ANALYTIC FOUNDATIONS: MEASURING THE REDISTRIBUTIVE IMPACT OF TAXES AND TRANSFERS

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The CEQ logo is a stylized graphical representation of a Lorenz curve for a fairly unequal distribution of income (the bottom part of the C, below the diagonal) and a concentration curve for a very progressive transfer (the top part of the C).

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ABSTRACT

This paper provides a theoretical foundation for analyzing the redistributive effect of taxes and transfers for the case in which the ranking of individuals by pre-fiscal income remains unchanged. We show that in a world with more than a single fiscal instrument, the simple rule that progressive taxes or transfers are always equalizing not necessarily holds, and offer alternative rules that survive a theoretical scrutiny. In particular, we show that the sign of the marginal contribution unambiguously predicts whether a tax or a transfer is equalizing or not.

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Introduction

Suppose we observe that income inequality after taxes and transfers is lower than pre-fiscal income inequality. Can this finding be related to the characteristics of the tax and transfer system in terms of the usual indicators of progressivity and size? As shown below, once one leaves the world of a single fiscal intervention, the relationship between inequality outcomes and the size and progressivity of fiscal interventions is complex and at times counter-intuitive. In particular, in a system of multiple taxes and transfers, the simple relationship between the size of a tax (or transfer) and its progressivity, on the one hand, and its impact on inequality, on the other, no longer holds.

We start this paper with a review of the simplest case: a single fiscal intervention. The first section shows the conditions for a tax or a transfer to be equalizing. We draw, primarily, on Lambert and Duclos and Araar.[1] The second section presents the conditions for the net fiscal system to be equalizing in the case of multiple fiscal interventions. We also derive the conditions that must prevail for a particular tax or transfer to be equalizing and see that in the world of multiple interventions, some of these conditions defy our preconceptions and intuition.

Both sections of this paper assume no reranking, that is, individuals do not change their position in the post-fiscal income ordering. In other words, the poorest individual in the pre-fiscal income scale will continue to be the poorest individual in the post-fiscal income scale, the second poorest individual in the pre-fiscal income scale will continue to be the second poorest individual in the post-fiscal income scale, and, so on, all the way up to the richest individual. These sections also assume that there is dominance: that is, the pre-fiscal and post-fiscal Lorenz curves do not cross. They also assume that, when comparing systems with different taxes and transfers, the respective post-fiscal Lorenz curves do not cross either. Finally, these sections assume a constant pre-fiscal income distribution, that is, that the conditions apply to a particular country at a particular point in time. Comparisons across countries and over time will usually feature different pre-fiscal income distributions and are not the subject of this paper.

Chapter 3 of the CEQ handbook discusses how the conditions derived in sections below change in the presence of reranking. The implications of relaxing the assumption of dominance or having different pre-fiscal income distributions will be the subject of future

[1] Lambert (2001); Duclos and Araar (2007).

work. Throughout this paper, the traditional Gini coefficient is used as our measure of inequality but the ideas presented here can be easily extended to all members of the S-Gini family. However, while the idea of "marginal" analysis (introduced in this paper) can be applied to other measures of inequality, the type of decomposition that we rely on in this paper and in Chapter 3 of the CEQ Handbook one may not be applicable for other measures of inequality, such as the Theil index.

1. The Fiscal System and Income Redistribution: The Case of a Single Tax or a Single Transfer

In this section, we focus on a fiscal system with a single tax or transfer. Here we define concepts that we use throughout this paper to analyze the effect of a tax or a transfer on the income distribution. We should first clarify that the word "single" does not mean that a system has only one tax but rather that the same conditions apply when all taxes are combined into a single category.[2]

1.1. A Single Tax

We start by presenting some notations and definitions that will be used throughout the paper:

x = pre-tax income

$f(x)$ = pre-tax income distribution

$T(x)$ = tax liability at income x

$x - T(x)$ = post-tax income

[2] This section draws from Lambert (2001) and Duclos and Araar (2007).

$t(x) = T(x)/x$ = tax rate at income x

$t'(x)$ = marginal tax rate at income x

Let's assume that the tax schedule adheres to a typical pattern of starting at a zero rate and that it follows a sequence of fixed and increasing marginal tax rates.[3] Let's also assume that both the tax liability and post-tax income increase with pre-tax income:

(1) $0 \leq T(x) < x$

(2) $0 \leq t'(x) < 1$

Condition 2 rules out reranking; that is, no pair of individuals switch places after the tax has been imposed.

Now, let's define the following terms:

T = total taxes paid= $\sum_i T(x_i)$

X = total pre-tax (and pre-transfers) income = $\sum_i x_i$

g = total tax ratio = T / X; thus, $(1 - g) = (X - T) / X$ and $g / (1 - g) = T / (X - T)$ = Total tax as a share of pre-tax income

[3] Lambert (2001).

$g = \frac{\int_i T(x_i) f(x_i) dx_i}{\int_i x_i f(x_i) dx_i}$ = total tax ratio (continuous version)

$L_X(p)$, $L_{X-T}(p)$= Lorenz curve of pre-tax income and post-tax income, respectively (ranked by original income)

$C_{X-T}(p)$, $C_T(p)$ = Concentration curve of post-tax income and taxes, respectively (ranked by original income)[4]

In all preceding formulas p has a value between zero and one and represents quantile p of income distribution in which $100p$% of individuals are below it.

It can be shown that the Lorenz curve of pre-tax income is the weighted average of the concentration curve of taxes and the concentration curve of post-tax income:

(3) $$L_X(p) = g\, C_T(p) + (1-g)\, C_{X-T}(p).$$

Because of conditions 2 and 2, the ranking of people by pre-tax and post-tax income is exactly the same. Thus, condition 3 can be re-written simply as the weighted average of the concentration curve of taxes and the Lorenz curve of post-tax income:

(3)' $$L_X(p) = g\, C_T(p) + (1 - g)\, L_{X-T}(p).$$

[4] Recall that concentration curves plot the cumulative shares of post-tax income and taxes by positions in pre-tax income distribution (in notational terms, if there is no superscript, they are ranked by pre-tax income). The reader should recall that a concentration coefficient is calculated in the same manner as the Gini coefficient. The difference is the same as that between the Lorenz and concentration curves: the cumulative distribution of the tax (in this case) is plotted against the cumulative distribution of the population ranked by original income and *not* the tax.

1.1.1. Equalizing, Neutral, and Unequalizing Net Fiscal Systems: Conditions for the One Tax Case

In this section, we review conditions that allow us to determine whether a fiscal system with only a single tax is equalizing, neutral, or uneqaulizing.

Concentration and Lorenz Curves

When the post-tax income Lorenz curve lies everywhere above the pre-tax income Lorenz curve, that is $L_{X-T}(p) \geq L_X(p)$, the tax is equalizing (and vice versa).

Equation 3 implies that the post-tax income Lorenz curve lies completely above the pre-tax income Lorenz curve if and only if the concentration curve of taxes lies completely below the pre-tax income Lorenz curve.[5]

(4) $L_{X-T}(p) \geq L_X(p) \Leftrightarrow C_T(p) \leq L_X(p)$ for all p, and with strict inequality for some p

In other words, the distribution of post-tax income is less unequal than the pre-tax income distribution if and only if the tax is distributed more unequally than the income to which it applies, or put another way, if and only if the concentration curve of taxes lies completely below the pre-tax income Lorenz curve. This condition is shown on figure 1, which features the Lorenz curves for pre-tax and post-tax income and the concentration curve for taxes.

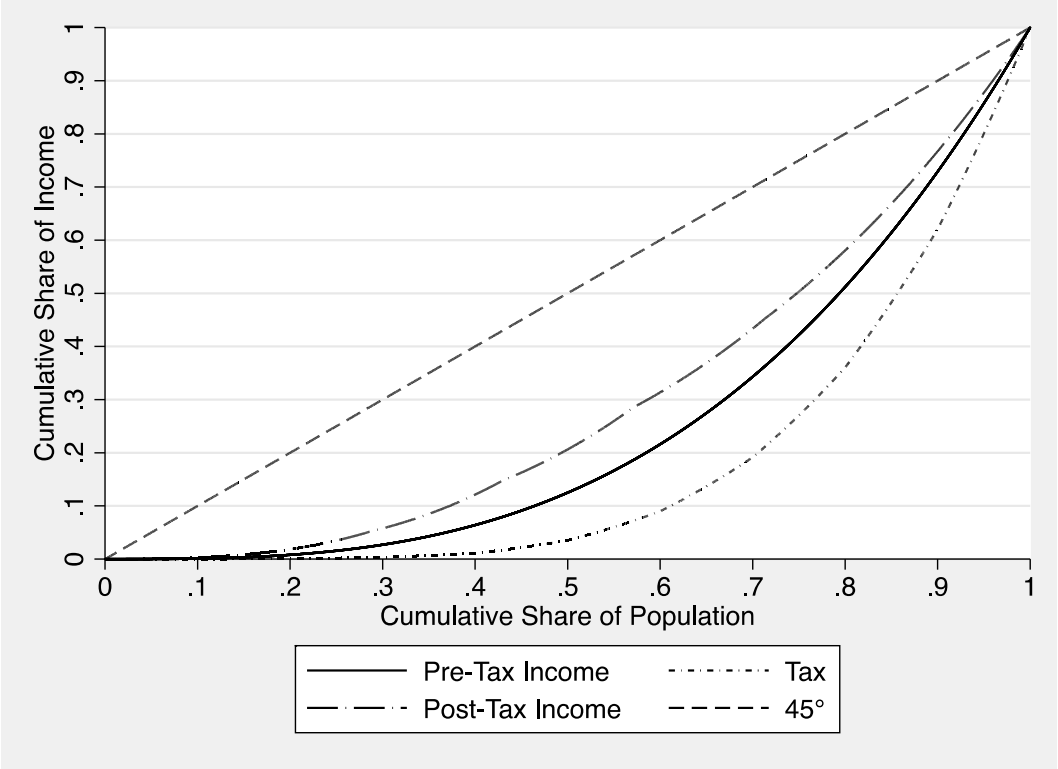
In other words, if the average tax rate $t(x)$ is increasing with income *everywhere,* then taxes are distributed more unequally than pre-tax income. Thus, an everywhere progressive tax will always be equalizing.

Given equation 4, it is easy to see that the condition for a tax to be unequalizing is $C_T(p) \geq L_X(p)$. This condition will occur if $t(x)$ decreases with income, that is, if taxes are regressive everywhere. However, just like in the case of progressive taxes, it is not necessary for taxes to be regressive everywhere to be unequalizing. Finally, in the case of a proportional tax--that is,

[5] This is true because if $0 < g < 1$, the weights by definition sum to one. Hence $L_X(p)$ must lie between $C_T(p)$ and $C_{X-T}(p)$ by necessity.

when $T(x)/x$ is the same for all x--the distribution of post-tax and pre-tax income will be exactly the same and $C_T(p) = L_X(p)$.

Figure 1. Lorenz Curve of Pre-Tax Income and Post-Tax Income and Concentration Curve of Tax

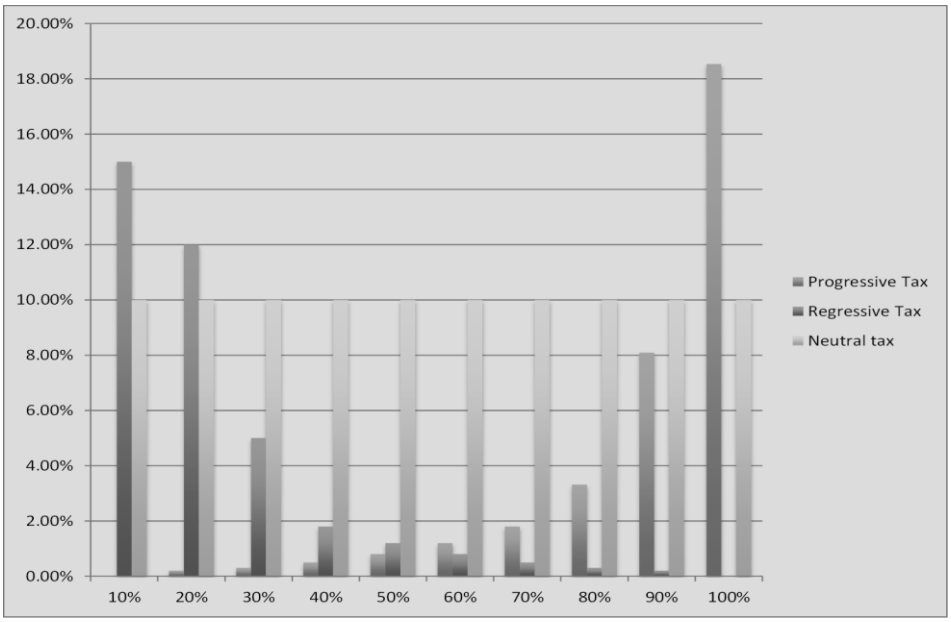


In sum, incomes are *less* unequal after a tax than before the tax if and only if the tax is distributed more unequally than the income to which it applies. Incomes are *more* unequal after a tax than before the tax if and only if the tax is distributed more equally than the income to which it applies. A proportional tax will have the same distribution as the pre-tax income and leave the distribution of income unchanged. A poll tax, which taxes all individuals by the same absolute amount, will feature a concentration curve coincidental with the diagonal, that is, it will be very unequalizing.[6]

[6] Although not impossible in principle, taxes in absolute terms (that is, per capita) rarely decline with income in the real world. If such a tax were to exist, its concentration curve would lie above the diagonal and be extremely unequalizing.

If condition 2 is *everywhere* observed, plotting the average tax rate $T(x) / x$ against values (or quantiles) of pre-tax income will be sufficient to determine whether a tax system is *everywhere* progressive (tax rates rise with income), neutral (tax rates are the same for all incomes--a flat tax), or regressive (tax rates decrease with income). For example, if we are sure that condition 2 is strictly observed within deciles, we can determine whether a tax system is progressive, regressive, or neutral by plotting the incidence of the tax by decile as we do in figure 2.

Figure 2. Average Tax Rate by Pre-Tax Income: A Progressive, Neutral, and Regressive Tax



Globally Progressive Taxes and Taxes That Are Everywhere Progressive

Note, however, that taxes do not have to be progressive everywhere for the distribution of post-tax income to be less unequal than the pre-tax income distribution. A necessary and sufficient condition for a tax to be equalizing is for it to be globally progressive, that is, that $C_T(p) \leq L_X(p)$ for all p and strict inequality for some p and for any distribution of pre-tax income.

The following toy example in table 1 illustrates the difference between a tax that is progressive everywhere and one that is globally progressive only.

Table 1. An Everywhere Progressive Tax and a Globally Progressive Tax

Everywhere Progressive Tax

Population	Pre-tax Income	Lorenz Curve Pre-Tax	Average Tax Rate (Everywhere Progressive)	Tax paid	Post-tax Income	Lorenz Curve Post-Tax	Difference Between Post and Pre-tax Lorenz Curves
1	$10.00	10.0%	0%	$0.00	$10.00	13%	2.50%
2	$20.00	30.0%	10%	$2.00	$18.00	23%	5.00%
3	$30.00	60.0%	20%	$6.00	$24.00	30%	5.00%
4	$40.00	100.0%	30%	$12.00	$28.00	35%	0.00%
Total	$100.00		20%	$20.00	$80.00	100%	

Globally Progressive Tax

Population	Pre-tax Income	Lorenz Curve Pre-Tax	Average Tax Rate (Not Everywhere Progressive)	Tax paid	Post-tax Income	Lorenz Curve Post-Tax	Difference Between Post and Pre-tax Lorenz Curves
1	$10.00	10%	0%	$0.00	$10.00	13%	2.50%
2	$20.00	30%	18%	$3.60	$16.40	21%	3.00%
3	$30.00	60%	12%	$3.60	$26.40	33%	6.00%
4	$40.00	100%	32%	$12.80	$27.20	34%	0.00%
Total	$100.00		20%	$20.00	$80.00	100%	

The Kakwani Index

To assess whether a tax is equalizing or not, one can also use the Kakwani index of progressivity.[7] Kakwani's index of progressivity of tax t is defined as the difference between the concentration coefficient (C_T) of the tax and the Gini coefficient of pre-tax income (G_X), or

[7] Kakwani was among the first to propose a measure of tax progressivity based on "disproportionality," that is, by the extent to which a tax distribution was not proportional to the distribution of pre-tax income. See Kakwani (1977).

(5) $$\Pi_T^K = C_T - G_X,$$

where C_T is the concentration coefficient of the tax t and G_X is the Gini coefficient of pre-tax income. The conditions for a tax to be equalizing, neutral, or unequalizing are $\Pi_T^K > 0$, $\Pi_T^K = 0$, and $\Pi_T^K < 0$, respectively.

Table 2 presents a summary of the conditions described above. Of course, if the tax meets the sufficient condition, it implies that the necessary condition is met too (but not vice versa). Since we assumed there is no reranking, the disproportionality measures such as the concentration curves and the Kakwani index translate immediately into measures of redistribution.

Table 2. Conditions of Equalizing, Neutral, and Unequalizing Taxes

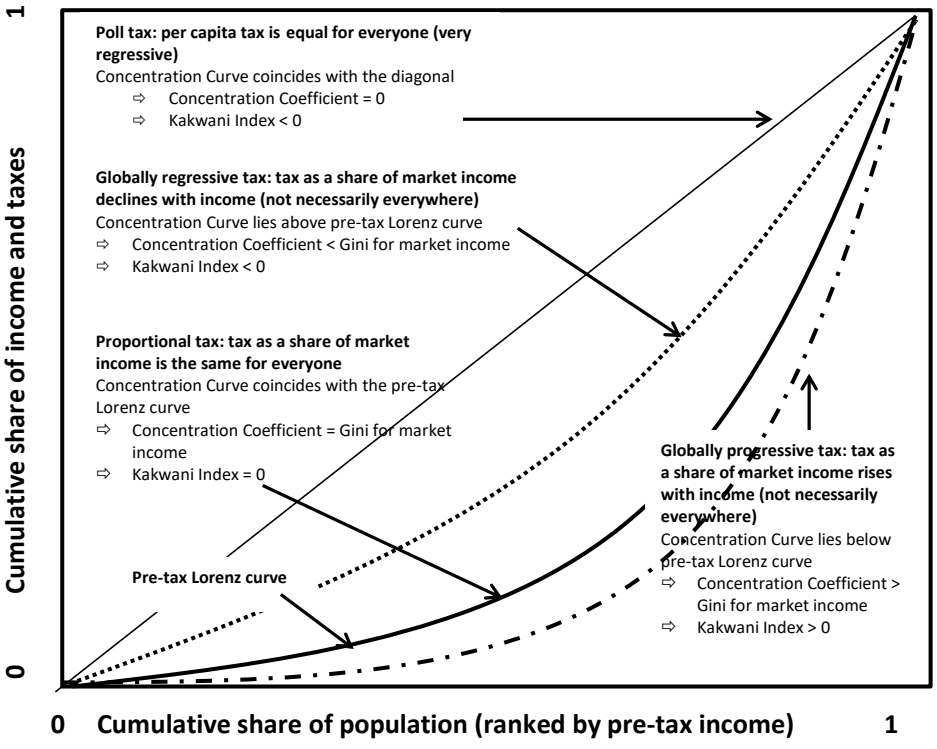
Tax	Sufficient	Necessary and Sufficient
Equalizing	$t'(x) \geq 0$ for all x with some $t'(x) > 0$	$C_T(p) \leq L_X(p)$ for all p and for any distribution of pre-tax income **OR** $\Pi_T^K > 0$
Neutral	$t'(x) = 0$ for all x	$C_T(p) = L_X(p)$ for all p and for any distribution of pre-tax income **OR** $\Pi_T^K = 0$
Unequalizing	$t'(x) \leq 0$ for all x with some $t'(x) < 0$	$C_T(p) \geq L_X(p)$ for all p and for any distribution of pre-tax income **OR** $\Pi_T^K < 0$

If there is reranking, the link between inequality and measures of disproportionality is no longer straightforward because with reranking we need to use equation 3, that is, $L_X(p) = g\ C_T(p) + (1-g)\ C_{X-T}(p)$ instead of equation 3'. Note that in equation 3, the post-tax income Lorenz curve has been replaced by the post-tax income concentration curve (the distribution of post-tax income with individuals ranked by pre-tax income). Because we are no longer comparing two income distributions with the presence of reranking, some of the "redistribution" will not be actual redistribution; instead, the tax will be reordering individuals. The consequences of reranking are further discussed in chapter 3 of the CEQ handbook.[8]

In addition, because we assume that the post-tax income Lorenz dominates the pre-tax income Lorenz curve, we can be sure that the Kakwani index will give an unambiguous ordering of different taxes in terms of progressivity (the implication of no dominance is left for future work). However, it is important not to extrapolate from progressivity to impact on inequality when comparing taxes of different sizes. We discuss this issue in the subsection on comparing taxes in section 1.1.

Measures of progressivity of a tax are presented diagrammatically in figure 3.

Figure 3. Progressive, Neutral, and Regressive Taxes



[8] See also Urban (2009).

1.1.2. Comparing Two Taxes of Different Sizes

We have just shown how progressivity determines whether a tax in a single tax system is equalizing or not. Does this mean that the more unequally distributed a tax is (that is, the more progressive), the more equalizing it is? The following example will show that this is not necessarily the case.[9] In table 3, we present two hypothetical taxes taken from Duclos and Tabi, A and A'.[10] We can see that tax A' is more unequally distributed (that is, more progressive) than tax A, or using the terminology presented in the previous section, that the concentration curve of tax system A lies completely above the concentration curve of tax system A' (that is, A is less disproportional than A'). Yet, the post-tax distribution is more unequal under tax system A'. How can that be? Notice that tax system A' collects a lower share of post-tax income than system A. The higher tax ratio in A more than compensates for its lower progressivity to the point that the redistributive effect in A is higher.

Table 3. Redistributive Effect and the Progressivity and Size of Taxes

Individual	Gross income		Tax A=50.5%		Net income under A		Tax A'=1%		Net income under A'	
	Income ($)	Distribution (%)	Tax ($)	Distribution (%)	Income ($)	Distribution (%)	Tax ($)	Distribution (%)	Income ($)	Distribution (%)
1	21	21	1	2	20	40	0	0	21	21
2	80	79	50	98	30	60	1	100	79	79
Total	101	100	51	100	50	100	1	100	100	100

Source: Duclos and Tabi, 1996, table 1.

The extent of disproportionality is not sufficient to compare the redistributive effect across different taxes. What indicators can we use? There are three options: comparing the post-tax Lorenz curves, comparing the residual progression functions, or comparing the Reynolds-Smolensky (R-S) indices if one wishes to use a scalar instead of a function. In the absence of reranking and if there is Lorenz dominance, the three approaches are equivalent.

[9] This section draws from Lambert (2001) and Duclos and Araar (2007).
[10] Duclos and Tabi (1996).

The first condition is straightforward. If the Lorenz curve of post-tax income A dominates the Lorenz curve of post-tax income A', inequality will be reduced more greatly under the former than the latter.

Residual progression is defined as the elasticity of post-tax income with respect to pre-tax income (that is, the percentage change in post-tax income per one percent change in pre-tax income) and can be written as follows:

(6) $$RP_{X\text{-}T} = [\partial\ (X - T\,(X))/\ \partial X]\ [X\ /\ ((X - T\,(X))],$$ and

(7) $$RP_{X\text{-}T} = (1 - T'\,(x))/(1 - T(x)/x).$$

If $RP_{X\text{-}T} < 1$ everywhere, the tax is progressive everywhere. To determine if tax A is more equalizing than tax A', compare the residual progression for tax A and A'. If $RP_{X\text{-}T}$ for tax A lies completely below the $RP_{X\text{-}T}$ of tax A', the former will generate a higher reduction in inequality than the latter.

Finally, the Reynolds-Smolensky (R-S) index is defined as

(8) $$\Pi_{T}^{RS} = G_X - C_{X\text{-}T} = g\ /\ (1 - g)\ (C_T - G_X) = [g\ /\ (1 - g)]\ \Pi_{T}^{K}$$

where $C_{X\text{-}T}$ is the concentration coefficient of post-tax income, G_X is the Gini coefficient of pre-tax income, C_T is the concentration coefficient of tax T, and Π_{T}^{K} is the Kakwani index of progressivity of tax T defined as $C_t - G_x$ (see the subsection on progressive taxes in section 1.1).

To see this equality, note the following. Lerman and Yitzhaki prove that

$$C_Q = \frac{2cov(Q, F_X)}{\mu_Q}$$

where $cov(Q, F_X)$ is the covariance between income concept or component Q and ranking of individuals with respect to the original income (that is, X).[11] Moreover, μ_Q is the average value of income concept or component Q among all individuals. Similarly,

$G_X = \frac{2cov(X,F_X)}{\mu}$.

Therefore, we have the following:

$$G_X - C_{X-T} = G_X - \frac{2cov(X-T,F_X)}{\mu(1-g)} = G_X - \frac{2cov(X,F_X)}{\mu(1-g)} + \frac{2cov(T,F_X)}{\mu(1-g)}$$

$$= G_X - \left(\frac{1}{1-g}\right)\frac{2cov(X,F_X)}{\mu} + \left(\frac{g}{1-g}\right)\frac{2cov(T,F_X)}{\mu g}$$

$$= G_X - \left(\frac{1}{1-g}\right)G_X + \left(\frac{g}{1-g}\right)C_T$$

$= \left(\frac{g}{1-g}\right)(C_T - G_X)$.

Under no re-ranking, it turns out that the R-S index is identical to the redistributive effect (RE), that is, the change in inequality between pre-tax and post-tax income distribution measured in Gini points.[12]

With no re-ranking,

$C_{X\text{-}T} = G_{X\text{-}T}$

(8)' $RE = G_X - G_{X\text{-}T} = g / (1 - g)\,(C_T - G_X) = \Pi_T^{RS} = [g / (1 - g)]\,\Pi_T^K$

[11] Lerman And Yitzhaki (1989).
[12] This result can be generalized to a wide range of inequality measures of the S-Gini family. See also Lambert (2001) and Duclos and Araar (2007).

The R-S Index, Π_T^{RS}, is greater than, equal to, or less than 0, depending on whether the tax is equalizing, neutral, or unequalizing, respectively. The larger the R-S index, the more equalizing the tax. Thus, we can use Π_T^{RS} to order different taxes individually based on their redistributive effects.

The R-S index (Π_T^{RS}) shows exactly how the redistributive effect does not depend *only* on the extent of progressivity. It is an increasing function of the latter *and* the tax ratio *g*.[13] Therefore, either making a given tax more progressive or raising the tax ratio of a progressive tax can increase the redistributive effect. In the case of a regressive tax, either making the tax less regressive or lowering the tax ratio will make its effect less unequalizing. We summarize these conditions in table 4.

Table 4. Conditions for the Redistributive Effect and Progressivity and Size of Taxes

	Necessary and sufficient conditions
Tax A is more **equalizing** than Tax A' *if*	$L^{A}_{X\text{-}T}(p) \geq L^{A'}_{X\text{-}T}(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income, **OR** $RP^{A}_{X\text{-}T}(p) \leq RP^{A'}_{X\text{-}T}(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income.
Tax A is more **unequalizing** than Tax A' *if*	$L^{A}_{X\text{-}T}(p) \leq L^{A'}_{X\text{-}T}(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income, **OR** $RP^{A}_{X\text{-}T}(p) \geq RP^{A'}_{X\text{-}T}(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income.

[13] See Lambert (2001).

We have developed table 4 assuming there is no reranking. If there is reranking, the link between the progressivity and size of a tax and its redistributive effect is no longer straightforward, and thus comparisons are no longer straightforward either. (The consequences of reranking in chapter 3 of the CEQ Handbook.) In addition, the three conditions in table 4 are equivalent under the assumption that the post-tax Lorenz curve under a specific tax dominates the post-tax Lorenz curve under another tax. We have left the discussion of the implications of no dominance for future work

Note also that the conditions for comparing the redistributive effect between different taxes characterized by different degrees of progressivity and size were defined for the case in which the pre-tax income distribution is always the same. The comparison of the redistributive effect of taxes (and transfers) in cases when the original income distributions are not the same is left for future work.[14]

More important, even without reranking, with dominance and keeping the original distribution constant, in the case of more than one intervention the neat relationship between the size and progressivity of a fiscal intervention and its redistributive effect no longer holds. As we will see in section 2 of this paper, a tax can be regressive using any of the necessary or sufficient conditions spelled out in table 2 and still exert an equalizing influence on the post-tax and transfer income distribution, by which we mean that, in the absence of such a tax, the reduction in inequality would be smaller than with the tax in place. Before we turn to this topic, however, we will present the analogous conditions for a single transfer.

1.2. A Single Transfer

The word "single" here does not mean that the conditions derived in this section apply to a system with only one transfer. In the case of multiple transfers, however, they need to be aggregated into one category in order for the conditions to apply.

Transfers here encompass a wide spectrum of benefits provided by the government such as cash transfers, school food programs, consumption subsidies, and access to free public services. We will use the words transfer and benefit interchangeably and use the abbreviation *B* for both.

[14] Interested readers can refer to Dardanoni and Lambert (2000).

We will also use the following definitions:

x = pre-tax income

$B(x)$ = transfer at income x

$x + B(x)$ = post-transfer income

$B(x) / x = b(x)$ = average benefit rate at income x

$b'(x)$ = marginal benefit rate

B = total transfers = $\sum_{i=1} B(x_i)$

b = total transfers ratio = B / X

=> $(1 + b) = (X + B) / X$

=> $b / (1 + b) = B / (X + B)$

$L_X(p)$, $L_{X+B}(p)$ = Lorenz curve of pre-transfer income and post-transfer income, respectively (ranked by original income)

$C_{X+B}(p)$, $C_B(p)$ = Concentration curve of post-transfer income and transfer, respectively (ranked by original income)

It can be shown that

$$L_X(p) = (1 + b)\, C_{X+B}(p) - b\, C_B(p) \tag{9}$$

which implies that

$$L_X(p) \geq C_{X+B}(p) \Leftrightarrow C_{X+B}(p) \geq C_B(p). \tag{10}$$

If we assume no re-ranking, that is,

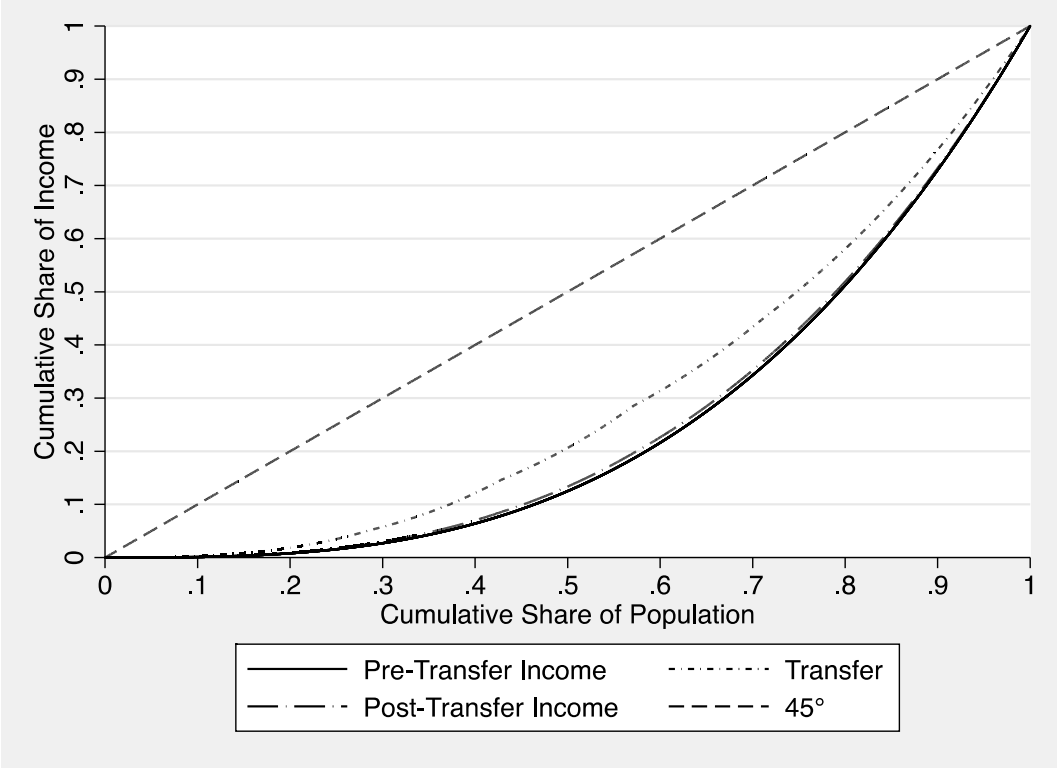
$$-1 \leq b'(x) \tag{11}$$

where $b'(x)$ is the increase in benefits that occurs as pre-transfer income X rises, the ranking of people by pre-transfer and post-transfer income does not change. Thus, equation 10 can be re-written as

$$L_X(p) \geq L_{X+B}(p) \Leftrightarrow L_{X+B}(p) \geq C_B(p). \tag{10'}$$

Under no reranking, incomes are less unequal after transfers than before if and only if transfers are distributed more *equally* than the income to which they apply. If the average transfer rate $b(x)$ decreases with income *everywhere*, then transfers are distributed more equally than pre-transfer income. This scenario is shown in figure 4.

Figure 4. A Progressive Transfer: Lorenz Curve of Pre-Transfer Income, Concentration Curve of an Equalizing Transfer, and Lorenz Curve of Post-Transfer Income



For instance, although cash transfers are very unlikely to be regressive, this is not the case with subsidies, contributory pensions, and spending on tertiary education, which are sometimes regressive in the real world. An everywhere regressive transfer will fulfill the following condition:

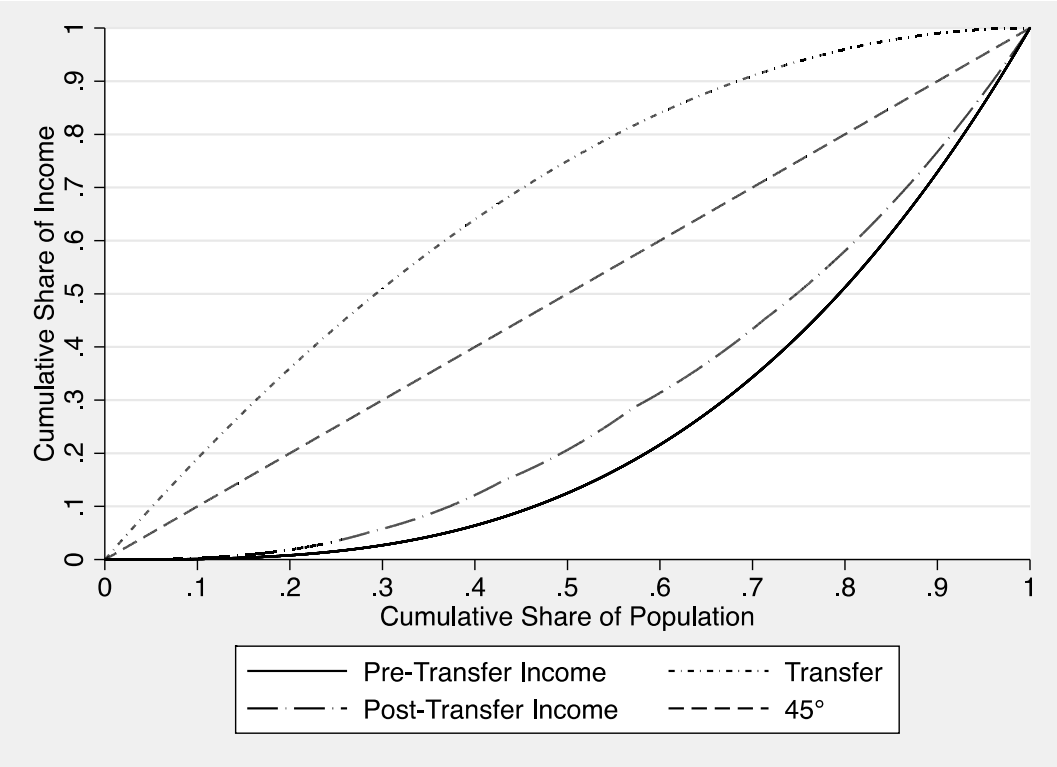
(10)" $$L_X(p) \leq L_{X+B}(p) <=> L_{X+B}(p) \leq C_B(p)$$

When 10" occurs, benefits will be unequalizing.

However, equalizing transfers may not be pro-poor. As long as the relative size of the transfer declines with income, a transfer will be equalizing. In order to be pro-poor, however, the *absolute* size of the transfer also needs to decline with income (although not so much that the marginal benefit is less than −1). That is, the share of a transfer going to the rich can be higher than the share going to the poor even if a transfer is equalizing (or progressive).

Figure 5 shows the concentration curve for a transfer that is both equalizing and pro-poor.

Figure 5. A Pro-Poor Transfer: Lorenz Curve of Pre-Transfer Income, Concentration Curve of an Equalizing Transfer and Lorenz Curve of Post-Transfer Income



1.2.1. Fiscal Systems: Comparing Two Single-Transfer Systems of Different Sizes

So far, we have shown that in a system with only one transfer and no reranking, a progressive transfer is equalizing. Does this mean that the more progressive a transfer is (that is, the more progressive or disproportional), the more equalizing it is? Table 5 shows that this need not be the case: transfer A is not only more progressive but also more pro-poor than A' yet the post-transfer distribution is considerably more equal with transfer A than with transfer A'.

Table 2-5. Redistributive Effect and the Progressivity of Transfers

Population	Gross Income		Transfer A		Net Income under A		Transfer A'		Net Income under A'	
	Income	Distribution	Transfer	Distribution	Income	Distribution	Transfer	Distribution	Income	Distribution
1	21	21%	50	98%	71	0.47	1	100%	22	22%
2	80	79%	1	2%	81	0.53	0	0%	80	78%
Total	101	100%	51	100%	152	1	1	100%	102	100%

As with taxes, the redistributive effect of a transfer depends not only on its progressivity but also on its relative size. That is, under no reranking,

(11) $$RE = G_X - G_{X+B} = b / (1 + b) [G_X - C_B] = \boldsymbol{\rho_B^{RS}} = [b / (1 + b)] \boldsymbol{\rho_B^K}$$

where $\boldsymbol{\rho_B^{RS}}$ and $\boldsymbol{\rho_B^K}$ are the R-S index and Kakwani index of the benefit *B*, respectively.[15] This equation highlights the fact that the redistributive effect does not depend on the extent of progressivity (disproportionality) of the transfer *only*. Rather, the redistributive effect depends on both the extent of progressivity *and* the relative size of the transfer, $b / (1 + b)$, which equals the total transfer divided by the post-transfer total income. Therefore, either making a given transfer more progressive or raising the relative size of a progressive transfer can increase the redistributive effect. The R-S index can also be used to compare the redistributive effect across transfers.

As in the case of taxes, the R-S is a summary index and thus will not alert us to cases in which a transfer is more redistributive in some parts of the distribution and less in others. Additionally, as with taxes, one can use the residual progression to compare the redistributive effect of transfers across the entire distribution.

We summarize these results and present the conditions under which a transfer exerts an equalizing force on the pre-transfer distribution of income in table 6.

Table 6. Conditions of Equalizing, Neutral, and Unequalizing Transfers

A transfer is	Sufficient	Necessary and Sufficient
Equalizing, if	$-1 < b'(x) \leq 0$ for all $\boldsymbol{x}$ and $b'(x) < 0$ for some $\boldsymbol{x}$	$C_B(p) \geq L_X(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income
Neutral, if	$b'(x) = 0$ for all $\boldsymbol{x}$	$C_B(p) = L_X(p)$ for all p and for any distribution of pre-tax income
Unequalizing, if	$b'(x) \geq 0$ for all $\boldsymbol{x}$ and $b'(x) > 0$ for some $\boldsymbol{x}$	$C_B(p) \leq L_X(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income

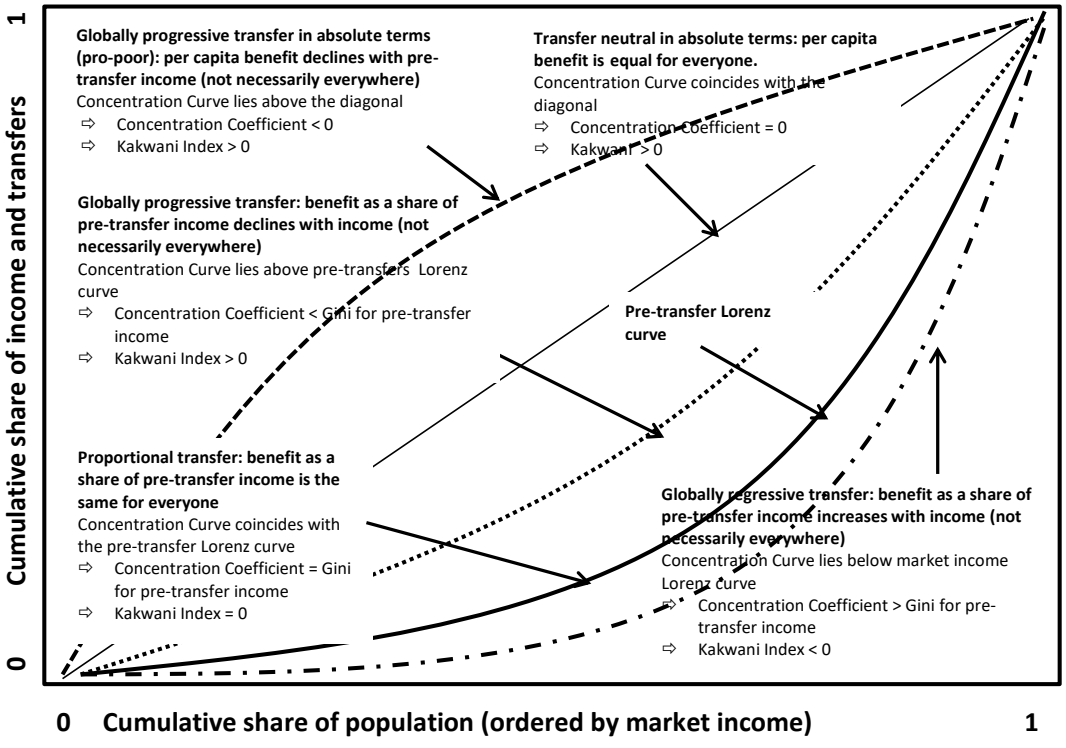
[15] The proof of this formula is similar to equation 2-8 explained earlier.

In the case of transfers, the literature tends to distinguish between a relatively progressive transfer and a transfer that is progressive in absolute terms.[16] The former is defined by the following condition $b'(x) \leq 0$ for all $\boldsymbol{x}$ and $b'(x) < 0$ for some $\boldsymbol{x}$. This condition is sufficient for a transfer to be equalizing. However, this condition does not need to be fulfilled in order for a transfer to be equalizing. As mentioned previously, the necessary and sufficient condition is $C_B(p) \geq L_X(p)$ for all p, with strict inequality for some p, and for any distribution of pre-tax income or for $\boldsymbol{\rho_B^K} > 0$.

In the case of a transfer that is progressive in absolute terms, the concentration curve $C_B(p)$ is not compared to the $L_X(p)$ but rather to the population shares or the diagonal. When the transfer tends to decline with income in per capita terms, that is $B(x)$, transfers are called progressive in absolute terms. They are also sometimes called "pro-poor."

In figure 6, we present hypothetical concentration curves for progressive, neutral (proportional), and regressive transfers. Among the progressive transfers, we distinguish between the transfers that are progressive in relative and in absolute terms. A simple way to identify a transfer that is progressive in absolute terms is by the sign of its concentration coefficient, which will be negative.

Figure 6. A Diagrammatic Representation of Progressivity of Transfers



[16] Such a distinction is not made in the case of taxes because no one expects per capita taxes to increase with income.

2. The Fiscal System and Income Redistribution: Multiple Taxes and Transfers

This section derives the conditions for fiscal redistribution in a world of multiple fiscal interventions.[17] We first derive the conditions for the simple one tax-one transfer case and, subsequently, for the case with multiple taxes and transfers. Suppose we observe that post-fiscal income inequality is lower than pre-fiscal income inequality. Can we relate this finding to the characteristics of specific taxes and transfers in terms of indicators of progressivity and size? As demonstrated in the following section, once we leave the world of a single fiscal intervention, the relationship between inequality outcomes and the size and progressivity[18] of fiscal interventions is complex and at times counter-intuitive. In particular, the relative size and progressivity of a fiscal intervention by itself can no longer tell us if inequality would be higher or lower without it. We will show that, under certain conditions, a fiscal system that includes a regressive tax can be *more* equalizing than a system that excludes it.[19] In the same vein, a fiscal system that includes a progressive transfer can be *less* equalizing than a system that excludes it.

The so-called "Lambert's Conundrum" helps to illustrate this point in the case of a regressive tax.[20] Table 7 below shows that "taxes may be regressive in their effect on original income…and yet the net system may exhibit more progressivity" than the progressive benefits alone.[21] The R-S index for taxes in this example is equal to −0.0517, highlighting their regressivity.[22] Yet, the R-S index for the net fiscal system is 0.25, higher than the R-S index for benefits equal to 0.1972. If taxes are regressive in relation to the original income[23] but progressive with respect to the less unequally distributed post-transfers (and subsidies) income, regressive taxes exert an equalizing effect over and above the effect of progressive transfers.[24]

[17] The word "multiple" is used as opposed to the word "single." In the case of a "single" tax or transfer, we either deal with only one tax or transfer or a group of taxes or transfers that are combined and treated as one incident.
[18] Using, for example, the Kakwani index of progressivity.
[19] See also Lambert (2001, p. 278), for the same conclusion.
[20] Lambert (2001, p.278).
[21] Lambert (2001, p.278).
[22] Since there is no reranking, the R-S index equals the difference between the Ginis before and after the fiscal intervention.
[23] Note that original income is in fact the "tax base" in this example.
[24] Note that Lambert uses the terms "progressive" and "regressive" differently than other authors in the theoretical and empirical incidence analysis literature. Thus, he calls transfers that are equalizing "regressive." See definitions in chapter 1 of the CEQ Handbook.

Table 7. Lambert's Conundrum

Individual	1	2	3	4	Total
Original income x	10	20	30	40	100
Tax liability (T)	6	9	12	15	42
Benefit level (B)	21	14	7	0	42
Post-benefit income	31	34	37	40	142
Final income	25	25	25	25	100

Source: Lambert, 2001, p. 278, table 11.1.

Note that Lambert's conundrum is not equivalent to the well-known (and frequently repeated) result that efficient regressive taxes can be fine as long as the net fiscal system is equalizing when combined with transfers.[25] The surprising aspect of Lambert's conundrum is that a net fiscal system *with* a regressive tax (in relation to pre-fiscal income) can be *more* equalizing than *without* the tax.[26]

The implications of Lambert's "conundrum" for real fiscal systems are quite profound. In order to determine whether a particular intervention (or a particular policy change) is inequality-increasing or inequality-reducing--and by how much--one must resort to numerical calculations that include the whole system. As Lambert mentions, his example is "not

[25] As Higgins and Lustig (2016) mention, "efficient taxes that fall disproportionately on the poor, such as a no-exemption value added tax, are often justified with the argument that "spending instruments are available that are better targeted to the pursuit of equity concerns" (quoted in Keen and Lockwood, 2010, p. 141). Similarly, Engel and others (1999, p. 186) assert that "it is quite obvious that the disadvantages of a proportional tax are moderated by adequate targeting" of transfers, because "what the poor individual pays in taxes is returned to her." Ebrill and others (2001, p. 105) argue that "a regressive tax might conceivably be the best way to finance pro-poor expenditures, with the net effect being to relieve poverty." For the interested reader, the paper appears as chapter 4 of the CEQ Handbook as well.

[26] It can also be shown that if there is reranking (a pervasive feature of net tax systems in the real world), making a tax (or a transfer) more progressive can *increase* post-tax and transfer inequality. In Lambert's example, regressive taxes not only enhance the equalizing effect of transfers, but making taxes more progressive (that is, more disproportional in the Kakwani sense) would result in *higher* inequality. Any additional change (towards more progressivity) in taxes or transfers would just cause reranking and an increase in inequality.

altogether farfetched."[27] For example, two renowned studies in the 1980s found this type of result for the United States and the United Kingdom.[28] Moreover, two recent studies for Chile found that although the value-added tax (VAT) is regressive, it is equalizing.[29] The conundrum, however, can occur with transfers as well: a transfer may be progressive but unequalizing, as was the case for contributory pensions in the CEQ Assessment for Colombia.[30] In this analysis, the Kakwani index for contributory pensions was positive but unequalizing in the sense that the reduction in inequality would have been higher without the contributory pensions (and the rest of the fiscal interventions) in place.

Estimating the sign and order of magnitude of the contribution of a particular intervention to the change in inequality will depend on the particular question one is interested in. For example, if one is interested in answering the question "what if we remove or introduce a particular intervention," one should estimate the "marginal" contribution by taking the difference in the indicators of interest (for example, the Gini coefficient) that would prevail with and without the specific intervention.[31]

Note, however, that the sum of all the marginal contributions will not equal the total redistributive effect (except by a fluke)[32] because there is path dependency in how interventions affect the net fiscal system and the marginal effect.[33] Essentially, the path in which the fiscal intervention of interest is introduced last is just one of the possible paths. To obtain the average contribution of a specific intervention, one would need to consider all the possible (and institutionally valid) paths and use an appropriate formula to average them. One commonly used approach is to calculate the Shapley value. The Shapley value fulfills the efficiency property: that is, the sum of all the individual contributions is equal to the total effect.[34] Moreover, if some particular paths are irrelevant, the Shapley formula can be modified to exclude them (without losing the efficiency property introduced earlier). We shall return to the Shapley value and its use in appendix A.[35]

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[27] Lambert (2001, p. 278).
[28] See O'Higgins and Ruggles (1981) for the United Kingdom and Ruggles and O'Higgins (1981) for the United States.
[29] See Martinez, Fuchs, and Ortiz-Juarez (2016) and Engel and others (1999). Although Engel and his coauthors were not aware of this characteristic of the Chilean system when they published their article, in a recent interaction with Engel, he concluded that the Chilean system featured regressive albeit equalizing indirect taxes.
[30] Lustig and Melendez (2016).
[31] The same applies to poverty indicators or any other indicator of interest.
[32] This is also the case for the vertical equity and reranking components of redistributive effect.
[33] Note that here we use the terms marginal contribution and marginal effect interchangeably.
[34] See the discussion on path dependency in chapter 7 of Duclos and Araar (2007). See also Bibi and Duclos (2010).
[35] For a review of the decomposition techniques in economics, see Fortin and others (2011). For a review of the Shapley decomposition, see also Shorrocks (2013).

In the next section, we first turn to deriving the conditions that ensure that a net fiscal system is equalizing. Next, we derive the conditions that must prevail in order for the marginal contribution of a tax or a transfer to be equalizing. As mentioned earlier, we first derive the conditions for the simple one tax-one transfer case and, subsequently, for the case with multiple taxes and transfers.

2.1. Equalizing, Neutral, and Unequalizing Net Fiscal Systems

The next two sub-sections discuss the conditions for a net system to have an equalizing marginal effect. We begin with the simple case of one tax and one transfer and then we extend it to the case of a system with multiple taxes and transfers.

2.1.1. Conditions for the One Tax-One Transfer Case

As shown by Lambert,[36] the redistributive effect (measured by the change in Gini coefficients) is equal to the weighted sum of the redistributive effect of taxes and transfers

$$\Pi_N^{RS} = \frac{(1-g)\Pi_T^{RS}+(1+b)\rho_B^{RS}}{1-g+b} \quad (12)$$

where Π_N^{RS}, Π_T^{RS}, and ρ_B^{RS} are the Reynolds-Smolensky indices for the net fiscal system, taxes and benefits, respectively; and *g* and *b* are the total tax and benefit ratios, that is, total taxes and total benefits divided by total pre-fiscal (original) income, respectively.[37] There are two features to note. First, the weights sum to more than unity so the redistributive effect is not a weighted *average*. This fact is not innocuous: it lies at the heart of Lambert's conundrum. Second, recall that in the absence of reranking, the Reynolds-Smolensky index is identical to the redistributive effect measured as the difference between the Gini coefficients. As it´s seen in chapter 3 of the CEQ handbook, if there is reranking, equation 12 will no longer be equal to the redistributive effect.

[36] Lambert (2001, p. 277, equation 11.29).

[37] It is important to note that the tax relative sizes or ratios have to be those that are calculated in the actual data of the fiscal incidence analysis, which are not necessarily equal to the ratios of taxes or transfers to GDP obtained from administrative accounts.

Using equation 12, we can derive the general condition for the case in which the combination of one tax and one transfer (that is, the net fiscal system) is equalizing, neutral, or unequalizing. As noted, when there is no reranking, $\boldsymbol{\Pi_N^{RS}}$ is equal to the change in the Gini coefficient (that is, $G_X - G_{X\text{-}T+B}$). If $G_X - G_{X\text{-}T+B} > 0$, the net fiscal system is equalizing, which simply means that equation 12 must be positive. Since the denominator is positive by definition, the condition implies that the numerator has to be positive. In other words,

(13) $$\boldsymbol{\Pi_N^{RS} = \frac{(1-g)\Pi_T^{RS}+(1+b)\rho_B^{RS}}{1-g+b} > 0 \Leftrightarrow (1-g)\Pi_T^{RS} + (1+b)\rho_B^{RS} > 0}$$

(14) $$\boldsymbol{\Leftrightarrow \Pi_T^{RS} > -\frac{(1+b)}{(1-g)}\rho_B^{RS}}$$

(15) $$\boldsymbol{\Leftrightarrow \Pi_T^{K} > -\frac{(b)}{(g)}\rho_B^{K}}$$

where $\boldsymbol{\Pi_T^K}$ and $\boldsymbol{\rho_B^K}$ are the Kakwani index of the tax and transfer, respectively, and $1-g$ is positive.

Therefore, we can state the following conditions.

Condition 16:

If and only if $\Pi_T^{RS} > -\frac{1+b}{1-g}\rho_B^{RS}$ $(\rho_B^{RS} > -\frac{1-g}{1+b}\Pi_T^{RS})$ or $\boldsymbol{\Pi_T^K > -\frac{(b)}{(g)}\rho_B^K}$ $\boldsymbol{(\rho_B^K > -\frac{(g)}{(b)}\Pi_T^K)}$, the net fiscal system reduces inequality.

Condition 17:

If and only if $\Pi_T^{RS} = -\frac{1+b}{1-g}\rho_B^{RS}$ $(\rho_B^{RS} = -\frac{1-g}{1+b}\Pi_T^{RS})$, or $\boldsymbol{\Pi_T^K = -\frac{(b)}{(g)}\rho_B^K}$ $\boldsymbol{\left(\rho_B^K = -\frac{(g)}{(b)}\Pi_T^K\right)}$, the net fiscal system leaves inequality unchanged.

Condition 18:

If and only if $\Pi_T^{RS} < -\frac{1+b}{1-g}\rho_B^{RS}$ $(\rho_B^{RS} < -\frac{1-g}{1+b}\Pi_T^{RS})$ or $\boldsymbol{\Pi_T^K < -\frac{(b)}{(g)}\rho_B^K}$ $\boldsymbol{(\rho_B^K < -\frac{(g)}{(b)}\Pi_T^K)}$, the net fiscal system increases inequality.

As shown in table 8, a system that combines a regressive tax with a regressive or neutral transfer or a neutral tax with a regressive transfer can never be equalizing. A system that combines a progressive tax with a neutral or progressive transfer or a neutral tax with a progressive transfer is always equalizing. Combining a neutral tax and a neutral transfer leaves inequality unchanged. A regressive tax combined with a progressive transfer or a progressive tax combined with a regressive transfer can be equalizing if and only if Condition 16 holds.

Table 8. Net Fiscal System: Conditions for the One Tax-One Transfer Case

		Transfer		
		Regressive $\boldsymbol{\rho_B^K < 0}$	**Neutral** $\boldsymbol{\rho_B^K = 0}$	**Progressive** $\boldsymbol{\rho_B^K > 0}$
Tax	**Regressive** $\boldsymbol{\Pi_T^K < 0}$	Always unequalizing	Always unequalizing	Equalizing if and only if Condition 16 holds
	Neutral $\boldsymbol{\Pi_T^K = 0}$	Always unequalizing	No change in equality	Always equalizing
	Progressive $\boldsymbol{\Pi_T^K > 0}$	Equalizing if an only if Condition 16 holds	Always equalizing	Always equalizing

2.1.2. Conditions for the Multiple Taxes and Transfers Case

Let's assume there are *n* taxes and *m* transfers in a fiscal system. Equation 12 can be written as

(19) $$\Pi_N^{RS} = \frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}$$

The condition for the net system to be equalizing is that the Reynolds-Smolensky index for the net fiscal system should be higher than zero, that is,

(20) $$\Pi_N^{RS} > 0$$

that is,

(21) $$\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j}>0$$

Assuming, of course, that the denominator is positive,

(22) $$\Leftrightarrow \sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}>-\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}$$

or equivalently,

(22b) $$\Leftrightarrow \sum_{i=1}^{n}g_i\Pi_{T_i}^{K}>-\sum_{j=1}^{m}b_j\rho_{B_j}^{K}.$$

Therefore, we can state the following conditions.

Condition 23:

If and only if $\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}>-\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}$ or $\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}>-\sum_{j=1}^{m}b_j\rho_{B_j}^{K}$, the net fiscal system reduces inequality.

Condition 24:

If and only if $\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}=-\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}$ or $\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}=-\sum_{j=1}^{m}b_j\rho_{B_j}^{K}$, the net fiscal system leaves inequality unchanged.

Condition 25:

If and only if $\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}<-\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}$ or $\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}<-\sum_{j=1}^{m}b_j\rho_{B_j}^{K}$, the net fiscal system increases inequality.

2.2. Equalizing, Neutral, and Unequalizing Taxes or Transfers

The previous section looked at the net system and provided conditions for the whole system to be equalizing, whereas this section focuses on only one tax or only one transfer in the system. The question is whether that specific component leads to a more equalizing total

system. The first case is a simple system with only one transfer (or one tax) in place and determines the conditions for the addition of a tax (or a transfer) to make the system more equal. In the following sub-section, a more general case with multiple taxes and transfer is analyzed.

2.2.1. Conditions for the One Tax-One Transfer Case

In a scenario where there is one tax and one transfer, conditions to assess whether adding a regressive (or progressive) transfer or tax exerts an unequalizing (or equalizing) effect do not necessarily hold as described in the section "The Fiscal System and Income Redistribution: The Case of A Single Tax or A Single Transfer," and introducing these interventions could even derive unintuitive results. For example, adding a regressive transfer to a regressive tax could result in a more equal system or adding a progressive transfer to a progressive tax could decrease equality. The following toy examples illustrate the two unintuitive cases just mentioned.[38] The main factor in these unintuitive examples is that progressivity is (usually) calculated with respect to the original income and it is perfectly possible for a transfer (for example) to be progressive with respect to the original income yet regressive with respect to the "original income plus tax." Such a transfer, therefore, would decrease equality if it were added to this system.

Table 9. Toy Example: Adding a Regressive Transfer to a Regressive Tax Can Exert an Equalizing Effect

Individual	1	2	3	4	Total	Gini
Original income	10.00	20.00	30.00	40.00	100.00	0.2500
Tax (regressive)	9.00	10.00	2.00	0.00	21.00	n.c.
Original income minus tax	1.00	10.00	28.00	40.00	79.00	0.4272
Benefit (regressive)	0.00	3.50	7.00	10.50	21.00	n.c.
Original income plus benefit	10.00	23.50	37.00	50.50	121.00	0.2789

[38] In the toy examples, we assume that the tax and transfer ratios are equal (it would be very easy to show that the results occur when the ratios are not equal so we chose the "most difficult" assumption).

Original income minus tax plus benefit	1.00	13.50	35.00	50.50	100.00	0.4250

n.c. Not calculated.

Table 10. Toy Example: Adding a Progressive Transfer to a Progressive Tax Can Exert an Unequalizing Effect

Individual	1	2	3	4	Total	Gini
Original income	10.00	20.00	30.00	40.00	100.00	0.2500
Tax (progressive)	0.00	1.55	3.10	4.65	9.30	n.c.
Original income minus Tax	10.00	18.45	26.90	35.35	90.70	0.2329
Benefit (progressive)	1.00	1.80	2.80	3.70	9.30	n.c.
Original income plus benefit	11.00	21.80	32.80	43.70	109.30	0.2495
Original income minus tax plus benefit	11.00	20.25	29.70	39.05	100.00	0.2340

n.c. Not calculated.

Given these results, we derive the conditions under which the marginal contribution of a single tax or benefit can be unequalizing, neutral, or equalizing.

Is the Marginal Contribution of a Single Tax Equalizing?

This section addresses the question of whether a tax is equalizing, unequalizing, or neutral, and if it is equalizing or unequalizing, by how much. To answer the question of whether the tax exerts an equalizing or unequalizing force over and above the one prevailing in the system without the tax, we must assess whether the marginal contribution of the tax is positive or negative.

Before continuing, it should be noted that there are three instances in which the word "marginal" is used in incidence analysis.[39]

1. *The marginal contribution or effect* of a fiscal intervention (or of a change in a particular intervention); this is the subject of this section of the paper. It is calculated as the difference between the indicator of choice (for example, the Gini) without the intervention of interest (or the change in the intervention of interest) and with the intervention. So, for example, if we are interested in the marginal contribution of direct taxes when going from market income to disposable income, we take the difference of, for example, the Gini without direct taxes and the Gini of disposable income (which includes the effect of direct taxes).

2. *The derivative of the marginal contribution* with respect to progressivity or size of the intervention. This is, so to speak, the marginal effect of progressivity or size on the marginal contribution. In the case of the derivative with respect to the relative size, this is also known as the *marginal incidence for the intensive margin*.

Both definitions one and two assume that the behavior of individuals is unchanged and unaffected by changes in the taxes or transfers.

3. *The extensive margin* is the last instance for the application of the phrase "margin." To calculate the extensive margin, one needs to estimate the predicted expansion in, for example, users of a service or beneficiaries of a cash transfer or payers of a tax, when the size of the intervention is increased. Researchers have followed different approaches in calculating this type of marginal effect. One way to estimate the effect of an expansion on the extensive margin is by comparing results of average incidence analyses over time. For example, in Mexico Lopez-Calva and others found that concentration curves for tertiary education moved conspicuously towards the diagonal from 1992 to 2010, that is, the extensive margin was progressive.[40] Because of identification problems, care must be taken not to ascribe a causal effect from the expansion of tertiary education to the fact that the extensive margin is progressive. However, one can argue that more spending has probably had something to do with the progressive extensive margin.

As shown by Lambert,[41] the general condition for the tax to be equalizing (when it is added to a system with a benefit in place) is derived from the following inequality:

(26) $$\Pi_N^{RS} > \rho_B^{RS}$$

[39] For an extensive review of the literature on analyzing the concept of tax incidence, see Fullerton and Metcalf (2002).
[40] Lopez-Calva and others (forthcoming).
[41] Lambert (2001, p. 278).

Substituting the expression in equation 2-12 for the left-hand side gives

(27) $$\frac{(\mathbf{1}-\boldsymbol{g})\Pi_{\boldsymbol{T}}^{\boldsymbol{RS}}+(\mathbf{1}+\boldsymbol{b})\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{RS}}}{\mathbf{1}-\boldsymbol{g}+\boldsymbol{b}} > \boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{RS}}$$

(28) $$\Leftrightarrow \Pi_{\boldsymbol{T}}^{\boldsymbol{RS}} > -\frac{\boldsymbol{g}}{\mathbf{1}-\boldsymbol{g}}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{RS}}$$

(29) $$\Leftrightarrow \Pi_{\boldsymbol{T}}^{\boldsymbol{K}} > -\frac{\boldsymbol{b}}{\mathbf{1}+\boldsymbol{b}}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{K}}$$

Therefore, we can state Condition 30.

Condition 30:

If and only if $\Pi_{\boldsymbol{T}}^{\boldsymbol{RS}} > -\frac{\boldsymbol{g}}{\mathbf{1}-\boldsymbol{g}}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{RS}}$ ($\rho_B^{RS} > -\frac{1-g}{g}\Pi_T^{RS}$) or $\Pi_{\boldsymbol{T}}^{\boldsymbol{K}} > -\frac{\boldsymbol{b}}{\mathbf{1}+\boldsymbol{b}}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{K}}$ ($\rho_B^K > -\frac{1+b}{b}\Pi_T^K$), adding the tax reduces inequality. This is exactly the condition derived by Lambert.[42]

Condition 31:

If and only if $\Pi_{\boldsymbol{T}}^{\boldsymbol{RS}} = -\frac{\boldsymbol{g}}{\mathbf{1}-\boldsymbol{g}}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{RS}}$ ($\rho_B^{RS} = -\frac{1-g}{g}\Pi_T^{RS}$) or $\Pi_{\boldsymbol{T}}^{\boldsymbol{K}} = -\frac{\boldsymbol{b}}{\mathbf{1}+\boldsymbol{b}}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{K}}$ ($\rho_B^K = -\frac{1+b}{b}\Pi_T^K$), adding the tax leaves inequality unchanged.

Condition 32:

If and only if $\Pi_{\boldsymbol{T}}^{\boldsymbol{RS}} < -\frac{(\boldsymbol{g})}{(\mathbf{1}-\boldsymbol{g})}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{RS}}$ (or $\rho_B^{RS} < -\frac{1-g}{g}\Pi_T^{RS}$) or $\Pi_{\boldsymbol{T}}^{\boldsymbol{K}} < -\frac{(\boldsymbol{b})}{(\mathbf{1}+\boldsymbol{b})}\boldsymbol{\rho}_{\boldsymbol{B}}^{\boldsymbol{K}}$ ($\rho_B^K < -\frac{(1+b)}{(b)}\Pi_T^K$), adding the tax increases inequality.

From Conditions 30, 31, and 32, we can immediately derive some conclusions, summarized in table 11. As expected, adding a regressive tax to a system with a regressive transfer can never be less unequalizing. Similarly, adding a progressive tax to a progressive transfer is always more

[42] Lambert (2001, p. 278, equation 11.30).

equalizing. However, the unexpected result--which goes back to Lambert's conundrum--is that adding a regressive tax to a system with a progressive transfer can be *more* equalizing if and only if Condition 30 holds. Note that all of the inequality comparisons are made with respect to a system without the tax (that is, a system that only has a transfer in place). The other example of an unintuitive result is that a neutral tax is unequalizing when it is added to a progressive tax. To understand the logic behind these cases, note that the progressivity is calculated with respect to the original income (without any tax or transfer), whereas for a tax to be equalizing when it is added to a system that has a transfer in place, it has to be progressive with respect to the "original income plus transfer."

Table 11. Marginal Contribution of a Tax

		System with a Transfer that is		
		Regressive $\boldsymbol{\rho_B^K < 0}$	**Neutral** $\boldsymbol{\rho_B^K = 0}$	**Progressive** $\boldsymbol{\rho_B^K > 0}$
Adding a Tax that is	**Regressive** $\boldsymbol{\Pi_T^K < 0}$	Always more unequalizing	Always unequalizing	More equalizing only if Condition 30 holds
	Neutral $\boldsymbol{\Pi_T^K = 0}$	Always more unequalizing	No change in inequality	Always more equalizing
	Progressive $\boldsymbol{\Pi_T^K > 0}$	More equalizing only if Condition 30 holds	Always equalizing	Always more equalizing

Is the Marginal Contribution of a Single Transfer Equalizing?

Adding a transfer to a system that has a tax in place is equalizing if

(33) $$\Pi_N^{RS} > \Pi_T^{RS}.$$

Substituting for the left-hand side and rearranging the preceding inequality we have

(34) $$\Leftrightarrow \Pi_T^{RS} < \frac{(1+b)}{b}\rho_B^{RS}$$

(35) $$\Leftrightarrow \Pi_T^{K} < \frac{(1-g)}{g}\rho_B^{K}.$$

Therefore, we can state the following conditions.

Condition 36:

If and only if $\Pi_T^{RS} < \frac{1+b}{b}\rho_B^{RS}$ ($\rho_B^{RS} > \frac{b}{1+b}\Pi_T^{RS}$) or $\Pi_T^{K} < \frac{1-g}{g}\rho_B^{K}$ ($\rho_B^{K} > \frac{g}{1-g}\Pi_T^{K}$) does adding the transfer reduce inequality.

Condition 37:

If and only if $\Pi_T^{RS} = \frac{1+b}{b}\rho_B^{RS}$ ($\rho_B^{RS} = \frac{b}{1+b}\Pi_T^{RS}$) or $\Pi_T^{K} = \frac{1-g}{g}\rho_B^{K}$ ($\rho_B^{K} = \frac{g}{1-g}\Pi_T^{K}$) does adding the transfer leave inequality unchanged.

Condition 38:

If and only if $\Pi_T^{RS} > \frac{1+b}{b}\rho_B^{RS}$ ($\rho_B^{RS} < \frac{b}{1+b}\Pi_T^{RS}$) or $\Pi_T^{K} > \frac{1-g}{g}\rho_B^{K}$ ($\rho_B^{K} < \frac{g}{1-g}\Pi_T^{K}$) does adding the transfer increase inequality.

Some conclusions can be immediately derived from conditions 36 through 38. Adding a progressive transfer to a system with a regressive tax always results in a less inequality. Similarly, adding a regressive transfer to a system with a progressive tax increases inequality. However, somewhat counterintuitively, adding a regressive transfer to a system with a regressive tax does not always increase inequality (see the toy example in table 9). Similarly, adding a progressive transfer to a system with a progressive tax does not always increase equality (see the toy example in table 10). These two results (as shown in table 12) are essentially similar to Lambert's conundrum discussed earlier. Note that when comparing the change in equality, the reference point is the system with only a tax and without any transfer and not the original distribution of income.

Table 12. Marginal Contribution of a Transfer

		Adding a Transfer that is		
		Regressive $\boldsymbol{\rho_B^K < 0}$	**Neutral** $\boldsymbol{\rho_B^K = 0}$	**Progressive** $\boldsymbol{\rho_B^K > 0}$
A system with a Tax that is	**Regressive** $\boldsymbol{\Pi_T^K < 0}$	Less unequalizing if and only if Condition 36 holds	Always less unequalizing	Always less unequalizing
	Neutral $\boldsymbol{\Pi_T^K = 0}$	Always unequalizing	No change in equality	Always equalizing
	Progressive $\boldsymbol{\Pi_T^K > 0}$	Always less equalizing	Always less equalizing	More equalizing if and only if Condition 36 holds

2.2.2. Conditions for the Multiple Taxes and Transfers Case

This section generalizes the preceding discussion for a system with only one tax and one transfer. In the following sub-sections, we focus on the conditions for a tax or transfer to have an equalizing marginal contribution in a system with multiple other taxes and transfers.

Is the Marginal Contribution of a Tax Equalizing?

Assuming no-reranking, for a tax to be equalizing (if it is added to a system with other taxes and transfers in place), the following inequality has to hold:

(39) $$\Pi_N^{RS} > \Pi_{N\setminus T_k}^{RS}.$$

In other words, the redistributive effect is larger with the tax of interest than without it.

The element on the right-hand side shows the change in the Gini coefficient (from pre-fiscal to post-fiscal income) when all taxes and transfers other than tax T_k are in place. Without loss of generality and for simplicity, we will set $k = 1$. Using equation 13, we have

$$\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j} > \frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}.$$

The analysis goes similarly. After some rearranging, we have

(40) $$\Pi_{T_1}^{RS} > \left(\frac{-g_1}{1-g_1}\right)\left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$$

or equivalently,

(40b) $$\Pi_{T_1}^{K} > -\left(\frac{\sum_{i=2}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=1}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right).$$

Therefore, for T_1 to be equalizing when $(n - 1)$ taxes and m benefits are already in place, the following conditions apply.

Condition 41:

If and only if $\Pi_{T_1}^{RS} > \left(\frac{-g_1}{1-g_1}\right)\left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$ (or $\Pi_{T_1}^{K} > -\left(\frac{\sum_{i=2}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=1}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$), then adding T_1 reduces the inequality.

Condition 42:

If and only if $\Pi_{T_1}^{RS} < \left(\frac{-g_1}{1-g_1}\right)\left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$ (or $\Pi_{T_1}^{K} = -\left(\frac{\sum_{i=2}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=1}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$), then adding T_1 increases the inequality.

Condition 43:

If and only if $\Pi_{T_1}^{RS} = \left(\frac{-g_1}{1-g_1}\right)\left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$ (or $\Pi_{T_1}^{K} < -\left(\frac{\sum_{i=2}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=1}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=2}^{n}g_i+\sum_{j=1}^{m}b_j}\right)$), then adding T_1 does not change the inequality.

Is the Marginal Contribution of a Transfer Equalizing?

Assuming no-reranking, the following inequality should hold:

(44) $$\Pi_{N}^{RS} > \Pi_{N\setminus B_k}^{RS}.$$

Assuming $k = 1$ and substituting for both sides of the inequality, we have

$$\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j} > \frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}.$$

After some rearranging, we have

(45) $$\rho_{B_1}^{RS} > \left(\frac{b_1}{1+b_1}\right)\left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)$$

or equivalently,

(45b) $$\rho_{B_1}^{K} > \left(\frac{\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=2}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right).$$

Therefore for B_1 to be equalizing when n taxes and $(m-1)$ benefits are already in place, the following conditions apply.

Condition 46:

If and only if

$$\boldsymbol{\rho_{B_1}^{RS} > \left(\frac{b_1}{1+b_1}\right)\left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)} \text{ (or } \rho_{B_1}^{K} > \left(\frac{\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=2}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)\text{)},$$

then adding B_1 reduces inequality.

Condition 47:

If and only if

$$\boldsymbol{\rho_{B_1}^{RS} = \left(\frac{b_1}{1+b_1}\right)\left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)} \text{ (or } \rho_{B_1}^{K} = \left(\frac{\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=2}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)\text{)},$$

then adding B_1 does not change inequality.

Condition 48:

If and only if

$$\boldsymbol{\rho_{B_1}^{RS} < \left(\frac{b_1}{1+b_1}\right)\left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS}+\sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)} \text{ (or } \rho_{B_1}^{K} < \left(\frac{\sum_{i=1}^{n}g_i\Pi_{T_i}^{K}+\sum_{j=2}^{m}b_j\rho_{B_j}^{K}}{1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j}\right)\text{)},$$

then adding B_1 increases inequality.

Table 13 presents the marginal contributions for broad categories of fiscal interventions for eight countries for which CEQ assessments were performed. The redistributive effect shown here is from market income to final income, which includes the monetized value of transfers in kind in the form of public spending on education and health.[43] The main results can be summarized as follows. Direct taxes and transfers as well as indirect subsidies are equalizing in all countries. Indirect taxes are equalizing in four countries: Brazil, Chile, Sri Lanka, and South Africa. Given that Indirect taxes are regressive in all countries, these four countries displaying a (Lambert) conundrum in which a regressive tax is equalizing and the fiscal system would be more unequal in the absence of it. Lambert's conundrum, thus, is much more common than one might anticipate. Education and health spending are always equalizing except for health spending in Jordan. In Jordan, health spending is progressive but unequalizing, demonstrating another example of the conundrum.

[43] For the definitions of income concepts and how they are calculated, see chapter 1 by Higgins and Lustig in the CEQ handbook.

Table 13. Marginal Contributions: Results from CEQ Assessments

	Lower-middle-income economies			Upper-middle-income economies				
	Georgia (2013)	Indonesia (2012)	Sri Lanka (2010)	Brazil (2009)	Chile (2013)	Jordan (2010)	Russia (2010)	South Africa (2010)
Redistributive effect (from Gini market income plus pensions) to *final* income)	0.1244	0.0238	0.0278	0.1221	0.0740	0.0230	0.0629	0.1758
Marginal contribution								
Direct taxes	0.0221	0.0000	0.0025	0.0143	0.0120	0.0071	0.0139	0.0430
Direct transfers	0.1002	0.0037	0.0041	0.0148	0.0190	0.0052	0.0203	0.0517
Indirect taxes	-0.0141	-0.0022	0.0006	0.0113	0.0040	-0.0014	-0.0009	0.0127
Indirect subsidies	0.0004	0.0014	0.0051	0.0005	0.0023	0.0042	0.0001	--
Education	0.0199	0.0194	0.0105	0.0509	0.0321	0.0155	0.0207	0.0490
Health	0.0077	0.0031	0.0056	0.0292	0.0135	-0.0087	0.0127	0.0433
Kakwani								
Direct taxes	0.1819	--	0.5458	0.2490	0.4520	0.5941	0.1042	0.1254
Direct transfers	0.7063	0.6397	0.7572	0.5069	0.8243	0.5497	0.5927	1.0421
Indirect taxes	-0.2298	-0.0420	-0.0063	-0.0179	-0.0273	-0.0664	-0.0724	-0.0828
Indirect subsidies	0.3716	0.0560	0.3056	0.8373	0.4969	0.1512	0.2128	--
Education	0.5414	0.3630	0.3892	0.7087	0.6641	0.4784	0.4978	0.8169
Health	0.6360	0.2730	0.3963	0.6914	0.5930	0.0557	0.3740	0.8275
Relative size								
Direct taxes	9.8%	--	0.5%	4.2%	2.3%	1.3%	0.0704	15.0%
Direct transfers	19.4%	0.7%	0.6%	5.1%	2.7%	1.3%	0.0467	5.4%
Indirect taxes	12.8%	6.8%	7.4%	12.9%	10.3%	3.1%	0.0803	14.1%
Indirect subsidies	0.4%	8.2%	2.0%	0.1%	0.5%	3.5%	0.0009	--
Education	4.3%	6.2%	3.2%	10.6%	5.2%	3.6%	0.0445	6.9%
Health	1.9%	1.6%	1.6%	4.8%	3.2%	3.4%	0.0419	5.5%

Sources: Higgins and Pereira (2014); Martinez-Aguilar et al., (2016); Alam et al. (forthcoming); Afkar et al., (forthcoming); Arunatilake et al., (forthcoming); Cancho and Bondarenko, (forthcoming); Inchauste et al., (forthcoming); Lopez-Calva, (forthcoming).

2.3. The Derivative of Marginal Contribution with Respect to Progressivity and Size

Section 2.2 showed the conditions that must prevail for the marginal contribution of a tax or a transfer to be equalizing, neutral, or unequalizing. How will the marginal contribution of a particular tax or transfer be affected if its progressivity or size is changed? This is a relevant question in terms of policymaking, especially in the realistic context where leaders want to

adjust the progressivity or relative size of an existing intervention given a pre-existing fiscal system--for example, making cash transfers more progressive or increasing the level of collection of a VAT, or more generally, expanding any pilot program.

This question can be answered by taking the derivative of the particular tax or transfer of interest with respect to progressivity and size. The reader should bear in mind that while the derivative yields the marginal effect of changing the progressivity or size of a particular intervention, the word *marginal* in this context does not have the same meaning or interpretation as the word *marginal* when one is talking about marginal contributions in a joint distribution. The marginal contribution or effect in the latter sense was discussed previously throughout this paper. This section presents the conditions for the marginal effect in the "partial derivative sense."

2.3.1. The Derivatives for the Case of a Marginal Change in Taxes

We will define $\boldsymbol{M_{T_i}}$ as the marginal contribution of tax T_i. The marginal contribution of a tax ($T_i = T_1$ is chosen without loss of generality) in the case of multiple taxes and benefits is defined as follows:

$$\boldsymbol{M_{T_1} = G_{N\setminus T_1} - G_N}$$

or

(49a)
$$\boldsymbol{M_{T_1} = G_{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j} - G_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}}$$

$$\boldsymbol{= \left(G_X - G_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}\right) - \left(G_X - G_{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j}\right)}$$

$$\underbrace{=}_{\textbf{Assuming no-reranking}} \boldsymbol{\Pi^{RS}_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j} - \Pi^{RS}_{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j}}$$

$$\boldsymbol{= \frac{\sum_{i=1}^{n}(1-g_i)\Pi^{RS}_{T_i} + \sum_{j=1}^{m}(1+b_j)\rho^{RS}_{B_j}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j} - \frac{\sum_{i=2}^{n}(1-g_i)\Pi^{RS}_{T_i} + \sum_{j=1}^{m}(1+b_j)\rho^{RS}_{B_j}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}}$$

$$= \frac{\sum_{i=1}^{n} g_i \Pi_{T_i}^{K} + \sum_{j=1}^{m} b_j \rho_{B_j}^{K}}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j} - \frac{\sum_{i=2}^{n} g_i \Pi_{T_i}^{K} + \sum_{j=1}^{m} b_j \rho_{B_j}^{K}}{1 - \sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}$$

or

(49b) $$= \frac{g_1\left[\left(1-\sum_{i=2}^{n} g_i+\sum_{j=1}^{m} b_j\right)\Pi_{T_1}^{K}+\left(\sum_{i=2}^{n} g_i \Pi_{T_i}^{K}+\sum_{j=1}^{m} b_j \rho_{B_j}^{K}\right)\right]}{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)\left(1-\sum_{i=2}^{n} g_i+\sum_{j=1}^{m} b_j\right)}.$$

What are the derivatives of the marginal contribution of a tax with respect to its progressivity and size? Manipulating equation 49b, we obtain[44]

(50) $$\frac{\partial M_{T_1}}{\partial \Pi_{T_1}^{K}} = \frac{g_1}{1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j}.$$

Note that the derivative 50 is always positive given the usual assumption about the total size of taxes and transfers, that is $1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j > 0$.

$$\frac{\partial M_{T_1}}{\partial g_1} = \frac{\left[\Pi_{T_1}^{K}\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)\right]-\left[(-1)\left(\sum_{i=1}^{n} g_i \Pi_{T_i}^{K}+\sum_{j=1}^{m} b_j \rho_{B_j}^{K}\right)\right]}{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)^2}$$

(51) $$= \frac{\left[\Pi_{T_1}^{K}\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)\right]+\left[\left(\sum_{i=1}^{n} g_i \Pi_{T_i}^{K}+\sum_{j=1}^{m} b_j \rho_{B_j}^{K}\right)\right]}{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)^2}$$

To sign derivative 51, please note that it is equal to[45]

$$= \frac{\Pi_{T_1}^{K}+\Pi_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}^{RS}}{1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j}.$$

Since the denominator is always positive, the sign depends only on the numerator, which is the Kakwani index of Tax ($\Pi_{T_1}^{K}$) and the R-S index of the net system with T_1 ($\Pi_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}^{RS}$), that is, the following condition assures the derivative is positive.

Condition MT1: $$\Pi_{T_1}^{K} > -\Pi_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}^{RS}$$

[44] Here we hold the relative size of T_1 and everything else constant.
[45] Here we hold the progressivity of T_1 and everything else constant.

The following table shows what the ultimate sign will be. Here the assumption is that there is no reranking, so the R-S index being positive is equivalent to the fiscal system being equalizing.

Table 14. The Sign of the Derivative of a Tax's Marginal Contribution with Respect to Its Relative Size

		The Tax of Interest: T_1		
		Regressive $\Pi^{K}_{T_1} < 0$	**Neutral** $\Pi^{K}_{T_1} = \mathbf{0}$	**Progressive** $\Pi^{K}_{T_1} > 0$
The Whole System (including T_1)	**Unequalizing** $\Pi^{RS}_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j} < 0$	Negative (more unequalizing)	Negative (more unequalizing)	Positive (less unequalizing), if and only if condition MT1 holds
	Neutral $\Pi^{RS}_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j} = \mathbf{0}$	Negative (more unequalizing)	Zero	Positive (more equalizing)
	Equalizing $\Pi^{RS}_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j} > 0$	Positive (more equalizing), if and only if condition MT1 holds	Positive (more equalizing)	Positive (more equalizing)

The following expression shows that when the marginal effect of progressivity on the marginal contribution of a tax is more than its relative size,

(52) $$\frac{\partial M_{T_1}}{\partial \Pi^{K}_{T_1}} > \frac{\partial M_{T_1}}{\partial g_1}$$

$$\Leftrightarrow \frac{g_1}{1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j} > \frac{\Pi^{K}_{T_1}\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)+\left(\sum_{i=1}^{n} g_i \Pi^{K}_{T_i}+\sum_{j=1}^{m} b_j \rho^{K}_{B_j}\right)}{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)^2}$$

$$\Leftrightarrow \boldsymbol{g_1} > \Pi^{\boldsymbol{K}}_{\boldsymbol{T_1}} + \frac{\sum_{i=1}^{n} \boldsymbol{g_i}\Pi^{\boldsymbol{K}}_{\boldsymbol{T_i}} + \sum_{j=1}^{m} \boldsymbol{b_j}\boldsymbol{\rho}^{\boldsymbol{K}}_{\boldsymbol{B_j}}}{\mathbf{1} - \sum_{i=1}^{n} \boldsymbol{g_i} + \sum_{j=1}^{m} \boldsymbol{b_j}}$$

(53) $$\Leftrightarrow \boldsymbol{g_1} > \Pi^{\boldsymbol{K}}_{\boldsymbol{T_1}} + \Pi^{\boldsymbol{RS}}_{\boldsymbol{X}-\sum_{i=1}^{n} \boldsymbol{T_i}+\sum_{j=1}^{m} \boldsymbol{B_j}}.$$

Formula 52 for the simple case of one tax and one transfer is

$$\boldsymbol{M_T} = \boldsymbol{G}_{\square+\boldsymbol{B}} - \boldsymbol{G}_{\boldsymbol{X}-\boldsymbol{T}+\boldsymbol{B}} \underbrace{=}_{\textit{Assuming no reranking}} \Pi^{\boldsymbol{RS}}_{\boldsymbol{X}-\boldsymbol{T}+\boldsymbol{B}} - \boldsymbol{\rho}^{\boldsymbol{RS}}_{\boldsymbol{B_j}}$$

$$= \frac{\boldsymbol{g}\Pi^{\boldsymbol{K}}_{\boldsymbol{T}} + \boldsymbol{b}\boldsymbol{\rho}^{\boldsymbol{K}}_{\boldsymbol{B}}}{\mathbf{1}-\boldsymbol{g}+\boldsymbol{b}} - \boldsymbol{\rho}^{\boldsymbol{RS}}_{\boldsymbol{B_j}}.$$

The derivatives with respect to progressivity and size are shown as follows:

$$\frac{\partial \boldsymbol{M_T}}{\partial \Pi^{\boldsymbol{K}}_{\boldsymbol{T}}} = \frac{\boldsymbol{g}}{\mathbf{1} - \boldsymbol{g} + \boldsymbol{b}}$$

and

$$\frac{\partial \boldsymbol{M_T}}{\partial \boldsymbol{g}} = \frac{\left[\Pi^{\boldsymbol{K}}_{\boldsymbol{T}}(\mathbf{1} - \boldsymbol{g} + \boldsymbol{b})\right] + \left[\boldsymbol{g}\Pi^{\boldsymbol{K}}_{\boldsymbol{T}} + \boldsymbol{b}\boldsymbol{\rho}^{\boldsymbol{K}}_{\boldsymbol{B}}\right]}{(\mathbf{1} - \boldsymbol{g} + \boldsymbol{b})^{\mathbf{2}}} = \frac{\Pi^{\boldsymbol{K}}_{\boldsymbol{T}} + \Pi^{\boldsymbol{RS}}_{\boldsymbol{X}-\boldsymbol{T}+\boldsymbol{B}}}{\mathbf{1} - \boldsymbol{g} + \boldsymbol{b}}$$

Equation 53a shows the condition under which the derivative of the marginal contribution of a tax with respect to its progressivity would be greater than the derivative with respect to its size:

$$\frac{\partial \boldsymbol{M_T}}{\partial \Pi^{\boldsymbol{K}}_{\boldsymbol{T}}} > \frac{\partial \boldsymbol{M_T}}{\partial \boldsymbol{g}}$$

$$\Leftrightarrow \frac{\boldsymbol{g}}{\mathbf{1} - \boldsymbol{g} + \boldsymbol{b}} > \frac{\left[\Pi^{\boldsymbol{K}}_{\boldsymbol{T}}(\mathbf{1} - \boldsymbol{g} + \boldsymbol{b})\right] + \left[\boldsymbol{g}\Pi^{\boldsymbol{K}}_{\boldsymbol{T}} + \boldsymbol{b}\boldsymbol{\rho}^{\boldsymbol{K}}_{\boldsymbol{B}}\right]}{(\mathbf{1} - \boldsymbol{g} + \boldsymbol{b})^{\mathbf{2}}}$$

(53a) $$\Leftrightarrow \boldsymbol{g} > \Pi^{\boldsymbol{K}}_{\boldsymbol{T}} + \Pi^{\boldsymbol{RS}}_{\boldsymbol{X}-\boldsymbol{T}+\boldsymbol{B}}$$

2.3.2. The Derivatives for the Case of a Marginal Change in Transfers

The marginal contribution $\boldsymbol{M_{B_i}}$ of a transfer B_i ($B_i = B_1$ is chosen without the loss of generality) in the case of multiple taxes and benefits can be similarly written in this format as

$$\boldsymbol{M_{B_1}} = \boldsymbol{G_{N\setminus B_1}} - \boldsymbol{G_N}$$

or

$$\boldsymbol{M_{B_1} = G_{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j} - G_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}}$$

$$\boldsymbol{= \left(G_X - G_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j}\right) - \left(G_X - G_{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j}\right)}$$

$$\underset{\text{Assuming no reranking}}{=} \boldsymbol{\Pi^{RS}_{X-\sum_{i=1}^{n} T_i+\sum_{j=1}^{m} B_j} - \Pi^{RS}_{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j}}$$

$$= \frac{\boldsymbol{\sum_{i=1}^{n}(1-g_i)\Pi^{RS}_{T_i} + \sum_{j=1}^{m}(1+b_j)\rho^{RS}_{B_j}}}{\boldsymbol{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}} - \frac{\boldsymbol{\sum_{i=1}^{n}(1-g_i)\Pi^{RS}_{T_i} + \sum_{j=2}^{m}(1+b_j)\rho^{RS}_{B_j}}}{\boldsymbol{1-\sum_{i=1}^{n} g_i + \sum_{j=2}^{m} b_j}}$$

$$= \frac{\boldsymbol{\sum_{i=1}^{n} g_i\Pi^{K}_{T_i} + \sum_{j=1}^{m} b_j\rho^{K}_{B_j}}}{\boldsymbol{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}} - \frac{\boldsymbol{\sum_{i=1}^{n} g_i\Pi^{K}_{T_i} + \sum_{j=2}^{m} b_j\rho^{K}_{B_j}}}{\boldsymbol{1-\sum_{i=1}^{n} g_i + \sum_{j=2}^{m} b_j}}$$

or

(54) $$= \frac{\boldsymbol{b_1\left[\left(1-\sum_{i=1}^{n} g_i+\sum_{j=2}^{m} b_j\right)\rho^{K}_{B_1} - \left(\sum_{i=1}^{n} g_i\Pi^{K}_{T_i}+\sum_{j=2}^{m} b_j\rho^{K}_{B_j}\right)\right]}}{\boldsymbol{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)\left(1-\sum_{i=1}^{n} g_i+\sum_{j=2}^{m} b_j\right)}}.$$

The derivatives with respect to progressivity and size are expressed in equations 55 and 56 respectively:

(55) $$\frac{\partial M_{B_1}}{\partial \rho^{K}_{B_1}} = \frac{b_1}{1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j}.$$

Note that the derivative 36 is always positive given the usual assumption about the total size of taxes and transfers, that is, $1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j > 0$

$$\boldsymbol{\frac{\partial M_{B_1}}{\partial b_1} = \frac{\left[\rho^{K}_{B_1}\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)\right] - \left[(+1)\left(\sum_{i=1}^{n} g_i\Pi^{K}_{T_i}+\sum_{j=1}^{m} b_j\rho^{K}_{B_j}\right)\right]}{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)^2}}$$

(56) $$= \frac{\boldsymbol{\left[\rho^{K}_{B_1}\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)\right] - \left(\sum_{i=1}^{n} g_i\Pi^{K}_{T_i}+\sum_{j=1}^{m} b_j\rho^{K}_{B_j}\right)}}{\boldsymbol{\left(1-\sum_{i=1}^{n} g_i+\sum_{j=1}^{m} b_j\right)^2}}.$$

To sign the preceding derivative, please note that it is equal to

$$= \frac{\rho^{K}_{B_1} - \Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j}}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}.$$

Because the denominator is always positive, the sign depends only on the numerator, which is the Kakwani index of transfer ($\rho^{K}_{B_1}$) and R-S index of the net system with B_1 ($\Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j}$). The following condition assures the derivative is positive.

Condition MB1: $$\rho^{K}_{B_1} > \Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j}$$

Table 15 shows what the ultimate sign will be. Here, we assume that there is no reranking, so the R-S index being positive is equivalent to the fiscal system being equalizing.

Table 15. The Sign of the Derivative of the Marginal Contribution of a Transfer with Respect to its Relative Size

		The Transfer of Interest: B_1		
		Regressive $\rho^{K}_{B_1} < 0$	**Neutral** $\rho^{K}_{B_1} = 0$	**Progressive** $\rho^{K}_{B_1} > 0$
The Whole System (including B_1)	**Unequalizing** $\Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j} < 0$	Positive (more equalizing), if and only if condition MB1 holds	Positive (more equalizing)	Positive (more equalizing)
	Neutral $\Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j} = 0$	Negative (more unequalizing)	Zero	Positive (more equalizing)
	Equalizing $\Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j} > 0$	Negative (more unequalizing)	Negative (more unequalizing)	Positive (more equalizing), if and only if condition MB1 holds

Expression 57 shows the scenario in which the effect of progressivity on the marginal effect of a benefit is more than its relative size:

(57) $$\frac{\partial M_{B_1}}{\partial \rho^K_{B_1}} > \frac{\partial M_{B_1}}{\partial b_1}$$

$$\Leftrightarrow \frac{b_1}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j} > \frac{\left[\rho^K_{B_1}\left(1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j\right)\right] - \left(\sum_{i=1}^{n} g_i \Pi^K_{T_i} + \sum_{j=1}^{m} b_j \rho^K_{B_j}\right)}{\left(1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j\right)^2}$$

$$\Leftrightarrow b_1 > \rho^K_{B_1} - \frac{\sum_{i=1}^{n} g_i \Pi^K_{T_i} + \sum_{j=1}^{m} b_j \rho^K_{B_j}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}$$

(58) $$\Leftrightarrow b_1 > \rho^K_{B_1} - \Pi^{RS}_{X-\sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j}.$$

In order to have an equivalent condition for the simple case of one tax and one transfer similar to equation 58, note the following equations introduced earlier:

$$\frac{\partial M_B}{\partial \rho^K_B} = \frac{b}{1-g+b}$$

and

$$\frac{\partial M_B}{\partial b} = \frac{[\rho^K_B(1-g+b)]-(g\Pi^K_T + b\rho^K_B)}{(1-g+b)^2} = \frac{\rho^K_B - \Pi^{RS}_{X-T+B}}{1-g+b}.$$

Equation 59 shows the condition under which the derivative of marginal contribution with respect to a transfer's progressivity would be greater than the derivative with respect to its size:

$$\frac{\partial M_B}{\partial \Pi^K_B} > \frac{\partial M_B}{\partial b}$$

(59) $$\Leftrightarrow b > \rho^K_B - \Pi^{RS}_{X-T+B}$$

2.4. The Sensitivity of Marginal Contribution Analysis to the Use of the Conventional Gini Index

Thus far, we have focused on the conventional Gini coefficient to determine whether a specific tax or transfer is equalizing. The application of this index implies a normative choice with regard to how individuals from different parts of an income distribution are weighted (Gini puts more weights on the middle of the income distribution). One may prefer to weight more heavily the gains that accrue to lower deciles (or the higher ones) and, therefore, can opt for the family of S-Gini indexes (or Extended Gini) to calculate the marginal contribution of the components of a fiscal system.[46] The final conclusion about a tax (or transfer) having a positive marginal contribution (that is, an equalizing effect) could change if the concentration curve of that tax (or transfer) crosses the Lorenz curve of the total system without that tax (or transfer). In other words, in the case of no dominance, one would expect the results to depend on the normative choice of how to weight individuals. In the following explanation, we clarify this issue further.

In section 1, we discussed the application of the concentration and Lorenz curves in determining whether a tax or transfer is (everywhere) progressive or not. A similar analysis can be applied to the concept of the marginal contribution. Suppose we define the Lorenz curve of "the final income without a specific tax (T_1)" as $L(p)_{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j}$. Then the specific tax that is being analyzed has an equalizing effect (in the marginal contribution sense), regardless of the normative choice of how to weigh individuals if and only if

$$(60)\qquad \begin{cases} L(p)_{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j} \geq C(p)_{T_1}^{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j} \;\; \forall p \; and, \\ L(p)_{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j} > C(p)_{T_1}^{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j} \;\; for \; some \; p \end{cases}$$

where $C(p)_{T_1}^{X-\sum_{i=2}^{n} T_i+\sum_{j=1}^{m} B_j}$ is the concentration curve of T_1 when individuals are ranked with respect to their final income without T_1.

Similarly, for the case of a transfer (B_1), we have the following condition:

$$(61)\qquad \begin{cases} L(p)_{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j} \leq C(p)_{B_1}^{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j} \;\; \forall p \; and, \\ L(p)_{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j} < C(p)_{B_1}^{X-\sum_{i=1}^{n} T_i+\sum_{j=2}^{m} B_j} \;\; for \; some \; p \end{cases}$$

If these conditions do not hold for some p, that is, if there is at least one crossing of the two curves, then the conclusion about whether a specific tax or transfer is equalizing depends on how one weights individuals in different parts of an income distribution. Therefore, it is

[46] See Yitzhaki and Schechtman (2005) for a mathematical review of these indicators.

important to use graphical representations and the sensitivity analysis (that is, using S-Gini indexes with different values for the normative parameter of weighting instead of the conventional Gini) in the context of the inequality (and poverty) analysis. These tools help to determine how much the results of an analysis using a specific index hinges on the underlying normative choice of using that specific indicator.

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Appendix: The Shapley Value

Introduction to the Shapley Value

Despite its seeming simplicity, the question "how much does inequality increase (or decrease) due to a particular source of income?" does not have a straightforward answer. In fact, the answer will be different depending on 1) what other sources of income are available to the society, 2) whether any particular meaningful order of allocating different sources of income exists. and 3) whether any theoretical basis for aggregating income sources exists.

To better understand why information about "the other sources of income" (regarding the first point) is important, imagine the following simple example. There are two individuals, *I* and *J*, who need to get a taxi. They live on the same street but at different distances from the place that they need to get in the taxi. If each of them gets a taxi separately, they will need to pay $10 and $15, respectively. But if they share the ride, they have to pay $15 together. How should they divide the cost? Now, assume a third person joins them, who lives between the two initial passengers and who would have to pay $12 if he were to get a taxi on his own. If they all three go together, their fare remains $15 and unchanged from the previous case when only *I* and *J* shared the ride. Going from the first case to the second case, individuals *I* and *J*'s share of the taxi fare should change because a third person has joined them. This example makes it clear that it is perfectly possible that based on a particular circumstance or depending on how an inequality index is defined, individual shares of each income source in creating or reducing inequality can depend on information about all other sources of income. This situation is why the Shapley value was initially formulated by Lloyd Shapley.[47]

Now, focusing on the second and third points of our original question, if there is no particular order for how the income sources are assigned and all income sources are perceived in the most disaggregated way (no aggregation hierarchy), then the "simple Shapley value" is the way to calculate the effect of each individual source. This formula is discussed later in this appendix in the section on the simple Shapley value.

If there is a particular order for how some sources of income will be allocated (for example, if taxes cannot be first), then the problem can be easily reduced to the case of simple Shapley. Imagine we have five sources of income and source numbers 1 and 2 are always first and the other sources (3, 4, and 5) are always last. The inequality will change in two steps. First, when sources 1 and 2 are added, the amount of change in inequality can be decomposed between these two sources using the simple Shapley formula. Then, in the second step, inequality will change due to the remaining sources. This change can be decomposed again between only the remaining sources using the simple Shapley formula. The total change will be then equal to the individual shares.

Finally, if there is no particular order but there is an aggregation scheme (for example, taxes, benefits, and so on), then a two-stage, or hierarchy-Shapley value should be used, which is

[47] See Shapley (1952).

discussed in the next section. The general idea of this two-stage methodology is to determine the contribution of different groups (such as a group of taxes versus a group of transfers) in the first step and then to determine the share of each individual fiscal incidence from the total contribution of its group.

Simple Shapley Value

There are two ways to calculate simple Shapley values that result in different outcomes and therefore have different theoretical implications. Sastre and Trannoy call these methods "zero income decomposition" (ZID) and "equalized income decomposition" (EID).[48] The difference between the two formulas is the way that they answer a simple but fundamental question: What should be considered the reference point? In ZID (as the word "zero" implies), we always calculate changes in inequality by using zero allocation of a particular source of income as the reference point. In EID, the reference point is a hypothetical state in which a particular source is divided evenly among all people, so here change in inequality occurs because we deviate from this (hypothetical) equalized distribution of income. To see this point more clearly, assume we have three individuals and their income from a specific source is $10, $20, and $30, respectively. In order to determine the contribution of this source of income to inequality, ZID compares the Gini after this source of income is added to the scenario when this source is not added. EID, on the other hand, compares the Gini after this source of income is added to the scenario when everybody would receive $20 from this source.

Sastre and Trannoy[49] prefer EID over ZID due to a major theoretical difference. To better understand the difference, discussing a simple question is enlightening: If there were a source of income that was distributed evenly among members of a society, what should be the share of this source in creating inequality? Sastre and Trannoy argue that the answer is zero because this particular source does not create any inequality. Only EID produces zero value for such a source and ZID would result in a non-zero value.

The preceding justification for preferring EID over ZID is, however, not as tenable if one deals with taxes and transfers as other types of income (using a broad definition of income to include negative sources as a type of income). An evenly distributed tax (that is, a lump-sum tax) is regressive or pro-rich (poor people pay the same tax as rich people so their tax rate is much higher given their lower income) and an evenly distributed transfer (that is, a lump-sum transfer) is progressive or pro-poor (because poor people get the same amount of money as rich people but relative to their lower income, they are receiving higher benefits). A regressive tax is considered a cause of increasing inequality and a progressive transfer is considered a cause of reducing inequality, so accordingly one would expect to see a negative Shapley value for a lump-sum tax and a positive Shapley value for a lump-sum transfer, which is only possible through the ZID approach. The EID method would give zero shares to these taxes and transfers.

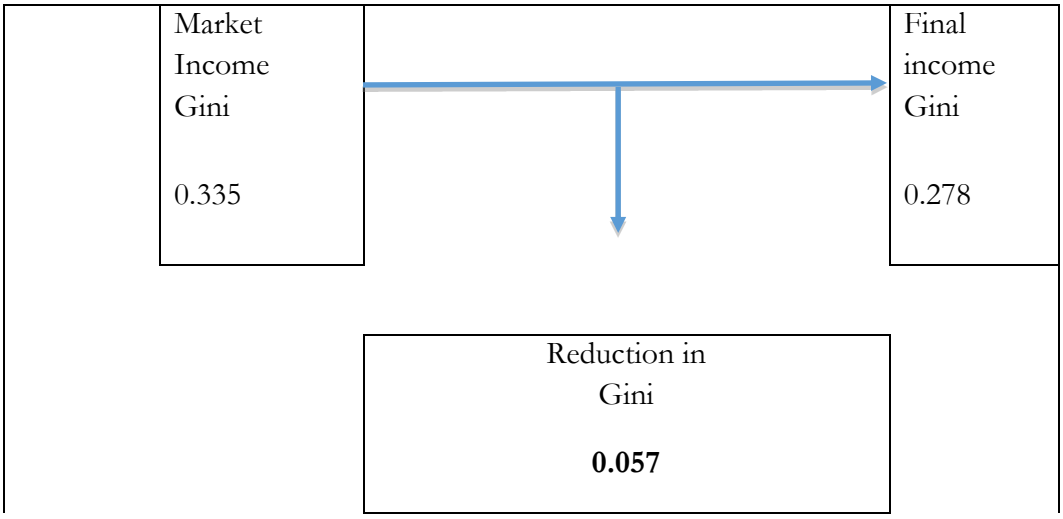
[48] Sastre and Trannoy (2002, p. 54).
[49] Sastre and Trannoy (2002).

The other problem with the EID approach is that it cannot be used to decompose changes in the inequality index if the starting value of the index is not zero and the sum of the total sources of income is not zero (for example, if taxes are not equal to transfers due to inefficiency in the fiscal system). This problem is explained in more detail when the EID formula is introduced.

The following example shows the simple Shapley value calculated using the ZID and EID approaches for a specific example of three sources of income: a market income (*M*), an equalized tax (*T*), and a (non-equalized) transfer (*R*). We assume that market income is always first, so we are only interested in the share of the tax and transfer in changing the Gini index (as a measure of inequality) between market income and total income.

Table A-1. Comparison of ZID and EID Approaches in Calculating the Shapley Value When an Equalized (Regressive) Tax Is Involved

Individual	Market income	Tax (equalized)	Transfer	Final income
1	1	−5	9	5
2	20	−5	7	22
3	30	−5	5	30
4	40	−5	3	38
5	50	−5	1	46
Total	**141**	**−25**	**25**	**141**
Average	**28.2**	**−5**	**5**	**28.2**



	Share of Tax in reducing inequality	Share of Transfer in reducing inequality
Shapley Value (ZID)	**−0.057**	**0.114**
Shapley Value (EID)	**0.000**	**0.057**

ZID. Zero income decomposition.
EID. Equalized income decomposition.

As is clear from table A-1, the ZID approach produces a negative share (that is, inequality increases) for a regressive (pro-rich) tax, which is in line with the literature. It seems reasonable to use these two different approaches in their appropriate contexts. When the sources of income do not include any form of income redistribution (taxes or transfers), using EID has more theoretical justification. On the other hand, if one is only performing an incidence analysis (that is, if only taxes and transfers are included in the analysis), then ZID is the better approach. In cases where both income and redistribution sources are involved, using a two-step approach in ordering different sources can solve the problem. If one can argue that all sources of earned income come first, after which taxes and transfers are added, then a two-step decomposition (as explained earlier) can be employed with the EID approach for the first step (when only earned incomes are considered), followed by the ZID approach for the second step (when only taxes and transfers are considered).

Because both approaches have merits depending on the circumstances, they are both introduced mathematically in the following sections.

Simple Shapley Value: ZID Approach

Define a value function V that uses different income sources as input and produces one value as output. The Gini coefficient is an example of such value function. If there are n sources of income and m individuals in the society, then V can be defined as $V: R^{m\times n} \rightarrow R$. The set of sources of income is $N = \{\underset{\sim}{I}_1, \underset{\sim}{I}_2, \dots, \underset{\sim}{I}_n\}$ where each $\underset{\sim}{I}_i$ is itself a $(m \times 1)$ vector of values for all individuals in the society. Therefore, $V(\underset{\sim}{I}_1, \underset{\sim}{I}_2, \dots, \underset{\sim}{I}_n)$ is, for example, the Gini coefficient when all sources of income are distributed in the society and $V(\underset{\sim}{I}_1, \underset{\sim}{0}, \dots, \underset{\sim}{0})$ is the Gini coefficient when only source I_1 (and none of the other sources) is distributed. The Shapley value is a weighted average of all possible cases in which we can demonstrate the effect of adding one source to the value function. For example, $V(\underset{\sim}{I}_1, \underset{\sim}{I}_2, \dots, \underset{\sim}{I}_n) - V(\underset{\sim}{0}, \underset{\sim}{I}_2, \dots, \underset{\sim}{I}_n)$ and $V(\underset{\sim}{I}_1, \underset{\sim}{0}, \dots, \underset{\sim}{0}) - V(\underset{\sim}{0}, \underset{\sim}{0}, \dots, \underset{\sim}{0})$ are two of many ways to measure the effect of adding $\underset{\sim}{I}_1$ to the

value function. If all of these different ways result in the same value, there is no need to use a complicated weighted average. But for many indexes, including the Gini, this is not the case. While it is easy to list all of the possible ways of calculating the effect of adding a particular source to the value function, determining the weights requires more attention. Before introducing the formula for the weights, let's start with an intuitive example.

Assume we are interested in determining the weight of path $V(\underset{\sim}{I}_1, \underset{\sim}{I}_2, \underset{\sim}{I}_3, \underset{\sim}{I}_4, \underset{\sim}{I}_5, \underset{\sim}{0}, \dots, \underset{\sim}{0}) - V(\underset{\sim}{0}, \underset{\sim}{I}_2, \underset{\sim}{I}_3, \underset{\sim}{I}_4, \underset{\sim}{I}_5, \underset{\sim}{0}, \dots, \underset{\sim}{0})$. This path determines how much V changes when we add I_1 given that I_2, I_3, I_4 and I_5 are already added and sources I_6 through $I_n$will not be added. The Shapley value is determined based on the permutation of sources, or put another way, order matters. In other words, we need to ask how many times we can permute sources I_2 through I_5 (which is 4! = 24) and then add I_1 and permute sources $\underset{\sim}{I}_6 = \underset{\sim}{0}$ through $\underset{\sim}{I}_n = \underset{\sim}{0}$ (which is $\big(n - (4+1)\big)!$). We have to multiply all these numbers to get the total number of permutations, that is, $(4!) \times \big[\big(n - (4+1)\big)!\big]$. Two important points should be noted. First, even though none of the sources from *6* through *n* would be added for this path, the number of their permutations matters. Second, for any path, we always calculate the permutation of previously-added sources (sources other than the one that we are interested in) together and then multiply it by the number of permutations of sources that are not added. For example, if we were calculating the weight of path $V(\underset{\sim}{I}_1, \underset{\sim}{0}, \underset{\sim}{0}, \underset{\sim}{I}_4, \underset{\sim}{I}_5, \underset{\sim}{I}_6, \underset{\sim}{I}_7, \underset{\sim}{0}, \dots, \underset{\sim}{0}) - V(\underset{\sim}{0}, \underset{\sim}{0}, \underset{\sim}{0}, \underset{\sim}{I}_4, \underset{\sim}{I}_5, \underset{\sim}{I}_6, \underset{\sim}{I}_7, \underset{\sim}{0}, \dots, \underset{\sim}{0})$, the number of permutations is exactly equal to the previous case, that is $(4!) \times \big[\big(n - (4+1)\big)!\big]$. One should note that 4 is the number of income sources that are added already and $[n - (4+1)]$ is the number of income sources that will not be added. Therefore, what matters is the number of added sources, not which source is added. The number of permutations is the weight of each path. The total number of permutations, *n!*, is used (as the denominator) so that the weights add up to one. With this explanation, the ZID formula can now be formally introduced.

Assume we are interested in finding the Shapley value of income source *i,* Define set S_{I_i} as the set of sub-sets of set $N - \{I_i\}$ (that is, a set that includes all sources of income except for source I_i). Note that the empty set, $\emptyset$, and $N - \{I_i\}$ itself are considered two-subsets of $N - \{I_i\}$ and therefore included in S_{I_i}. Each element in S_{I_i}represents a different path through which the effect of adding I_i to V can be measured. These elements (which are themselves a set) represent income sources that are added before I_i is added. Because all of the possible paths are represented by elements of S_{I_i}, a summation over these elements with appropriate weights would result in the Shapley value. The resulting formula is therefore

(A-1) $$Sh_{I_i}^{ZID} = \sum_{S \in S_{I_i}} \left(\frac{(s!) \times ((n-s-1)!)}{n!} \left(V^{ZID}(S \cup I_i) - V^{ZID}(S) \right) \right).$$[50]

[50] Sastre and Trannoy (2002).

First, note that in this formula, *S* represents an element of set S_{I_i}. Second, *s* is the dimensionality of each element of *S* that enters in the summation and *n* is the dimensionality of set *N*. It should be noted that *s* is the number of income sources that are already added and $n-s-1$ is the number of sources that will not be added. Third, $V^{ZID}(S \cup \underset{\sim}{I_i})$ means the value function *V* allocates zero to any income source that is not included in set *S* (and it is not I_i). For example, if $S = \{I_2, I_3, I_4, I_5\}$ then

$$V^{ZID} = V\left(\underset{\sim}{0}, \underset{\sim}{I_2}, \underset{\sim}{I_3}, \underset{\sim}{I_4}, \underset{\sim}{I_5}, \underset{\sim}{0}, \dots, \underset{\sim}{0}\right)$$

Simple Shapley Value: EID Approach

Using the same notation as in the previous section, the Shapley formula using the EID approach is

(A-2) $$Sh_{I_i}^{EID} = \sum_{S \in S_{I_i}} \left(\frac{(s!) \times ((n-s-1)!)}{n!} \left(V^{EID}(S \cup I_i) - V^{EID}(S) \right) \right).$$[51]

The only difference here is that $V^{EID}(S)$ means the value function *V* allocates the average income to all individuals in the society for any income source that is not included in *S*. For example, if $S = \{I_2, I_3, I_4, I_5\}$, then the corresponding value function is

$$V^{EID} = V\left(\left(\mu_{I_1} \times \underset{\sim}{1}\right), \underset{\sim}{I_2}, \underset{\sim}{I_3}, \underset{\sim}{I_4}, \underset{\sim}{I_5}, (\mu_{I_6} \times \underset{\sim}{1}), \dots, (\mu_{I_n} \times \underset{\sim}{1}) \right)$$

where $\underset{\sim}{1}$ is a $(m \times 1)$ vector of ones and μ_{I_i} is the average value of income source *i*.

Note how the EID formula would run into problems if one tried to use it to explain a change in a value function (for example, the Gini coefficient) between a reference point that is not zero and an end point that has a different per-capita income in comparison to the reference point (that is, the sum of taxes and transfers is not zero). Assume the same example that is shown in table A-1. When total taxes and transfers are the same, the per-capita values are also equal and they cancel each other out, so the reference point remains the market income, that is,

$$V\left(Market\ income, (\mu_{Tax} \times 1), \left(\mu_{Transfer} \times 1\right)\right) = V(Market\ income, 0,0)$$

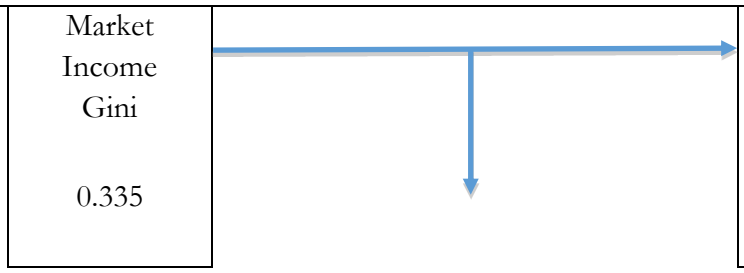
$$when\ \mu_{Tax} = -\mu_{Transfer}$$

If the sum of taxes and transfers is not zero, the reference point is no longer market income and has a different value for the Gini coefficient, which results in the decomposition differing from the value we want to explain. Table A-2 shows this problem in a simple example. The sum of the EID Shapley values does not add up to the change in the Gini coefficient that we would like to explain.

[51] Sastre and Trannoy (2002).

Table A-2. Example of EID Failing to Decompose the Change in Gini

Individual	Market Income	Tax	Transfer	Final income
1	1	−1	6	6
2	20	−2	4	22
3	30	−3	3	30
4	40	−4	2	38
5	50	−5	1	46
Total	**141**	**−15**	**16**	**142**
Average	**28.2**	**−3**	**3.2**	**28.4**



	Share of Tax in reducing inequality	Share of Transfer in reducing inequality	Sum of the shares of tax and transfer
Shapley Value (ZID)	−0.004	0.069	**0.065**
Shapley Value (EID)	0.028	0.034	**0.062**

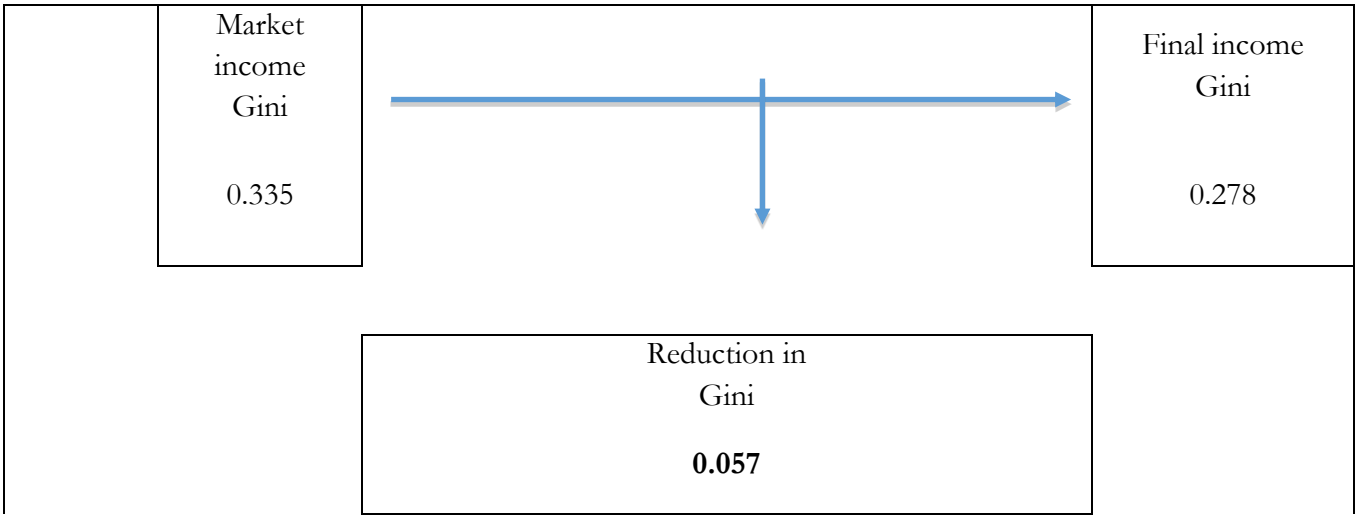
Gini value of the market income is not zero; the sum of taxes and transfers is not zero.
ZID. Zero income decomposition.
EID. Equalized income decomposition.

Hierarchy-Shapley Value

According to Sastre and Trannoy, "Shapley value does not satisfy the principle of independence of the aggregation level." [52] The following example demonstrates this shortcoming. Assume in our previous example in table A1, the equalized tax is in fact the combination of two independent taxes and we recalculate the simple (ZID) Shapley values for two taxes and one transfer. As is clear from table A-3, the Shapley values for these taxes would not add up to the Shapley value of the equalized tax in table A1. Moreover, the Shapley value of the transfer is different.

Table A-3. New Shapley Values (ZID) When Taxes Are Divided into Two Groups

Individual	Market income	Tax1	Tax2	Transfer	Final income
1	1	0	−5	9	5
2	20	−1	−4	7	22
3	30	−2	−3	5	30
4	40	−3	−2	3	38
5	50	−4	−1	1	46
Total	**141**	**−10**	**−15**	**25**	**141**
Average	**28.2**	**−2**	**−3**	**5**	**28.2**



[52] Sastre and Trannoy (2002, p. 54).

	Share of Tax1 in reducing inequality	Share of Tax2 in reducing inequality	Share of Transfer in reducing inequality	
Shapley Value (ZID)	**0.006**	**−0.063**	**0.114**	

ZID. Zero income decomposition. Given that no new tax has been added and that the only change is that some additional information about the sources of taxes has been included in the analysis, it is inconvenient that the Shapley value for transfers has also changed. Different solutions have been suggested to solve this problem. Sastre and Trannoy in particular introduce two methods, "Nested Shapley" and "Owen Decomposition."[53] Both of these solutions use a type of hierarchy, which is why they are called hierarchy-Shapley values here. In the following sections, unless otherwise specified, no distinction between ZID and EID approaches is made and the formulas can be used for both cases.

Hierarchy-Shapley Value: Nested Shapley

Using notations from the previous section, now assume each source of income I_i is the summation of a sub-set of sources, that is, $I_i = I_{i1} + I_{i2} + \cdots + I_{ik}$. It is assumed that this hierarchy has a particular theoretical basis. Define set $N_{I_i} = \{I_{i1}, I_{i2}, \dots, I_{ik}\}$ as the set of all incomes that comprise income source I_i. We are particularly interested in one of these sub-sources, the nested Shapley value of I_{ij}. Define set $NS_{I_{ij}}$as the set of sub-sets of set $N_{I_i} - \{I_{ij}\}$ (analogous to set S_{I_i}, defined in previous sections). According to Sastre and Trannoy, nested Shapley can be viewed as a two-step procedure. In the first step, we assume that the second layer does not exist and we calculate the simple Shapley value for all sources I_i. In the second step, we decompose the Shapley value of each source I_i between its sub-sources. The nested Shapley value of source I_{ij} (which is an element of I_i) is then equal to

(A-3) $$NSh_{I_{ij}} = \sum_{S \in NS_{I_{ij}}} \left(\frac{(s!)\times((k-s-1)!)}{k!} \Big(V(S \cup I_{ij}) - V(S) \Big) \right) + \frac{1}{k} \Big(Sh_{I_i} + V(I_i) - V(0) \Big).$$

Elements of this formula are either introduced above or in the previous sections. The only remaining item is k, which is the dimensionality of set N_{I_i}. Equation A2-3 is different from Sastre and Trannoy[54] because we do not assume that the value of $V(0)$ is zero, which is crucial when the inequality in the starting point is not zero (for example, the Gini value of the market income is not zero in our previous examples). The first term is exactly the same formula introduced for simple Shapley that is only applied to the set of sources that are part of N_{I_i} to

[53] Sastre and Trannoy (2002, p. 54).
[54] Sastre and Trannoy (2002).

explain the change in the value function between $V(0)$ and $V(I_i)$. The second term is the difference between the Shapley value of the aggregated source I_i and the value of function V when only aggregated source I_i is added. It is clear to see that

(A-4) $$\sum_{j=1}^{k} NSh_{I_{ij}} = Sh_{I_i}.$$

The proof is as follows:

$$\sum_{j=1}^{k} NSh_{I_{ij}} = \sum_{j=1}^{k}\left\{\sum_{S\in NS_{I_{ij}}}\left(\frac{(s!)\times\big((k-s-1)!\big)}{k!}\Big(V\big(S\cup I_{ij}\big)-V(S)\Big)\right)\right\} + \sum_{j=1}^{k}\frac{1}{k}\Big(Sh_{I_i}+V(I_i)-V(0)\Big)$$

$$\rightarrow \sum_{j=1}^{k} NSh_{I_{ij}} = \sum_{j=1}^{k}\left\{\sum_{S\in NS_{I_{ij}}}\left(\frac{(s!)\times\big((k-s-1)!\big)}{k!}\Big(V\big(S\cup I_{ij}\big)-V(S)\Big)\right)\right\} + Sh_{I_i} + V(I_i) - V(0).$$

Note that in the second term, the summation over *k* and (1/ *k)* cancel each other. Now note that the term inside the braces is equal to $Sh_{I_{ij}}$ if one decomposes the change in V between $V(0)$ and $V(I_i)$. The summation over the Shapley value of all *j* income concepts that are part of I_i is simply equal to the total change in the value function between $V(0)$ and $V(I_i)$. This means the preceding equation could be written as follows:

$$\rightarrow \sum_{j=1}^{k} NSh_{I_{ij}} = V(I_i) - V(0) + Sh_{I_i} + V(I_i) - V(0)$$

and therefore,

$$\rightarrow \sum_{j=1}^{k} NSh_{I_{ij}} = Sh_{I_i}.$$

Note that the value of *j* has to be at least 1 (that is, one income inside each income group) and if all income groups have *j* = 1, then the nested Shapley is reduced to the simple Shapley.

This nested Shapley formula, however, suffers from a few theoretical problems. First, the choice of decomposing $V(I_i) - V(0)$ between sub-elements of I_i (the first term in equation A-3) is arbitrary. One can choose any element of set S_{I_i}. Let's call it O_j and then decompose $V\big(I_i \cup O_j\big) - V\big(O_j\big)$ between elements of I_i and the decomposition also satisfies equation A-4. Equation A-3 can then be generalized as

(A-5) $$NSh_{I_{ij}} = \sum_{S \in NS_{I_{ij}}} \left(\frac{(s!)\times((k-s-1)!)}{k!} \left(V(O_j \cup S \cup I_{ij}) - V(O_j \cup S) \right) \right) + \frac{1}{k} \left(Sh_{I_i} - V(O_j \cup I_i) + V(O_j) \right) \quad For\ any\ arbitrary\ chosen\ O_j \in S_{I_i}.$$

The value of $NSh_{I_{ij}}$ would change with the choice of O_j. The second theoretical problem with equation A-3 is that $Sh_{I_i} + V(I_i) - V(0)$ is divided evenly between all k sub-elements of I_i. There is no particular reason to do so and any weighting scheme works as long as the weights add up to unity. In fact, one might argue that assigning similar weights is not in line with the idea of decomposition, which tries to allocate an appropriate share to each element depending on how important the element is. Using a weighting scheme that gives more weight to more important elements results in equation A-6:

(A-6) $$NSh_{I_{ij}} = \sum_{S \in NS_{I_{ij}}} \left(\frac{(s!)\times((k-s-1)!)}{k!} \left(V(O_j \cup S \cup I_{ij}) - V(O_j \cup S) \right) \right) + \left(\frac{\sum_{S \in NS_{I_{ij}}} \left(\frac{(s!)\times((k-s-1)!)}{k!} \left(V(O_j \cup S \cup I_{ij}) - V(O_j \cup S) \right) \right)}{V(I_i \cup O_j) - V(O_j)} \left(Sh_{I_i} + V(O_j \cup I_i) - V(O_j) \right) \right) \quad For\ any\ arbitrary\ chosen\ O_j \in S.$$

The weighting scheme in equation A-6 uses the relative importance of element I_{ij} in explaining the gap between $V(O_j \cup I_i)$ and $V(O_j)$, that is $V(O_j \cup I_i) - V(O_j)$. While this modified weighting scheme has a much better theoretical ground, the fact that $NSh_{I_{ij}}$ depends on the choice of O_j is still problematic. The following example helps to better visualize this problem. We use the same example as in table A-3 but the results should be compared to table A-1. Regardless of how we decompose the Shapley value of the total tax between its elements, the Shapley value of the transfer remains unchanged and equal to the value in table A-1 (the ZID Shapley value). However, depending on which formula is used for the decomposition for taxes, the Shapley values of Tax 1 and Tax 2 change, though they always add up to the Shapley value of total tax. Among the four different methods, A-6' is preferred to A-3' and A-6" is preferred to A-5' because of their modified weighting scheme, but there is no theoretical basis for any preference between A-6' and A-6". Note that in table A-4, values for A-5' and A-6' happen to be the same by pure luck and that this is not a general rule.

In the following formulas, $N_{Tax} = \{Tax1, Tax2\}$ and NS_{Taxj} is the set of all sub-sets of $N_{Tax} - \{Taxj\}$. Moreover, M represents the market income and $V(.)$ represents the Gini coefficient function. The following formulas are derived from the original formulas discussed in the specific example in table A-4.

(A-3)' $$NSh_{Taxj} = -\left[\sum_{S\in NS_{Taxj}}\left(\frac{\left(V(M\cup S\cup Taxj)-V(M\cup S)\right)}{2}\right)+\frac{1}{2}\left(Sh_{Tax}+V(M\cup Tax)-V(M)\right)\right]$$

(A-5)' $$NSh_{Taxj} = -\left[\sum_{S\in NS_{Taxj}}\left(\frac{\left(V(O_j\cup S\cup Taxj)-V(O_j\cup S)\right)}{2}\right)+\frac{1}{2}\left(Sh_{Tax}+V(O_j\cup Tax)-V(O_j)\right)\right] \quad Where\ O_j=\{Market\ Income+Transfer\}$$

(A-6)' $$NSh_{Taxj} = \sum_{S\in NS_{Taxj}}\left(\frac{\left(V(M\cup S\cup Taxj)-V(M\cup S)\right)}{2}\right) + \left(\frac{\sum_{S\in NS_{Taxj}}\left(\frac{\left(V(M\cup S\cup Taxj)-V(M\cup S)\right)}{2}\right)}{V(M\cup Tax)-V(M)}\left(Sh_{Tax}+V(M\cup Tax)-V(M)\right)\right)$$

(A-6)" $$NSh_{Taxj} = \sum_{S\in NS_{Taxj}}\left(\frac{\left(V(O_j\cup S\cup Taxj)-V(O_j\cup S)\right)}{2}\right) + \left(\frac{\sum_{S\in NS_{Taxj}}\left(\frac{\left(V(O_j\cup S\cup Taxj)-V(O_j\cup S)\right)}{2}\right)}{V(O_j\cup Tax)-V(O_j)}\left(Sh_{Tax}+V(O_j\cup Tax)-V(O_j)\right)\right)$$

$$Where\ O_j=\{Market\ Income+Transfer\}$$

Table A-4. Nested Shapley Values (ZID) Using Different Methods of Weighting and Reference Points

Individual	Market Income	Tax1	Tax2	Transfer	Final income
1	1	0	−5	9	5
2	20	−1	−4	7	22
3	30	−2	−3	5	30
4	40	−3	−2	3	38

5	50	−4	−1	1	46
Total	**141**	**−10**	**−15**	**25**	**141**
Average	**28.2**	**−2**	**−3**	**5**	**28.2**

	Market Income Gini		Final income Gini
	0.335		0.278

	Reduction in Gini		
	0.057		
	Share of Tax1 in reducing inequality	Share of Tax2 in reducing inequality	Share of Transfer in reducing inequality
Nested Shapley Value: equation A-3' (ZID)	**0.010**	**−0.067**	**0.114**
Nested Shapley Value: equation A-5' (ZID)	**0.002**	**−0.059**	**0.114**
Nested Shapley Value: equation A-6' (ZID)	**0.002**	**−0.059**	**0.114**
Nested Shapley Value: equation A-6" (ZID)	**0.013**	**−0.070**	**0.114**

ZID. Zero income decomposition.

Hierarchy-Shapley Value: Owen Decomposition

In order to avoid the problem of the reference point in the nested Shapley value, one can use the Owen value.[55] Intuitively, the Owen value can be viewed as a Shapley value of different nested Shapley values, that is, all possible reference points are included. Therefore, the Owen value is not subject to the theoretical shortcomings of the nested Shapley and accordingly, it has some advantages. Sastre and Trannoy disagree with this argument because they believe that reference points other than $V(0)$ imply that income elements are combined at a different aggregation level.[56] This argument loses its ground, however, as soon as we try to use the nested Shapley value to explain, for example, changes in the Gini index between market and final income. Because market income is on the same aggregation level as total tax but not Tax 1, using the nested Shapley implies the combination of two elements from two different aggregation levels. In other words, unless the reference point is "null," the combination of different aggregation levels is inevitable and therefore the Owen method is a theoretically better way of calculating the Shapley value since it incorporates all possible reference points.[57]

To better understand the Owen value, consider equation A-1 and particularly $V^{ZID}(S \cup I_i) - V^{ZID}(S)$ in that formula. This argument is calculated for each element of the summation. Owen decomposes this argument (for every element of the summation) to determine the share of each sub-element. The formula for the Owen decomposition is therefore

(A-7) $$OSh_{I_{ij}}^{ZID} = \sum_{S \in S_{I_i}} \left(\frac{(s!)\times((n-s-1)!)}{n!} \sum_{S' \in NS_{I_{ij}}} \left\{ \frac{(s'!)\times\left((k-s'-1)!\right)}{k!} \left(V^{ZID}(S \cup S' \cup I_i) - V^{ZID}(S \cup S') \right) \right\} \right).$$

All elements of this formula have been introduced previously. Note that the second summation (the inside summation) determines the share of I_{ij} in filling the gap $V^{ZID}(S \cup I_i) - V^{ZID}(S)$. Because the coefficient outside the second summation can move inside, the formula can be simplified to a formula similar to what Sastre and Trannoy suggest:

(A-7)' $$OSh_{I_{ij}}^{ZID} = \sum_{S \in S_{I_i}} \left(\sum_{S' \in NS_{I_{ij}}} \left\{ \frac{(s'!)(s!)\left((k-s'-1)!\right)((n-s-1)!)}{k!n!} \left(V^{ZID}(S \cup S' \cup I_i) - V^{ZID}(S \cup S') \right) \right\} \right)$$

Note that one can easily use V^{EID} in the preceding formula. Using the same example as in table A-3, the Owen values for the case of two taxes and one transfer are calculated in table A-5 and can be compared with the values in tables A-1 and A-4.

[55] Owen (1977).
[56] Sastre and Trannoy (2002).
[57] Sastre and Trannoy (2002) use a formula similar to equation 2-43, which suffers from a second theoretical problem (assigning equal weights to all sub-elements of one source) that is discussed in previous sections.

Table A-5. Owen Values (ZID)

Individual	Market Income	Tax1	Tax2	Transfer	Final income
1	1	0	−5	9	5
2	20	−1	−4	7	22
3	30	−2	−3	5	30
4	40	−3	−2	3	38
5	50	−4	−1	1	46
Total	**141**	**−10**	**−15**	**25**	**141**
Average	**28.2**	**−2**	**−3**	**5**	**28.2**

	Market Income Gini		Final income Gini
	0.335		0.278
		Reduction in Gini **0.057**	
		Share of Tax1 in reducing inequality	Share of Tax2 in reducing inequality	Share of Transfer in reducing inequality	
Owen Value (ZID)		**0.006**	**−0.063**	**0.114**	

ZID. Zero income decomposition.

It should be noted that the Owen value of the transfer is the same as in table A-1, as expected. Comparing Owen values from table A-5 to those in table A-4, the Owen value of each tax

component is between its nested Shapley value for equation A-6' and A-6". This outcome is expected because the Owen value incorporates all possible reference points and is intuitively a type of (weighted) average value. As a result, the Owen value is a more conservative estimate than the nested Shapley values for the share of each component.

Concluding Remarks

Of the different methods for estimating the Shapley value for income sources, there are better theoretical justifications for using the ZID approach than EID and for using the Owen value instead of the nested Shapley for performing an incidence analysis (which is mainly focused on different sources of taxes and transfers). This conclusion stands in contrast to the suggestions by Sastre and Trannoy and Duclos and Araar.[58] ZID is preferred over EID for two main reasons. First, ZID allocates a negative (or positive) value to a lump-sum tax (or transfer) that is by definition regressive (or progressive) and therefore increases (or decreases) inequality. EID will assign a zero value to such a tax (or transfer). Second, ZID decomposition is always exact; in contrast, EID will not be exact if we decompose a change in inequality between states A and B where inequality in the beginning point (that is, A) is not zero and average income in states A and B are different (that is, taxes and transfers do not add up to zero).

The Owen value is preferred over the nested Shapley value for two reasons. First, the simple nested Shapley formula (that is, equation A-3), which is used more often in the literature, assigns identical weights to different sub-items of a particular source of income. Second, even the modified version of nested Shapley (that is, equation A-6), which does not have the weighting problem, still suffers from the reference point dependency problem. This problem results in different Shapley values for sub-items depending on which reference point is chosen. The Owen value, on the other hand, solves this problem by using all reference points (and weighting them equally). The only critique made by Sastre and Trannoy for this technique (mixing items from different aggregation levels) is not unique to the Owen value.[59] Moreover, nested Shapley is also subject to this critique if it is used to explain a change in inequality between points A and B when point A is not the null case[60] such as, for instance, changes in the Gini coefficient between market income and total income.

Given these theoretical arguments, the Owen value with the ZID approach is the best option when the fiscal system under study mainly includes taxes and transfers, which is true for most cases. This method assures that the decomposition is exact and every single source of income receives its appropriate share based on how much it contributes to the reduction (or escalation) of inequality. Moreover, using the Owen value, there is no problem regarding the choice of the reference point.

[58] See Sastre and Trannoy (2002) and Duclos and Araar (2007).
[59] See Sastre and Trannoy (2002).
[60] The null case is where no source of income is distributed in the society.