponential function, and the Gini ratio for the population is exactly equal to one-half.[20]

Since decreasing social insurance increases *s*, we can examine the effect of social security on wealth inequality by evaluating $\partial G/\partial s$, the qualitative effect of which is given by

$$\operatorname{sgn}\left(\frac{\partial G}{\partial s}\right) = \operatorname{sgn}\left(\frac{\partial}{\partial s}\left(\frac{s - s^R}{s(s + s^R\theta^E)}\right)\right) = \operatorname{sgn}\left(-(s - s^R)^2 + \left(s^R\right)^2\left(1 + \theta^E\right)\right). \tag{38'}$$

From this expression we see that, given $s^R \neq 0$, when s^R is close to *s*, $\partial G^N/\partial s > 0$. Thus, a decrease in social insurance from the full insurance point, $s = s^E = s^R$, increases wealth inequality. But when insurance is very incomplete, $s > s^R$ and the negative term $-(s - s^R)^2$ can dominate, implying $\partial G^N/\partial s < 0$. In other words, when social insurance is very incomplete, reducing it further

[18] We may note that the fact that $G^N > 1/2$ is consistent with empirical evidence. While in the US, for example, the Gini coefficient for income inequality is something greater than 0.4, the corresponding wealth coefficient is much higher, around 0.8 by some estimates; see e.g. Wolf (1996).

[19] Comparing (36) and (38) we can easily show that $G^R > G^N$ so that wealth inequality among retirees exceeds that of the total population, and in turn that of workers (1/2).

[20] Paradoxically, $G^R > 1/2$ for retirees in this case (unless $s^R = 0$). This seemingly counter-intuitive result is explained by the fact that, in this model, age disparities across the retiree population are greater than for the whole population.

can reduce wealth concentration. Thus we see that the relationship between social insurance and wealth inequality for the population as a whole is non-monotonic.

The causes of wealth inequality in this economy are as follows. There are two independent savings motives in this economy, the first described by s^R, and the second by the excess of s^E over s^R. We call the latter, the "lifecycle motive for saving". The two motives vary across the population, with the first causing wealth disparities according to age, and the second causing wealth disparities according to years worked (retention of human capital). When the two saving rates are the same, or only one is operative, wealth is distributed according to a pure exponential distribution and the Gini ratio is ½ for the whole population. When savings rates are not the same, which occurs when $s^R \neq 0$ and $s^E > s^R$ (because social insurance is incomplete), the disparity in years worked as well as age adds to disparity in wealth levels, increasing the Gini ratio.

The non-monotonicity of (38') arises for the following reason. When social insurance is complete or nearly so, the lifecycle motive for saving is small or zero, so everyone in the economy accumulates wealth at the same rate. In this case, wealth concentration is determined solely by age, which is distributed exponentially. When lifecycle saving is present, decreasing social insurance increases the saving rate of workers, so disparity in accumulation depends also on years worked, and wealth concentration initially increases. However, if social insurance is very incomplete, the lifecycle motive for saving dominates the first motive, and disparity in accumulation is reduced by further decreases in social insurance.

We now summarize our findings on the effect of social insurance on wealth concentration.

Proposition 6: (The effect of social insurance on wealth concentration)

If households save for reasons in addition to lifecycle reasons, wealth concentration is greater among retirees than among the whole population, which in turn is greater than among workers. If lifecycle motives are the only motive for saving, social insurance increases both the mean and variance of the wealth distribution, but has no impact on the degree of wealth concentration. Decreases in social insurance at the full insurance point will increase wealth concentration across the retiree and whole populations. If

social insurance is very incomplete to begin with, decreases in social insurance may decrease wealth concentration across these populations.

6. Aggregate Consumption

Having derived savings and wealth behavior for various categories of agents, it is straightforward to determine the consequences for aggregate consumption. Again the details of the aggregation are provided in the Appendix, A.3.

Recalling (12) and (29), the equilibrium consumption of a worker of cohort *v* at time *t* is

$$c^E(t,v) = (r+p)\cdot(a_0 + s^E(t-v)) + h - \tau - s^E \tag{39}$$

where s^E is given by (15). The aggregate consumption of all workers, $C^E(t) = E(t)\cdot c^E$ where c^E, the average per capita consumption of workers, is

$$c^E = (r+p)\left(a_0 + \frac{s^E}{p+n+\pi}\right) + (h-\tau-s^E) = (r+p)a^E + (h-\tau-s^E). \tag{40}$$

Recalling (5'), and summing over cohorts and working horizons, the aggregate consumption of all retirees is $C^R(t) = R(t)\cdot c^R$ where c^R, per capita consumption of retirees, is

$$c^R = (r+p)\cdot\left[a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{n+p}\right] + b - s^R - \bar{\mu} = (r+p)a^R + b - s^R - \bar{\mu} \tag{41}$$

Using the fact that aggregate consumption, $C(t) = C^E(t) + C^R(t)$, average per capita consumption is simply the weighted average, $c = \theta^E c^E + \theta^R c^R$, and can be derived immediately from (40) and (41).

An important issue concerns the impact of social insurance on working-period and retirement consumption. Proper assessment of this requires care. From (40) and (41) we see that social insurance policy affects per capita consumption of different groups through its impact on the savings rate, s^E. This affects consumption in two ways, first through its impact on accumulated wealth (wealth effect), second through its impact on resources available for consumption from current income (income effect). If, when computing quantities such as $\partial c/\partial s^E$, we allow *both* components to change, we are comparing the structure of two economies, after the full impact on wealth accumulation has taken place. This is, in effect, a long-run comparison.

In the Appendix we show that a PAYG increase in social insurance changes the short-run and long-run per capita consumption (denoted by the respective subscripts *s*, *l*) of workers by[21]

$$\left.\frac{dc^E}{db}\right|_s = (1+\theta)\frac{\kappa(.)}{1+\kappa(.)} - \theta = \frac{\kappa(.)-\theta}{1+\kappa(.)} \tag{42a}$$

$$\left.\frac{dc^E}{db}\right|_l = -\frac{\theta}{(1+\kappa(.))}\left[1+\kappa(.)\left(\frac{r-n}{\pi}\right)\right] < 0 \tag{42b}$$

As we shall show in Section 7, the optimal social insurance system requires $\kappa(.) = \theta$, in which case an increase in the social security benefit has no effect on the short-run consumption of workers; their reduction in savings will exactly offset their higher tax rate. If social insurance is less than optimal, $\kappa(.) > \theta$, and a dollar increase in social security benefits causes workers to reduce their savings by more than a dollar, thus raising their short-run consumption. In the long run wealth declines and overall consumption declines provided $r \geq n$.

Similarly, from (41) we see that the short-run and long-run effects of a PAYG increase in social insurance on the per capita consumption of retirees is

$$\left.\frac{dc^R}{db}\right|_s = 1 \tag{43a}$$

$$\left.\frac{dc^R}{db}\right|_l = 1 - \left(\frac{\kappa}{1+\kappa}\right)\cdot\frac{r+p}{n+p} \tag{43b}$$

In the short-run, consumption by retirees increases by the increase in benefits, while the long run consumption increases providing the level of social insurance is not very incomplete.[22]

The short-run and long-run effects on total per capita consumption of social insurance are

$$\left.\frac{dc}{db}\right|_s = \frac{\kappa(.)}{1+\kappa(.)} > 0 \tag{44a}$$

$$\left.\frac{dc}{db}\right|_l = -\left(\frac{\kappa}{1+\kappa}\right)\cdot\frac{r-n}{\pi} \leq 0 \text{ as } r \geq n. \tag{44b}$$

Thus total per capita consumption rises in the short run, but fall over time as long as long as $r \geq n$.

[21] We rule out dynamic inefficiency so that $r \geq n$.

[22] The formal condition is that $\chi < (r+p)/(\theta(r-n))$.

We summarize these results as follows

Proposition 7: (Effect of social insurance on aggregate consumption)

In the short run, a PAYG increase in social insurance raises the consumption of retirees. While its effect on the consumption of workers is ambiguous, depending upon the level of social insurance relative to the optimum, overall per capita consumption rises. In the long run, the effect on the consumption of retirees is ambiguous, but this is dominated by the negative effect on workers so that overall per capita consumption declines.

7. Optimal Policy

We now use the results obtained thus far to derive the optimal level of social insurance. To do so, we recall the expected utility of a new household of vintage t , given by equation (9a). Substituting for $c^E(z,t)$ from equation (12), and for $a^E(z,t)$ from equation (29), (where z is the time index), and integrating (recalling that the optimal s^E and s^R are constants) we obtain

$$E\left[u(t)\middle|t\right]=\frac{-\beta\cdot\left\{1+\frac{\pi}{r+p}\cdot D\left(\hat{\mu}\right)\cdot e^{\gamma\cdot\left(h-\tau+b+\bar{\mu}+s^R-s^E\right)}\right\}\cdot e^{-\gamma\cdot\left[(r+p)\cdot a_0+h-\tau-s^E\right]}}{p+\delta+\pi+\gamma\left(r+p\right)s^E}. \tag{45}$$

Finally, substituting for s^R and for s^E from (15) into $E\left[u(t)\middle|t\right]$, we can express (45) as

$$W^*\left(b,\tau,\bar{\mu},\hat{\mu}\right)=\frac{-\beta\cdot e^{-\gamma\cdot\left[(r+p)\cdot a_0+h-\tau-s^E\left(s^R,D(\hat{\mu}),h-\tau-b+\bar{\mu},\pi/(r+p)\right)\right]}}{r+p}. \tag{46}$$

This expression is the maximum value of the expected lifetime utility of a new household of vintage, *t,* at that same initial instant, *t*. Note that it is independent of the vintage, *t*. It is clear that maximizing $W^*\left(b,\tau,\bar{\mu},\hat{\mu}\right)$ is equivalent to maximizing the expression in square brackets in the exponent of (46). This quantity is the agent's initial rate of consumption. Equation (46) states that the maximum utility is the utility associated with this consumption level, capitalized at the interest rate adjusted for mortality risk. Thus, social insurance policy affects lifetime expected utility through its effect on initial consumption, and through the savings rate chosen at that time.

We now consider social insurance policies that are optimal in the sense of maximizing the expected lifetime utility of a new household entering the economy. We can show that this *ex ante* optimality is also Pareto improving starting from the no-policy state of the world. All policies are subject to the government budget constraint. Substituting for (25), we see that maximizing $W^*\left(b,\tau,\bar{\mu},\hat{\mu}\right)$ is equivalent to maximizing:

$$c(t)=(r+p)\cdot a_0+h-\tau-s^E(.)=(r+p)\cdot a_0+h-\theta\cdot\left(b+(r+p)\cdot\bar{m}-\bar{\mu}\right)-s^E(.)$$

where $s^E(.)=s^E\left(s^R,D(\hat{\mu}),h-(1+\theta)\cdot(b-\bar{\mu})-\theta\cdot(r+p)\cdot\bar{m},\dfrac{\pi}{r+p}\right)$.

Using (17a), the optimal PAYG social security policy requires[23]

$$\frac{dc(t)}{db}=-\theta+(1+\theta)\cdot\frac{\kappa(.)}{1+\kappa(.)}=0$$

which implies

$$\frac{\kappa(.)}{1+\kappa(.)}=\frac{\theta}{1+\theta},\quad\text{or}\quad\kappa(.)=\theta. \tag{47}$$

Recalling the relation $\kappa(.)=\left(\pi/(r+p)\right)\chi(.)$, (47) can be expressed in the equivalent form

$$\chi(.)=\frac{r+p}{n+p} \tag{48}$$

Equations (47) and (48) provide two perspectives on the optimal social security policy. First, we see from (9a') that $\kappa(.)$ measures the expected present value of retirement utility at a point in time relative to the flow utility from consumption while employed at that point in time. Equation (47) states that the optimal social insurance scheme requires that this utility ratio be equal to the ratio of retirees to workers (the dependency rate) in the economy[24]. Alternatively, (48) states that the optimal policy risk should be equated to the return to capital relative to that of social security, both adjusted by mortality risk.

Substituting (48) into (15) yields the corresponding optimal savings rate for workers,

[23] We can also easily verify that the second order condition is met, so that (47) does indeed determine a maximum.

[24] More generally, optimal social insurance requires the ratio of the respective marginal utilities be equal to the dependency rate. The ratio can be expressed in terms of total utilities only in the CARA case.

$$s^E = s^R + \frac{\theta}{\gamma}\cdot\frac{(r-n)}{(r+p)} = s^R + \frac{\pi}{\gamma(n+p)}\cdot\frac{(r-n)}{(r+p)}. \qquad (49)$$

Optimal Medicare policy requires setting $\partial c(t)/\partial\bar{\mu} = \partial c(t)/\partial\hat{\mu} = 0$. Note that $\partial c(t)/\partial\bar{\mu} = 0$ requires (47) [or (48)] so this condition is redundant when social security is set optimally, and vice versa. Further, $\partial c(t)/\partial\hat{\mu} = 0$ requires $\hat{\mu} = 0$, which minimizes $D(\hat{\mu})$ at unity. Note that changes in $\hat{\mu}$ have no budgetary consequences because deviations of medical expenses from the mean are independent across households and can be fully pooled.[25]

We now derive the optimal benefit itself, finding it convenient to focus on two cases.

7.1 The case *r=n* (Golden Rule).

If $r = n$, then $s^E = s^R$. That is, working households do not save for retirement because social security is complete. To obtain an expression for the optimal social security benefit, note that $s^E = s^R$ implies $\chi(.) \equiv D(\hat{\mu})\cdot e^{\gamma[h-\tau-b+\bar{\mu}]} = 1$, or taking logarithms

$$b - (h-\tau) = \bar{\mu} + \frac{1}{\gamma}\ln\left(D(\hat{\mu})\right).$$

The left-hand side is the excess of the social security benefit over the loss in after-tax earnings. The second term on the right-hand side is positive if $\hat{\mu} > 0$ (private retirement medical expenses are variable). Hence, with optimal social security, the benefit should exceed the loss in earnings by the average private medical expense of the retiree, plus a premium for any medical cost variability.

Substituting the government budget constraint, we can express the optimal social security benefit when $r = n$ as

$$b^* = \frac{1}{1+\theta}\cdot\left(h - \theta\cdot\left[(r+p)\cdot\bar{m} - \bar{\mu}\right] + \frac{1}{\gamma}\ln D(\hat{\mu})\right). \qquad (50a)$$

The optimal social security benefit increases in human capital earnings, *h*, decreases with the dependency rate, θ, decreases with average (social) medical costs per capita, $\bar{m}$, and increases with medical expenses borne by the retiree, $(\bar{\mu}, \hat{\mu})$. If Medicare is set optimally, $\bar{\mu} = \hat{\mu} = 0$ and $b^* = (1/(1+\theta))(h - \theta(r+p)\bar{m})$. Together, optimal Social Security and optimal Medicare fully insure consumption levels of agents so that $c^R = c^E$.

[25] As a consequence, (49) is independent of $\hat{\mu}$.

7.2 The case *r*>*n*.

To sharpen results, we consider the case of no retirement medical expenses so that $\bar{\mu} = \hat{\mu} = \bar{m} = 0$. Now (49) implies $s^E > s^R$, so social security should not fully insure retirement consumption when $r > n$. Further, in the absence of medical expenses, the optimality condition (48) simplifies to

$$e^{\gamma\left(h-\tau-b-\frac{\theta\cdot(r-n)}{\gamma\cdot(r+p)}\right)} = \frac{r+p}{n+p}.$$

Taking logarithms and using the government budget constraint, (25), we can express the optimal social security benefit as

$$b^* = \frac{1}{1+\theta}\cdot\left[h - \frac{1}{\gamma}\cdot\left(\ln\left(\frac{r+p}{n+p}\right) + \theta\cdot\frac{(r-n)}{(r+p)}\right)\right]. \qquad (50b)$$

For $\frac{r+p}{n+p}$ near unity, $\ln\left(\frac{r+p}{n+p}\right) \cong \frac{r-n}{n+p}$, and the optimal insurance benefit can be approximated by

$$b^* \cong \frac{h}{1+\theta} - \frac{(r-n)}{\gamma(1+\theta)}\cdot\left[\frac{1}{n+p} + \frac{\theta}{r+p}\right]. \qquad (50b')$$

We saw that if *r*=*n* and there are no medical expenses, then optimal social security insurance would be complete and household consumption would not change upon retirement. With *r*>*n*, optimal social security is incomplete, so consumption falls upon retirement. The fall in consumption upon retirement will be greater the smaller is the coefficient of risk aversion γ (or, equivalently, the larger is the degree of intertemporal substitution).

Proposition 8: (Optimal social insurance policies)

In an economy where $r = n$ and the risk of human capital loss and medical expenses are independent across households and no moral hazard exists, the expected lifetime utility of a household entering the economy is maximized by policies that fully insure the individual's consumption. This requires fully insuring retirement medical expenses and providing a sufficient social security benefit. If $r > n$, complete social insurance is not optimal. The optimal social insurance benefit increases with human

capital earnings (fractionally), the degree of risk aversion, the population growth rate and the mortality rate, and decreases with the dependency rate and the real interest rate.

7.3 Pareto Optimality

The above analysis considers optimal policy from the point of view of a new entrant into society. Beginning from the state where $b < b^*$ and Medicare is incomplete, such policies are also Pareto optimal and so there is no generational conflict. Using expression (46), the maximum expected lifetime utility of another worker of arbitrary generation $v < t$ is equal to

$$W^*(t,v) = \frac{-\beta \cdot e^{-\gamma \cdot \left[(r+p)\cdot a(t,v)+h-\tau-s^E\left(s^R, D(\hat{\mu}), h-\tau-b+\bar{\mu}, \frac{\pi}{r+p}\right)\right]}}{r+p} = e^{-\gamma \cdot \left[(r+p)\cdot \bar{s}^E \cdot (t-v)\right]} W^*(t) \qquad (46')$$

using $a(t,v) = a_0 + \bar{s}^E \cdot (t-v)$. The bar over s^E indicates that saving before time *t* is fixed. Hence, at time *t* the expected maximum utilities of workers born earlier than *t* are proportional to the expected maximum utility of a new entrant, so the policies are optimal for all workers. These policies maximize the utilities of households who enter the economy later than *t* also.[26]

The maximum expected lifetime utility for a retired worker of arbitrary generation *v* and who retired at an arbitrary date $v \le y \le t$ is given by:

$$W^R\left(a(t,y,v)\right) = \frac{-\beta}{r+p} \cdot e^{-\gamma\left[(r+p)\cdot a^R(t,y,v)+b-\mu-s^R\right]} \qquad (46'')$$

Any increase in *b* or reduction in μ is a windfall gain for an existing retiree. Hence the adoption of optimal policy instruments, starting from a state of the world where $b < b^*$ and Medicare is incomplete, must increase the utility of retirees. Thus we can conclude that there is no generational conflict in adopting optimal policies; such policies are therefore Pareto improving. Note that increasing *b* beyond b^* at time *t* increases the welfare of retirees but decreases the welfare of workers and future generations. While such policies may increase social welfare as defined by a social welfare function, they involve generational conflict and are not Pareto improving.

[26] This statement is conditional on taxes and benefits being the same for all generations. Clearly, living generations can be made better off at the expense of future generations through deficit policies that grant benefits to current generations at the expense of higher taxes on future generations.

This result relates to the current debate over the welfare effects of "investment-based" social security systems over PAYG systems. Feldstein and Liebman (2002, p. 2258) argue that PAYG social security is not Pareto improving if the rate of return on capital exceeds the implicit rate of return (the growth rate) on PAYG social security, and that this case underlies the welfare gain to "investment-based" social security (compulsory retirement savings accounts to replace social insurance). In our model, there is no need for "investment-based" social security because individuals save on their own accord whenever social insurance is absent, or when it offers less than the optimal benefit.[27]

Moreover, it is never optimal in our model to substitute saving, compulsory or otherwise, for PAYG social security when the benefit level is less than its optimal value b^*. If the rate of return on capital exceeds the growth rate, the correct implication is that the optimal social security insurance program should not fully insure the loss of earnings and that private saving should be positive. The condition that $r > n$ does not imply that the optimal PAYG benefit is zero and that social security should be investment-based (private saving should be substituted for social insurance). The reason we can draw this conclusion is that in our model, unlike models that treat social security strictly as rate of return on saving issue, social security performs an important insurance function. It insures people against an unexpectedly long period of retirement for which they will have inadequate resources, or an unexpectedly short period of retirement for which individuals will have over-saved. Such risk is idiosyncratic to the individual, so there is a welfare benefit to pooling it through a PAYG system. This pooling benefit is not available through an investment-based system, at least one that is based on personal accounts. Other authors, notably Rust (1999), Shiller (1998) and Storesletten et al (2004), have also emphasized the risk sharing aspects of social security.[28]

8. Conclusion

The main contribution of this paper is to develop a tractable model of overlapping

[27] Feldstein and others often assume a form of Samaritan's dilemma to justify the need for a compulsory saving plan.

[28] Both intra- and intergenerational risks can be shared. In the exponential hazard model of this paper, all risk is independent so the distinction is not important. Ball and Mankiw (2001) consider the design of social security in the context of the optimal allocation of intergenerational risk in an over-lapping generations economy. Soares (2005) argues that actuarially fair calculations of the benefits of risk sharing through social security can be misleading when households are borrowing-constrained.

generations in which working households face the risks of uncertain retirement periods and medical costs in the future. Like the Blanchard model, the key to the model's tractability is the assumption that the hazard rate for retirement, like death, is constant over the household's life and the same for all households at risk. An advantage of the model is the fact that both the life-cycle consumption shifting and the idiosyncratic risk sharing features of PAYG social insurance policies are represented. Despite the unrealism of the constant hazard rates, the model can replicate realistic demographic features of the economy.

Savings motives in the economy reflect life-cycle consumption smoothing, precautionary saving, and the productivity of capital. Demographic, economic, and policy variables affect individual saving rates in plausible ways. Simple expressions for optimal PAYG social insurance policies that relate the optimal values of policy variables to demographic and economic parameters are derived. Aggregate values of saving, wealth and consumption are readily obtained, and the effects of social insurance on the aggregate variables are easily analyzed. Crowding out of aggregate saving by PAYG social insurance is typically complete or more than complete. The model is also used to derive wealth distributions. Social insurance is found to decrease wealth concentration in the population as a whole, and among retirees.

Finally, we conclude with two caveats. First, the model is obviously stylized. This has the advantage of transparency, enabling us to obtain strong analytical results that provide insights into the impact of social insurance on consumption and savings behavior. While these results must be viewed with caution, they serve as useful benchmarks for more realistic models which, with their added complexity, must be analyzed using numerical simulation methods. Second, by taking the rate of return on capital as given, the model abstracts from the production side of the economy. Clearly it is important to extend the model to include production and capital accumulation, enabling us to analyze the relationship between social security and economic growth.

Appendix

In this Appendix we provide the details underlying the derivations in Sections 5 and 6.

A.1 Aggregate and Per Capita Wealth

Workers

At time t there are $E(t,v) \equiv (n+p)e^{nv}e^{-(p+\pi)(t-v)}$ households belonging to cohort v. Given initial wealth a_0 and savings rate s^E, at time t a cohort v worker has accumulated wealth

$$a^E(t,v) = a_0 + s^E(t-v). \tag{A.1}$$

Total wealth of all cohort v workers at time t is

$$A^E(t,v) = E(t,v)a^E(t,v) = (n+p)e^{nv}e^{-(p+\pi)(t-v)}\left(a_0 + s^E(t-v)\right)$$

and total wealth of all workers at time t is

$$A^E(t) = \int_{-\infty}^{t} A^E(t,v)dv = \theta^E e^{nt}\left(a_0 + \frac{s^E}{p+n+\pi}\right) = E(t)\left(a_0 + \frac{s^E}{p+n+\pi}\right).$$

Average per-capita wealth of workers is thus constant, given by

$$\frac{A^E(t)}{E(t)} \equiv a^E = a_0 + \frac{s^E}{p+n+\pi}. \tag{A.2}$$

Retirees

The number of retirees at time t of vintage v who worked y units of time is

$$R(t,y,v) = \pi\cdot(n+p)\cdot e^{nv}\cdot e^{-(p+\pi)\cdot y}\cdot e^{-p\cdot(t-v-y)} = \pi\cdot(n+p)\cdot e^{nv}\cdot e^{-\pi y}\cdot e^{-p\cdot(t-v)}$$

and each has accumulated wealth

$$a^R(t,y,v) = a_0 + s^E\cdot y + s^R\cdot(t-y-v). \tag{A.3}$$

The total wealth of this group is $A^R(t,y,v) = a^R(t,y,v)\cdot R(t,y,v)$, and the wealth of retirees of all cohorts who worked y units of time is

$$A^R(t,y) = \int_{-\infty}^{t-y} A^R(t,y,v)dv = \left[a_0 + s^E y + \frac{s^R}{n+p}\right]\pi e^{nt}e^{-(n+p+\pi)y}. \tag{A.4}$$

Per capita wealth of the average retiree in this group is

$$\frac{A^R(t,y)}{R(t,y)} = a^R(y) = \left[a_0 + s^E y + \frac{s^R}{n+p} \right] \tag{A.5}$$

where $R(t,y) = \int_{-\infty}^{t-y} R(t,y,v)dv = \pi e^{nt} e^{-(n+p+\pi)y}$ is the number of retirees of all vintages who work exactly *y* units of time. Aggregating over *y*, the wealth of all retirees at time *t* is

$$A^R(t) = \int_0^{\infty} A^R(t,y)dy = R(t)\left(a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{p+n} \right) \tag{A.6}$$

and the average per-capita wealth of retirees is a constant

$$\frac{A^R(t)}{R(t)} \equiv a^R = a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{p+n}. \tag{A.7}$$

Total Population

Total wealth of both workers and retirees is $A(t) = A^E(t) + A^R(t)$. Per capita wealth at *t* is

$$\frac{A(t)}{N(t)} = a = \theta^E a^E + \theta^R a^R = a_0 + \frac{s^E}{p+n+\pi} + \frac{\theta^R s^R}{p+n} = a_0 + \frac{1}{p+n} s. \tag{A.8}$$

The share of all wealth held by retirees, obtained by dividing (A.7) by (A.8), is

$$\frac{A^R(t)}{A(t)} = \theta^R \left(\frac{a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{p+n}}{a_0 + \frac{s^E}{p+n+\pi} + \frac{\theta^R s^R}{p+n}} \right). \tag{A.9}$$

A.2 Distribution of Wealth

We now derive expressions describing the distribution of wealth, reported in Section 5.2. A worker at time *t* has wealth less than or equal to $\hat{a}$ if he or she is born at time $\hat{v}$ or later, where $\hat{v} = t - \hat{a}/s^E$. Aggregating over cohorts, the number of such workers, denoted by $E\left(t, a \le \hat{a}\right)$, is

$$E\left(t, a \le \hat{a}\right) = \int_{\hat{v}}^{t} \left(n+p\right) \cdot e^{n \cdot v} \cdot e^{-\left(p+\pi\right)\cdot(t-v)} dv = E\left(t\right) \cdot \nu^E\left(a \le \hat{a}\right) \tag{A.10}$$

where $E\left(t\right)$ is the number of workers at time *t* and $\nu^E\left(a \le \hat{a}\right) = 1 - e^{-\left(n+p+\pi\right)\cdot\frac{\hat{a}}{s^E}}$ is the fraction of the worker population having wealth less than, or equal to, $\hat{a}$.

A retiree at time *t* who retired at $z \geq t - \hat{a}/s^R$ has wealth less than or equal to $\hat{a}$ if he/she was born on or later than $\hat{v} = \left(1 - \frac{s^R}{s^E}\right) \cdot z + \frac{s^R}{s^E} \cdot t - \frac{\hat{a}}{s^E}$. (We ignore households who retired before $z = t - \hat{a}/s^R$ because their wealth exceeds $\hat{a}$ from retirement saving alone.) The number of such retirees denoted $R(t, z, a \leq \hat{a})$ is

$$\begin{aligned} R(t,z,a \leq \hat{a}) &= \int_{\hat{v}}^{z} \pi \cdot (n+p) \cdot e^{nv} \cdot e^{-(p+\pi)\cdot(z-v)} \cdot e^{-p\cdot(t-z)} dv \\ &= (n+p) \cdot R(t) \cdot e^{-(n+p)\cdot(t-z)} \cdot \left[1 - e^{(n+p+\pi)\cdot\left[\frac{s^R}{s^E}\cdot(t-z) - \frac{\hat{a}}{s^E}\right]}\right] \end{aligned} \tag{A.11}$$

where $R(t)$ is the total number of retirees at time *t*. The total number of retirees at time *t*, with wealth less than or equal to $\hat{a}$, denoted by $R(t, a \leq \hat{a})$, is

$$\begin{aligned} R(t,a \leq \hat{a}) &= \int_{t-\frac{\hat{a}}{s^R}}^{t} R(t,z,a \leq \hat{a}) \cdot dz \\ &= (n+p) \cdot R(t) \cdot \int_{t-\frac{\hat{a}}{s^R}}^{t} e^{-(n+p)\cdot(t-z)} \left[1 - e^{(n+p+\pi)\cdot\frac{s^R}{s^E}\cdot(t-z)} \cdot e^{-(n+p+\pi)\cdot\frac{\hat{a}}{s^E}}\right] \cdot dz \\ &= R(t) \cdot \nu^R(a \leq \hat{a}) \end{aligned} \tag{A.12}$$

where

$$\nu^R(a \leq \hat{a}) = \xi^R \cdot \left(1 - e^{-(n+p+\pi)\cdot\frac{\hat{a}}{s^E}}\right) + (1 - \xi^R) \cdot \left(1 - e^{-(n+p)\cdot\frac{\hat{a}}{s^R}}\right) \tag{A.13}$$

is the fraction of the retiree population with wealth less than or equal to $\hat{a}$, and $\xi^R \equiv \frac{\theta^E s^E}{\theta^E s^E - s^R}$.

The total population having wealth less than or equal to $\hat{a}$, can be expressed in the form $N(t, a \leq \hat{a}) = E(t, a \leq \hat{a}) + R(t, a \leq \hat{a}) = \nu^N(a \leq \hat{a}) \cdot N(t)$ where

$$\nu^N(a \leq \hat{a}) = \xi^N \cdot \left[1 - e^{-(n+p+\pi)\cdot\frac{\hat{a}}{s^E}}\right] + (1 - \xi^N) \cdot \left[1 - e^{-(n+p)\cdot\frac{\hat{a}}{s^R}}\right] \tag{A.14}$$

is the fraction of the population with wealth less than or equal to $\hat{a}$, and $\xi^N \equiv \frac{\theta^E (s^E - s^R)}{(\theta^E s^E - s^R)}$.

The expression for the Gini coefficient for a continuous distribution can be obtained from Atkinson (1970). Normalizing the population to unity, we can derive the expression for the Gini

coefficient of wealth concentration in this economy as follows. For the mixed distribution $\nu(a) = \xi \cdot \nu_1(a) + (1-\xi) \cdot \nu_2(a)$, where ξ is a constant and $A = \int_0^\infty a \cdot \nu'(a) \cdot da$, let

$$f\left(\nu(\hat{a})\right) = \int_0^{\hat{a}} a \cdot \nu'(a) \cdot da = \frac{1}{A} \cdot \int_0^{\hat{a}} a \cdot \left[\xi \cdot \nu_1'(a) + (1-\xi) \cdot \nu_2'(a)\right] \cdot da \qquad \text{(A.15)}$$

denote the fraction of total wealth held by households with wealth less than or equal to $\hat{a}$. The fraction of the population holding this wealth is $\nu(\hat{a})$. The area under the Lorenz curve with ordinate $f(.)$ is denoted L and is equal to

$$L = \int_0^1 f(v) \cdot dv = \int_0^\infty f\left(\nu(\hat{a})\right) \cdot \nu'(\hat{a}) \cdot d\hat{a}. \qquad \text{(A.16)}$$

The Gini coefficient is $G = 1 - 2 \cdot L$.

Let $\nu_i(a) = 1 - e^{-g_i \cdot a}$ and $\nu_i'(a) = g_i \cdot e^{-g_i \cdot a}$ for *i=1,2,* where $g_1 = (n + p + \pi)/s^E$ and $g_2 = (n + p)/s^R$. Now, $A = \frac{\xi \cdot s^E}{n + p + \pi} + \frac{(1-\xi) \cdot s^R}{n + p}$, and we can integrate (A.15) and (A.16) to obtain

$$L = 1 - \left(g_1 + g_2(1 + g_1)\right) \cdot \frac{\hat{\xi} \cdot (1-\xi) \cdot g_2 - \left(1-\hat{\xi}\right) \cdot \xi \cdot g_1}{\left(g_1 + g_2\right)^2} - \frac{3}{4} \cdot \left(\hat{\xi} \cdot \xi - \left(1-\hat{\xi}\right) \cdot (1-\xi)\right) \qquad \text{(A.17)}$$

where $\hat{\xi} \equiv \xi / A \cdot g_1$. After substitution and some manipulation, the Gini coefficient for the mixed exponential distribution function can be expressed in the form:

$$G = \frac{1}{2}\left\{1 - \frac{\xi \cdot (1-\xi)}{\xi + (1-\xi) \cdot \sigma} \cdot \left[\frac{(1-\sigma)^2}{(1+\sigma)}\right]\right\} \qquad \text{(A.18)}$$

where $\sigma = \frac{s^R}{\theta^E \cdot s^E}$.

In the case of the retiree population, $\xi = \xi^R = \frac{\theta^E \cdot s^E}{\theta^E \cdot s^E - s^R} = \frac{1}{1-\sigma}$ and we can derive the Gini ratio $G^R = \frac{1}{2}\left\{1 - \frac{\xi^R \cdot \left(1-\xi^R\right)}{\xi^R + \left(1-\xi^R\right) \cdot \sigma} \cdot \left[\frac{(1-\sigma)^2}{(1+\sigma)}\right]\right\}$, which simplifies to

$$G^R = \frac{1}{2}\left\{1 + \frac{\sigma}{(1+\sigma)^2}\right\}. \qquad \text{(A.19)}$$

For the whole population, $\xi = \xi^N = \frac{\theta^E \cdot \left(s^E - s^R\right)}{\theta^E \cdot s^E - s^R} = \frac{1-\theta^E \cdot \sigma}{1-\sigma}$ and we can derive the Gini ratio $G^N = \frac{1}{2}\left\{1 - \frac{\xi^N \cdot \left(1-\xi^N\right)}{\xi^N + \left(1-\xi^N\right)\cdot\sigma} \cdot \left[\frac{(1-\sigma)^2}{(1+\sigma)}\right]\right\}$, which simplifies to

$$G^N = \frac{1}{2}\left\{1 + \frac{\theta^R \cdot \sigma \cdot \left(1-\theta^E \cdot \sigma\right)}{\left(1+\sigma\right)\cdot\left(1+\sigma(1-\theta^E)\right)}\right\} = \frac{1}{2}\left[1 + \frac{\theta^R s^R (s - s^R)}{s(s + s^R\theta^E)}\right] \tag{A.20}$$

where $s = \theta^E s^E + \theta^R s^R$.

A.3 Aggregate and Average Consumption Levels, and the Effects of Social Insurance

Workers

The equilibrium of a worker of cohort *v* at time *t* is

$$c^E(t,v) = \left(r+p\right)\cdot\left(a_0 + s^E(t-v)\right) + h - \tau - s^E \tag{A.21}$$

so total consumption of all workers of cohort v at time *t* is

$$C^E(t,v) = \left[\left(r+p\right)\cdot\left(a_0 + s^E(t-v)\right) + (h - \tau - s^E)\right]E(t,v).$$

Total consumption of all workers is

$$C^E(t) = \int_{-\infty}^{t} C^E(t,v)dv = \left[(r+p)\left(a_0 + \frac{s^E}{p+n+\pi}\right) + (h-\tau-s^E)\right]E(t) \tag{A.22}$$

and per capita consumption of the average worker is

$$\frac{C^E(t)}{E(t)} \equiv c^E = (r+p)\left(a_0 + \frac{s^E}{p+n+\pi}\right) + (h-\tau-s^E) = (r+p)a^E + (h-\tau-s^E). \tag{A.23}$$

Retirees

The consumption at time *t* of a retiree of cohort *v* who worked exactly *y* years is

$$c^R(t,y,v) = \left(r+p\right)\cdot a^R(t,y,v) - \mu + b - s^R \tag{A.24}$$

so the consumption of all retirees of cohort *v* who work *y* years is

$$C^R(t,y,v) = \left(r+p\right)\cdot A^R(t,y,v) - \overline{\mu}\cdot R(t,y,v) + (b - s^R)R(t,y,v)$$

where $\bar{\mu}$ is average medical payments. Summing over cohorts and years worked, we obtain the total consumption by retirees as

$$C^R(t) = \left\{ (r+p) \cdot \left[a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{n+p} \right] + \left[(b - s^R) - \bar{\mu} \right] \right\} R(t) \qquad \text{(A.24)}$$

and per capita consumption by the average retiree as

$$\frac{C^R(t)}{R(t)} \equiv c^R = (r+p) \left[a_0 + \frac{s^E}{p+n+\pi} + \frac{s^R}{n+p} \right] + \left[(b - s^R) - \bar{\mu} \right] = (r+p) a^R + (b - s^R) - \bar{\mu} \,. \qquad \text{(A.25)}$$

Total Population

Total consumption $C(t) = C^E(t) + C^R(t) = c \cdot N(t)$ where

$$c = \left[(r+p) \left(a_0 + \frac{s^E}{p+n+\pi} + \frac{\theta^R s^R}{p+n} \right) + (h - \tau - s^E) \cdot \theta^E + (b - \bar{\mu} - s^R) \cdot \theta^R \right] \qquad \text{(A.26)}$$

is average per capita consumption and is constant.

Changes in Social Insurance Levels

Consider the impact of a change in social insurance on workers. Suppose the change in policy causes s^E to change by $ds^E \equiv s_2^E - s_1^E$ at time *T*. The wealth of a worker at time $t > T$ is:

$$a_0 + s_1^E (T - v) + s_2^E (t - T) \qquad v < T \qquad \text{(A.27)}$$

$$a_0 + s_2^E (t - v) \qquad v \geq T \qquad \text{(A.27')}$$

Workers born before *T* accumulate at s_1^E until *T* and at s_2^E thereafter, while workers born after *T* accumulate only at s_2^E. The total wealth of workers is

$$A^E(t,v) = \int_{-\infty}^{T} (n+p) e^{nv} e^{-(p+\pi)(t-v)} \left[a_0 + s_1^E (T - v) + s_2^E (t - T) \right] dv$$

$$+ \int_{T}^{t} (n+p) e^{nv} e^{-(p+\pi)(t-v)} \left[a_0 + s_2^E (t - v) \right] dv \,.$$

Evaluating these integrals, the per capita wealth of the average worker is

$$\frac{A^E(t)}{E(t)} \equiv a^E = a_0 + \frac{1}{n+p+\pi} \left[\left(1 - e^{-(n+p+\pi)(t-T)} \right) s_2^E + e^{-(n+p+\pi)(t-T)} s_1^E \right]. \qquad \text{(A.28)}$$

When $t = T$, (A.28) reduces to

$$\frac{A^E(t)}{E(t)} \equiv a^E = a_0 + \frac{s_1^E}{n+p+\pi}$$

and as $t \to \infty$

$$\frac{A^E(t)}{E(t)} \equiv a^E = a_0 + \frac{s_2^E}{n+p+\pi}$$

Similarly, the per-capita wealth of the average retiree is:

$$\frac{A^R(t)}{R(t)} \equiv a^R = a_0 + \frac{1}{n+p+\pi}\left[\left(1-e^{-(n+p+\pi)(t-T)}\right)s_2^E + e^{-(n+p+\pi)(t-T)}s_1^E\right] + \frac{s^R}{p+n}. \qquad \text{(A.29)}$$

Equations (A.28) and (A.29) are used to derive the short-run and the long-run effects of social insurance on consumption. From (A.23) and (A.28), (evaluated as $t \to T, t \to \infty$), and (25) of the text, we see that a PAYG increase in social insurance changes the short-run and long-run per capita consumption (denoted by the respective subscripts *s*, *l*) of employed workers by

$$\left.\frac{\partial c^E}{\partial b}\right|_s = \frac{1}{\theta^E}\left[\frac{\kappa}{1+\kappa} - \theta^R\right] \qquad \text{(A.30a)}$$

$$\left.\frac{\partial c^E}{\partial b}\right|_l = -\frac{\theta^R}{\theta^E(1+\kappa)}\left[1+\kappa\left(\frac{r-n}{\pi}\right)\right] < 0 \qquad \text{(A.30b)}$$

Similarly, from (A.25) and (A.29) we see that the short-run and long-run effects of a fully funded increase in social security on the per capita consumption of retirees is

$$\left.\frac{\partial c^R}{\partial b}\right|_s = 1 \qquad \text{(A.31a)}$$

$$\left.\frac{\partial c^R}{\partial b}\right|_l = 1 - \left(\frac{\kappa}{1+\kappa}\right)\cdot\frac{r+p}{n+p} \qquad \text{(A.31b)}$$

The short-run and long-run effects on total per capita consumption of social security are

$$\left.\frac{\partial c}{\partial b}\right|_s = \frac{1}{\theta^E}\frac{\kappa}{1+\kappa} > 0 \qquad \text{(A.32a)}$$

$$\left.\frac{\partial c}{\partial b}\right|_l = -\frac{\theta^R}{\theta^E}\left(\frac{\kappa}{1+\kappa}\right)\cdot\frac{r-n}{\pi} \le 0 \text{ provided } r \ge n. \qquad \text{(A.32b)}$$

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