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Threshold Bank-run Equilibrium in Dynamic Games

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Abstract

This study sets a bank-run equilibrium analysis in a dynamic and incomplete information environment where agents can reconsider attempts to run on the bank over time. The typical static bank-run model is extended in this paper to capture the learning dynamics of agents through time, giving bank-run analysis a more realistic feature. Apart from employing a self-fulfilling framework in this model, where agents' actions are strategic complements, we allow agents to update over time their beliefs on the strength of the fundamentals that is not commonly known. In particular, we extend the bank-run model analyzed by Goldstein and Pauzner (Journal of Finance 2005) and build it on a dynamic global games framework studied by Angeletos *et.al.* (Econometrica 2007). We present here how a simple recursive setup can generate a unique monotone perfect Bayesian Nash equilibrium and show how the probability of bank-run is affected through time by the inflow of information and the knowledge of previous state outcome. Finally, it is also shown that when an unobservable shock is introduced, multiplicity of equilibria can result in this dynamic learning process.

Keywords: threshold bank-run, monotone perfect Bayesian Nash equilibrium, dynamic global games
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1. Introduction

A widely recognized feature of crisis models with self-fulfilling character is the presence of strategic complementarity. Agents tend to rationally cluster their beliefs towards a certain action which eventually induces an economic outcome that it favors. In the case of bank-runs, this is observed when one argues that if depositors anticipate that others will run against the bank for fear of bankruptcy, their action in conforming to such belief provokes bankruptcy itself. Consequently, when a sufficiently large number of them entertain such expectation, a bad outcome is soon realized. For many of these studies on crisis[1], strategic complementarity offers a compelling reason why some large fluctuations in the economy could become so sudden and penetrating that they eventually tosses the market into some kind of *liquidity black hole*. Models with multiple equilibria then becomes a natural norm in explaining that a crisis is nothing but a shift to a lower equilibrium point of the economy.

While the origin of crisis is normally attributed to some behavioral and policy issues that are typically considered outside the formal theory (e.g. *irrational exuberance*, regulations), there is a growing literature now that explains its evolution through some endogenous responses made by rational agents themselves. Morris and Shin (2004), for example, argue that a feedback mechanism that relies on changing incentives of market participants can create financial distress that feeds on itself. In the recent past, attempts to provide an integrated model for this have made use of the tools on coordination games, particularly on the so-called global games framework introduced by Carlsson and van Damme (1993)[2]. In this approach, an equilibrium is seen as a cut-off point between crisis and no-crisis events, rather than a point of convergence induced by the changes in agents' beliefs. This, in turn, achieves a uniqueness result which resolves the selection problem inherent in models with multiple equilibria.

However, one of the most important critiques in the application of global games to crisis events like bank-runs is its confinement to static structures. Agents maintain only a one-time discernment to either "run" or "remain" and cannot use any updated information they have received over time which could reinforce their incentive to

[1]For currency crises, see studies of Maurice Obstfeld (1996), Flood and Garber (1984), Morris and Shin (1998, 2004); for debt crises, see Calvo (1988), Cole and Kehoe (1996); and for contagion, Goldstein and Pauzner (2004).

[2]For a survey on the literature of global games and its application, see Morris and Shin (2003).

withdraw prematurely in the future. Normally, when crisis is regarded as a dynamic event, agents must have that option to withdraw at any stage if the payoff for doing so becomes higher than the expected returns from remaining. Moreover, apart from the private information that one obtains through time, the fact that a bank has not failed in the past is a good signal that either it is strong or that there was really no potent belief among the other agents to abandon it. Thus, this type of learning-through-time which is crucial in an environment with strategic uncertainty is not incorporated in the static model.

In this paper, we address this need of extending the static bank-run model into a dynamic form. We take off from the model analyzed by Goldstein and Pauzner (2005) and show that bank-run threshold is a function of interest rates. Then, we set this model on a dynamic global games framework studied by Angeletos *et.al.* (2007), using monotone perfect Bayesian-Nash as a solution concept. We establish here how a simple recursive setup can generate a unique equilibrium strategy. Consequently, comparative statics is studied to show how the probability of bank-run is affected over time by the inflow of private information and the knowledge that the bank has survived from the bad speculations in the past. Finally, we will also show that when an unobservable shock is introduced, multiplicity of equilibria can result in this dynamic learning process.

1.1. Dynamic Global Games Literature

Global games under static framework normally admits unique equilibrium results in its various applications. It is the presence of a sufficiently small noise in the private signal about the fundamentals that is responsible for pinning down a threshold equilibrium. This does not happen when the noise reaches zero or when there is common knowledge since any outcome becomes possible within the range of values of the strength of the fundamentals θ. However, the results are not unanimous in a dynamic global game framework as they vary depending on how dynamic features are utilized. Giannitsarou and Toxvaerd (2003), for example, generated a uniqueness result on the basis of dynamic intertemporal complementarities that make use of stochastic state variable. Morris and Shin (1999) established the same result but by allowing the fundamentals to follow a random walk over time. Similarly, Goldstein and Pauzner (2004) maintains also this uniqueness in their study of crises contagion. On the other

hand, there are advocates of multiplicity results in dynamic global games like Angeletos *et.al* (2007) who focus on the learning dynamics that is based on updated signals and previous regime outcome. There is also Chassang (2008) who shows in his study on the role of miscoordination in the robustness of cooperation that a range of perfect Bayesian equilibria is generated when information becomes sufficiently precise. In essence, our study here contributes to this growing debate by asserting that uniqueness and multiplicity results can be both obtained in a certain dynamic framework, depending on the presence of unobservable shocks in the economy.

1.2. Uniqueness and Multiplicity Results

The dynamic feature we employ in this paper is depicted by a learning process derived from an agent's private information and the past period's state outcome. Although this is basically the heart of Angeletos *et. al.* model, their result does not coincide with ours as we generate here a unique equilibrium strategy. The main reason is that the "state of the world" threshold θ_t^* is only used implicitly in our model while the threshold on the mass of early withdrawals n_t^* takes the center stage in determining whether bank-run will occur at time t. As n_t^* is derived from θ_t^* and x_t^* (i.e. the private signal threshold), it carries a more summarized information that simplifies the payoff analysis for each agent and in a way that induces a unique equilibrium outcome. More specifically, we argue (in Lemma 2) that both thresholds θ_t^* and n_t^* can be used interchangeably in determining the probability of bank-run incidence at any time t. That is, intuitively, when bank-run has not occurred, we can say that either the fundamental θ is so strong that it has not gone down below θ_t^* or that the potential level of early withdrawals has not reached its threshold level n_t^*. Put differently, if bank-run occurs, then it is certain that both of these thresholds are breached simultaneously. Thus, it is the novel use of threshold n_t^* that plays the major role in the uniqueness result of our model.

The outcome in a setting with unobservable shocks is quite different. Multiple equilibria can now be generated since the noise about the shocks interferes with the learning process that makes the expected payoff to any agent at t to be non-monotonic in n_t^*. That is, even at a high and accommodating threshold measure of early withdrawals, bank-run can still persist. This further means that even if it is publicly known that the bank has survived in the past, the uncertainty on shocks

"perturbs" the information about the previous period's threshold, and is therefore supported fully by a probability distribution.

The rest of the paper is organized as follows. Section 2 deals with the characterization of unique equilibrium in a static bank-run model. Section 3 presents how the uniqueness result prevails even in a dynamic setup and discusses the effect on bank-run's probability over time. Finally, Section 4 incorporates the impact of shocks in the model which results to multiple equilibria.

2. The Static Bank-run Model

A continuum of agents, indexed by i, is uniformly distributed over $[0, 1]$. Everyone deposits at the start an amount of one unit and decides simultaneously at a specific moment whether to withdraw or retain the investment. We denote the agent's action as $s_i \in \{0, 1\}$, where $s_i = 0$ is to withdraw and $s_i = 1$ is to wait. An agent receives a payoff $P(>1)$ when she decides to withdraw and expects to receive $R(>P)$ if she decides to wait until maturity. Since waiting until maturity carries a risk and subjects one's payoff to the bank's residual resources, one receives at the moment of discernment the expected payoff for waiting as $\int R\frac{1-nP}{1-n}dF(n)$, where $n \in [0, 1]$ is the proportion of agents who have chosen to withdraw and $F(n)$ is the c.d.f. of n which will be further discussed later. Notice that whenever n reaches $\frac{1}{P}$, the bank runs out of resources and leaves the remaining agents with zero utility[3]. Thus, an agent chooses to wait provided that there is no bank-run and in turn, bank-run is prevented if and only if there is no sufficient proportion of agents who runs. We see here that along with the lack of coordination, there exists strategic complementarity of agents' actions which is central to Goldstein and Pauzner's (2005) application of global games to bank-run models. We summarize now each agent's best response strategy as follows:

[3]This is a consequence of the sequential-service constraint mechanism that is normally employed in the literature of bank runs. Under this mechanism, any agent who wishes to withdraw at a specified time receives P even if he is the last one to be accommodated by the bank just before its bankruptcy. See Peck and Shell (2003) for details. The above set-up is similar to Goldstein and Pauzner (2005) except that it does not ascribe anymore a probability of getting a positive payoff for those who remain when $n \geq 1/P$. For simplicity, we relax this assumption that residual depositors can claim from the bank's illiquid assets after its bankruptcy.

$$s_i(n) \in \underset{s_i \in \{0,1\}}{\arg\max} \left\{ s_i \left(U_{wait} - U_{withdraw} \right) \right\}, \tag{1}$$

$$\text{where } U_{wait} = \begin{cases} \int R \frac{1-nP}{1-n} dF(n) & \text{if } 0 \leq n < \frac{1}{P} \\ 0 & \text{if } \frac{1}{P} \leq n \leq 1 \end{cases} \quad \text{and } U_{withdraw} = \begin{cases} P & \text{if } 0 \leq n < \frac{1}{P} \\ 0 & \text{if } \frac{1}{P} \leq n \leq 1 \end{cases}$$

Consequently, any remaining agent would find it rewarding to stick with the bank if and only if $U_{wait} > U_{withdraw}$. This clearly depends on the level of n that each one thinks at the time when everyone contemplates simultaneously whether to withdraw or wait.

To model how n is realized[4], we note first that it is dependent on the state of the economy θ, which is not commonly known to all agents. Thus, an agent i observes a private signal $x_i = \theta + \varepsilon_i$, where ε_i is the error term that is independently and uniformly distributed over $[-\varepsilon, \varepsilon]$. Assume that there are known thresholds $\underline{\theta}$ and $\overline{\theta}$ $\left(i.e.\ \underline{\theta} < \overline{\theta}\right)$, where agent i decides to surely run when θ is observed below $\underline{\theta}$ and to surely wait when above $\overline{\theta}$. For simplicity, we assume that *Nature* draws first θ from a uniform distribution over $\left[\underline{\theta}, \overline{\theta}\right]$ which defines the initial common prior about θ. Since θ is observed with certain level of noise, the range of fundamentals where i will neither have the full conviction to withdraw nor to wait is within the interval $\left[\underline{\theta} - \varepsilon, \overline{\theta} + \varepsilon\right]$. This is the intermediate regime where one's action is not immediately determined since it can be influenced by the expectations on others' actions, *i.e.* if one believes that others are withdrawing, she may also think of withdrawing before the bank gets bankrupt; whereas, if one believes that others will not run, then she will also do the same.

From the two extreme regimes where actions to withdraw and to wait are predictable for a given x_i, it is natural to look into the intermediate regime for an equilibrium that "tears" the agent between withdrawing and waiting. In this case,

[4]The realization of n does not strictly mean that a proportion of n depositors have actually withdrawn from the bank. Since the game is symmetric and each one follows the same strategy, it can be understood that each one believes that there are n depositors who already *want* to withdraw early at a specified time. Although this difference does not matter much in a one-shot static game, the latter explanation is used for the dynamic game setup in the next section.

we consider a threshold Bayesian-Nash equilibrium wherein the agent's strategy is monotonic in x_i. Suppose for a given θ, agents follow a threshold $x^* \in [\theta - \varepsilon, \theta + \varepsilon]$, where $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, such that each one withdraws if and only if $x_i \leq x^*$. The proportion of agents withdrawing is then decreasing in θ and is given by:

$$n(\theta, x^*) = \Pr(x_i \leq x^* | \, \theta) = \frac{1}{2\varepsilon} \int_{\theta-\varepsilon}^{x^*} dx_i = \frac{x^* - \theta + \varepsilon}{2\varepsilon} \tag{2}$$

It follows then that there is a θ^* that solves $\theta^* = n(\theta^*, x^*)$, where bank-run occurs if and only if $\theta \leq \theta^*$. Thus, we obtain:

$$\theta^* = n(\theta^*, x^*) \Longleftrightarrow x^* = (1 + 2\varepsilon)\, n(\theta^*, x^*) - \varepsilon \tag{3}$$

As the parameter θ^* depicts the threshold strength of fundamentals that determines bank-run incidence, $n(\theta^*, x^*)$ which represents the mass of agents who withdraw at state θ^* can also be regarded as a threshold. Indeed, bank-run occurs if and only if $n \geq n(\theta^*, x^*)$ since

$$\Pr(\theta \leq \theta^* \, | x^*) = \frac{1}{2\varepsilon} \int_{x^*-\varepsilon}^{\theta^*} d\theta = \frac{\theta^* - x^* + \varepsilon}{2\varepsilon} = \int_{n(\theta^*, x^*)}^{1} dn = \Pr(n \geq n(\theta^*, x^*)) \tag{4}$$

Now, given that bank-run does not occur for as long as $n < n(\theta^*, x^*)$, the expected payoff advantage of retaining ones investment over withdrawing it is given by:

$$U(n(\theta^*, x^*), x^*) = P \int_0^{\frac{1}{P}} R\left(\frac{1 - n(\theta^*, x^*)P}{1 - n(\theta^*, x^*)}\right) dF(n(\theta^*, x^*)) - P \tag{5}$$

where $F(n(\theta^*, x^*)) \equiv F(n(\theta^*) \, | \, x^*)$ is the c.d.f. of n conditional on threshold x^* being followed. $F(n(\theta^*, x^*)$ is interpreted as the probability that bank-run has not

yet been induced (*i.e.* $n < n(\theta^*, x^*)$), given that every one follows the threshold x^* for their signals. It is clear from the utility representation of (5) that an agent i waits if $U(n(\theta^*, x^*), x^*) > 0$ and withdraws if $U(n(\theta^*, x^*), x^*) \leq 0$.

By setting $U(n(\theta^*, x^*), x^*) = 0$, we derive a unique monotone equilibrium using the threshold x^* such that if $x_i \leq x^*$, i will withdraw since $n \geq n(\theta^*, x^*)$ and bank-run is sure to come.

Proposition 1: *In the static game, there is a unique bank-run equilibrium using the threshold x^*. This equilibrium is characterized by $n(\theta^*, x^*)$ through the following equations:*

$$(i) \qquad (P-1)\ln\left(\frac{P}{P-1}\right) = \frac{R-1}{R}$$

$$(ii) \qquad n(\theta^*, x^*) = \frac{R-P}{P(R-1)} = 1 - \frac{1}{P\ln\left(\frac{P}{P-1}\right)} \in (0,1)$$

Proof: A direct way of solving the equilibrium is to solve first for $F(n(\theta^*, x^*))$ which was the method used by Morris and Shin (2003) in their analysis of liquidity in the traders' market. Note that for any threshold signal $\acute{x}$,

$$\Pr(\theta > \theta^* | \, \acute{x}) = \frac{\acute{x} + \varepsilon - \theta^*}{2\varepsilon} = \Pr(n < n(\theta^*, \acute{x})) = F(n(\theta^*, \acute{x})).$$

By using (3) and substituting the value of x^* on $\acute{x}$, we obtain $F(n(\theta^*, x^*)) = n(\theta^*, x^*)$ which implies that $dF(n(\theta^*, x^*)) = 1$. Applying this on (5) and noting that $\frac{1-nP}{1-n}$ is decreasing in n, for all $n \in [0,1]$, we see that $U(n(\theta^*, x^*), x^*)$ is monotonic in $n(\theta^*, x^*)$, giving a uniqueness result whenever a solution exist.

To solve $U(n(\cdot), x^*) = 0$, we use integration by parts on (5) with $dF(n(\theta^*, x^*)) = 1$, hence, we obtain

$$R\left[-(1 - n(\theta^*, x^*)P)\ln(1 - n(\theta^*, x^*)) + P\int -\ln(1 - n(\theta^*, x^*))dn(\theta^*, x^*)\right] = 1$$

$$\iff \quad R[(n(\theta^*, x^*)P - 1)\ln(1 - n(\theta^*, x^*)) + P(1 - n(\theta^*, x^*))\ln(1 - n(\theta^*, x^*)) + Pn(\theta^*, x^*)] \Big|_{n(\cdot)=0}^{n(\cdot)=1/P} = 1$$

Simplifying,

$$R\left((P-1)\ln\left(\frac{P-1}{P}\right)+1\right) = 1 \iff (P-1)\ln\left(\frac{P}{P-1}\right) = \frac{R-1}{R}$$

By substituting R on the equation $R\frac{1-n(\cdot)P}{1-n(\cdot)} = P$, we have

$$n(\theta^*, x^*) = 1 - \frac{1}{P\ln\left(\frac{P}{P-1}\right)}$$

Finally, to see that i indeed prefers to wait when $x_i > x^*$, recall that this implies $\theta > \theta^* \Leftrightarrow n(\theta^*, x_i) < n(\theta^*, x^*)$. Thus, from (5), we have $U(n(\theta^*, x_i), x_i) > 0$. Analogously, when $x_i \leq x^*$, an agent strictly prefers to withdraw since $U(n(\theta^*, x_i), x_i) \leq 0$. ■

In Figure 1, the equilibrium is depicted at $n(\theta^*, x^*)$ where the expected returns from waiting is equal to that of withdrawing, *i.e.* area A = area B. At this point, everyone starts thinking of withdrawing from the bank eventhough everyone knows that only $n(\theta^*, x^*)$ agents are willing to do so. This is because the expected residual payoff at this moment for those who may still want to wait is just equal to P, which is the amount received by withdrawing prematurely. Above $n(\theta^*, x^*)$, agents who remain will obtain expected payoff lower than P.

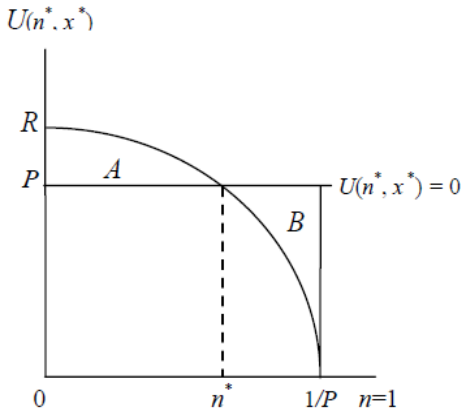

Figure 1: Unique threshold equilibrium in a static framework

One can also explain this equilibrium from a different perspective. Observe that in the region where $n > n(\theta^*, x^*)$, agents will preempt their actions for fear of being preempted by others and so each one simply wants to withdraw ahead of the others. Each one wants to withdraw as soon as possible to avoid being caught in a bankruptcy by being overtaken by others in withdrawing. This preemptive behavior continues until it is just below $n(\theta^*, x^*)$ where the mass of withdrawals is just sufficient not to provoke a bank-run. The perception of each one that others may withdraw early is a result of the assumption that economic fundamentals are not commonly known.

As regards bank's deposit interest rates, the above proposition has the following implication.

Corollary. *The threshold equilibrium measure of early withdrawals is a function of interest rates, i.e.*

$$n(\theta^*, x^*) = 1 - \frac{1}{(1+r)\ln\left(\frac{1+r}{r}\right)}.$$

Notice that the threshold $n(\theta^*, x^*)$ is decreasing in r which means that as interest rate increases, the threshold that triggers a crisis becomes less and less relaxed. This seems not to square well with the common idea that high interest rate promotes greater deposits in the bank. The main reason for this apparent paradox is that once the money is in the bank the situation becomes different. The very same high interest rate provides better incentive for agents to withdraw whenever fear of impending bankruptcy starts to appear. The higher therefore is the payoff at the exit stage, the

more will the agents be induced to withdraw when there is already uncertainty about the fundamentals.

3. The Dynamic Bank-run Model

Within a dynamic game framework, the static benchmark previously discussed is modified in such a way that agents can receive information over time so that they can choose to run on the bank at any time if so needed. Clearly, the static model is designed only for a one-shot decision to either withdraw or wait and does not capture the learning process of agents through time which gives bank-run phenomena a more realistic feature. In what follows, we apply on the static benchmark the dynamic regime change framework studied by Angeletos *et.al.* (2007).

Here, we set all $t \in \{1, 2, 3, ..., T\}$ as moments for making decisions for every agent to choose $s_{it} \in \{0, 1\}$ for as long as the bank remains viable after $t-1$; once bank–run occurs, the game ends. Denote $n(\theta)$ as the measure of potential early withdrawals at a given state θ. Given that bank-run has not occurred at date $t-1$, bank-run could occur only at t if and only if $n(\theta) \geq n(\theta_t^*, x_t^*)$, where $n(\theta_t^*, x_t^*)$ is the threshold size of early withdrawals that triggers a bank-run at t. Over time, agents receive noisy signals $x_{it} = \theta + \varepsilon_{it}$ about θ which is never commonly known to all as in the static model. This form of learning through time which provides agents the possibility of withdrawing at any period is the main ingredient in the mechanics of this intertemporal coordination problem in dynamic bank-run.

3.1. Monotone Perfect Bayesian-Nash Equilibrium

Let $\hat{x}_t = \{\hat{x}_{it}\}_{i\in[0,1]}$ where each $\hat{x}_{it} = \{x_{i1}, x_{i2}, ..., x_{it}\}$ is the history of i's signals until t. Denote $\hat{s}_t(\hat{x}_t) = \{s_1(\hat{x}_1), s_2(\hat{x}_2), ...s_t(\hat{x}_t)\}$ as the complete strategy profile of all players up to date t where $s_t(\hat{x}_t)$ represents the strategy of everyone at t. The decision for i to either withdraw or wait at any time t is depicted by $s_{it}(\hat{x}_{it}, \hat{s}_{t-1}(\hat{x}_{t-1})) \in \{0, 1\}$ since it is contingent only on i's own history of private signals until t and the knowledge that bank-run has not occurred in the past.

To characterize an equilibrium, recall that an agent chooses an action that maximizes her expected payoff difference between withdrawing and waiting, such that

she chooses to withdraw ($s_{it} = 0$) if the expected payoff from withdrawing is higher, otherwise she chooses to wait ($s_{it} = 1$). As in the static model, the expected payoff difference depends mainly on the measure of agents who would want to withdraw which is implied by the level of fundamentals. Thus, we let $F(n(\theta) \mid \hat{x}_{it}, \hat{s}_{t-1})$ as the c.d.f. of $n(\theta)$, conditional on the knowledge of one's own past and present signals and that bank-run has not yet occurred in the past as represented by agents' strategy $\hat{s}_{t-1}$. Note that this characterization does not necessarily require an infinite horizon setting and is valid even with only a finite length of time T. Thus, we define our equilibrium concept as follows:

Definition: The symmetric strategy $\{s_{it}\}_{t=1}^{T}$ for each agent i is an equilibrium if and only if for all $\hat{x}_{it}$ and for all $t \in \{1, 2, 3, ..., T\}$:

$$s_{it}(n(\theta), \hat{x}_{it}) \in \arg\max_{s_i \in \{0,1\}} \left\{ s_i \left[\int R(n(\theta))\, dF(n(\theta) \mid \hat{x}_{it}, \hat{s}_{t-1}) - P \right] \right\}.$$

The above definition is a perfect Bayesian-Nash equilibrium since at every subgame, every player chooses the optimal payoff given her own signal and the knowledge that bank-run has not occurred in the past. The fact that the bank continues to exist makes agent's actions sequentially rational as they do not depend on any other period. Moreover, any "off-the-equilibrium" action of any agent is negligible since apart from being unobservable, it forms part only of the final and summarized information that bank-run has not occurred in the past[5]. Therefore, an agent's expected payoff for waiting is determined only by Baye's rule at any relevant history of the game.

For $t = 1$, we have $\hat{x}_{i1} = x_{i1}$ and $\hat{s}_0 = 1$ *i.e.* trivially, no bank-run has occurred before period 1. This dynamic game played only in period 1 is analog to the static model that admits a unique equilibrium characterized by the thresholds for x_{i1} and θ. The result of Proposition 1 therefore applies also for this case of dynamic game with $t = 1$. For $t \geq 2$, the equilibrium strategy $s_{it}(\cdot)$ is conditioned by the profile of i's signals over time ($\hat{x}_{it}$) and the knowledge whether bank-run has already happened

[5]This concept has semblance of open-loop strategies in the sense that agents are nonatomic (*i.e.* no single agent can influence other player's payoff). The only difference is that each agent still reacts to the collective previous actions of others at every time t based on the summary of information obtained until time $t - 1$.

before the present time ($\hat{s}_{t-1}$). Given a uniform distribution about θ and errors ε_{it} being independent of θ and serially uncorrelated across time, the simple average $\bar{x}_{it} = \theta + \bar{\varepsilon}_{it}$ is a sufficient statistic for the profile of signals $\hat{x}_{it}$, where $\bar{\varepsilon}_{it} = \frac{1}{t}\sum_{\tau=1}^{t} \varepsilon_{i\tau}$. Thus, we summarize i's history of private information into a single parameter at each time t. Where no confusion may arise, we simply denote $\bar{x}_{it}$ and $\bar{\varepsilon}_{it}$ in this symmetric game as $\bar{x}_t$ and $\bar{\varepsilon}_t$, respectively.

Lemma 1: *Any monotone perfect Bayesian-Nash equilibrium strategy* $\{s_t\}_{t=1}^{T}$ *is characterized by a sequence* $\{x_t^*, \theta_t^*\}_{t=1}^{T}$, *where* $x_t^* \in \left(\underline{\theta} - \bar{\varepsilon}_t, \overline{\theta} + \bar{\varepsilon}_t\right)$ *and* $\theta_t^* \in \left[\underline{\theta}, \overline{\theta}\right]$, *such that:*

(i) if $\theta_t^* > \theta_{t-1}^*$, *then bank-run occurs at* $t \geq 2$ *with probability* $1 - \frac{\bar{x}_t + \bar{\varepsilon}_t - \theta_t^*}{\bar{x}_t + \bar{\varepsilon}_t - \theta_{t-1}^*}$;

(ii) at any $t \geq 1$, *an agent withdraws* $(s_{it} = 0)$ *if* $\bar{x}_t \leq x_t^*$ *and waits* $(s_{it} = 1)$ *if* $\bar{x}_t > x_t^*$.

Proof: The proof is similar to the induction procedure adopted by Angeletos, *etc.* For $t = 1$, the equilibrium is characterized following Proposition 1. Now for any $t \geq 2$, we consider two possibilities: Case 1: Suppose $\theta_t^* \leq \theta_{t-1}^*$, then if bank-run has not occurred at $t-1$(*i.e.* $\theta > \theta_{t-1}^*$), then it is with probability 1 that bank-run will also not occur at t (*i.e.* $\theta > \theta_t^*$). Hence, $\bar{x}_t \geq \overline{\theta} + \bar{\varepsilon}_t$ for any $t \geq 2$. Case 2: Suppose $\theta_t^* > \theta_{t-1}^*$, then if bank-run has not occurred at $t-1$, the probability that bank-run will also not occur at time t is given by:

$$\Pr(\theta > \theta_t^* |\, \bar{x}_t, \theta > \theta_{t-1}^*) = \frac{\frac{1}{2\varepsilon}\int_{\theta_t^*}^{\bar{x}_t + \bar{\varepsilon}_t} d\theta}{\frac{1}{2\varepsilon}\int_{\theta_{t-1}^*}^{\bar{x}_t + \bar{\varepsilon}_t} d\theta} = \frac{\bar{x}_t + \bar{\varepsilon}_t - \theta_t^*}{\bar{x}_t + \bar{\varepsilon}_t - \theta_{t-1}^*} \tag{6}$$

The above posterior probability is continuous and strictly increasing in $\bar{x}_t \in (\theta_t^* - \bar{\varepsilon}_t, \infty)$. Initially, we know that $\bar{x}_t \in \left(\underline{\theta} - \bar{\varepsilon}_t, \overline{\theta} + \bar{\varepsilon}_t\right)$, however as $\overline{\theta} \to \infty$,the posterior approaches 1; while as $\underline{\theta} \to \theta_t^*$ it approaches to 0. (Lower than θ_t^*, the probability becomes either negative or greater than 1.) This implies therefore that there exist $x_t^* \in (\theta_t^* - \bar{\varepsilon}_t, \infty)$ such that $R(n(\theta)) \cdot \Pr(\theta > \theta_t^* |\, \bar{x}_t, \theta > \theta_{t-1}^*) \leq P_t$ whenever $\bar{x}_t \leq x_t^*$ and so $s_{it}(n(\theta), \bar{x}_t) = 0$ and $R(n(\theta)) \cdot \Pr(\theta > \theta_t^* |\, \bar{x}_t, \theta > \theta_{t-1}^*) > P_t$ whenever $\bar{x}_t > x_t^*$ and therefore $s_{it}(n(\theta), \bar{x}_t) = 1$. ■

Remark: Note that Lemma 1 (i) allows the possibility of bank-runs to occur at any time $t \geq 2$. If $\theta_s^* \leq \theta_{s-1}^*$, for any $s \geq 2$, then it is immediate that no one thinks of running against the bank in period $t \geq s$.

Since Lemma 1 claims the existence of a threshold x_t^* from which an agent bases her decision to withdraw, one can now measure the mass of agents who will withdraw following that threshold. This measure at a given period t, which is continuous and decreasing in θ, is given by $n(\theta, x_t^*) = \Pr(\bar{x}_t \leq x_t^* \,|\theta) = \frac{1}{2\bar{\varepsilon}} \int_{\theta-\bar{\varepsilon}_t}^{x_t^*} d\,\bar{x}_t = \frac{x_t^*-\theta+\bar{\varepsilon}_t}{2\bar{\varepsilon}_t}$. This implies that bank-run occurs if and only if $\theta \leq \theta_t^*$, where θ_t^*, like in the static model, is the unique fixed point of n, that is,

$$\theta_t^* = \frac{x_t^* - \theta_t^* + \bar{\varepsilon}_t}{2\bar{\varepsilon}_t} = n(\theta_t^*, x_t^*) \tag{7}$$

Moreover, to say that the level of fundamentals is above its bank-run threshold (*i.e.* $\theta > \theta_t^*$) is similar to saying that the proportion of agents who wish to withdraw at period t is still below its own bank-run threshold (*i.e.* $n < n(\theta_t^*, x_t^*)$). This assertion which has its analog in the static game is presented in the following lemma by stating that the c.d.f. of the posterior belief on n at any period t is equal to the probability that bank-run will *not* occur at t, given that it has not occurred in the past.

Lemma 2: *Given thresholds* $x_t^* \in \left(\underline{\theta} - \bar{\varepsilon}_t, \overline{\theta} + \bar{\varepsilon}_t\right)$, $n(\theta_t^*, x_t^*) \in (0,1)$ *and* $\theta_t^* \in (\underline{\theta}, \overline{\theta})$ *and that* $n(\theta_t^*, x_t^*) < n(\theta_{t-1}^*, x_t^*)$, *the c.d.f. of the posterior beliefs on* n *at period* t *is equivalent to* $\frac{n(\theta_t^*, x_t^*)}{n(\theta_{t-1}^*, x_t^*)}$ *and to* $\Pr(\theta > \theta_t^* | \, x_t^*, \theta > \theta_{t-1}^*)$.

Proof:

$$\begin{aligned}
F(n(\theta_t^*, x_t^*) | \, \hat{s}_{t-1}) &= \Pr(n < n(\theta_t^*, x_t^*) \,|\, n < n(\theta_{t-1}^*, x_t^*)) \\
&= \frac{F(n(\theta_t^*, x_t^*)) \,\cap\, F(n(\theta_{t-1}^*, x_t^*))}{F(n(\theta_{t-1}^*, x_t^*))} \\
&= \frac{n(\theta_t^*, x_t^*)}{n(\theta_{t-1}^*, x_t^*)} \\
&= \frac{\Pr(\bar{x}_t \leq x_t^* \mid \theta_t^*)}{\Pr(\bar{x}_t \leq x_t^* \,\big|\, \theta_{t-1}^*)} \\
&= \frac{x_t^* + \bar{\varepsilon}_t - \theta_t^*}{x_t^* + \bar{\varepsilon}_t - \theta_{t-1}^*} \\
&= \Pr(\theta > \theta_t^* | \, x_t^*, \theta > \theta_{t-1}^*), \text{ from Lemma 1}
\end{aligned}$$

■

Even if the size of potential early withdrawals is not observable to anyone at any point in time, the fact that bank-run did not occur at $t-1$ is enough to be ascertained that this size has not reached its threshold level at $t-1$. In other words, despite the absence of information about n, one can also describe the probability of bank-run at t in terms of threshold $n(\theta_t^*, x_t^*)$ since this can be derived from how bank-run incidence is determined using θ_t^* and the signal $\bar{x}_t$. In what follows, Lemma 2 shall allow us to simplify our payoff analysis by using the parameter $n(\theta_t^*, x_t^*)$ instead of θ_t^* in pinning down an equilibrium. We note from the proof of Lemma 2 that $n(\theta_t^*, x_t^*)$ has a uniform distribution over $(0,1)$ since $f(n(\theta_t^*, x_t^*)|\,\hat{s}_{t-1})$ is equal to a constant $\frac{1}{n(\theta_{t-1}^*, x_t^*)}$.

Similar to (5) of the static model, we define below the expected payoff advantage in retaining one's investment over withdrawing it for an agent who holds a threshold signal x_t^* at time t.

$$\Pi\left(n_t^*, n_{t-1}^*, x_t^*\right) = P\int_0^{\frac{1}{P}} R\left(\frac{1-n(\theta_t^*, x_t^*)P}{1-n(\theta_t^*, x_t^*)}\right) dF\left(n(\theta_t^*, x_t^*)\,|\hat{s}_{t-1}\right) - P \tag{8}$$

The value of $n(\theta_t^*, x_t^*)$, as defined in (7), is contingent on θ_t^* and x_t^*. Thus, even if $n(\theta_t^*, x_t^*)$ is unobservable as a threshold, this does not pose any problem in determining the probability of bank-run since it can be indirectly drawn from any realizable private signal and the knowledge that bank-run has not happened at $t-1$, *i.e.* $\theta > \theta_{t-1}^*$. It follows therefore that equilibrium can be characterized by:

$$\Pi\left(n_t^*, n_{t-1}^*, x_t^*\right) = 0 \tag{9}$$

Proposition 2: *In a dynamic game, there is a unique monotone bank-run equilibrium using the threshold x_t^* at time t. This equilibrium is characterized by the sequence $\{n(\theta_t^*, x_t^*), x_t^*\}_{t=1}^T$ through the following equations:*

$$(P_t - 1)\ln\left(\frac{P_t}{P_t - 1}\right) = \frac{R - n(\theta_{t-1}^*, x_t^*)}{R} \tag{10}$$

$$n(\theta_t^*, x_t^*) = \frac{R - P_t n(\theta_{t-1}^*, x_t^*)}{P_t(R - n(\theta_{t-1}^*, x_t^*))} = 1 - \frac{1}{P_t \ln\left(\frac{P_t}{P_t - 1}\right)} \tag{11}$$

Proof.

From Lemma 2, we have $F\left(n(\theta_t^*, x_t^*) \,|\hat{s}_{t-1}\right) = \frac{n(\theta_t^*, x_t^*)}{n(\theta_{t-1}^*, x_t^*)}$. Thus, we have (9) as:

$$\Pi\left(n_t^*, n_{t-1}^*, x_t^*\right) = \frac{R}{n(\theta_{t-1}^*, x_t^*)} \int_0^{\frac{1}{P_t}} \left(\frac{1 - n(\theta_t^*, x_t^*)P_t}{1 - n(\theta_t^*, x_t^*)}\right) dn(\theta_t^*, x_t^*) - 1 = 0 \tag{12}$$

The derivation of (10) from (12) follows the same steps as in the proof of Proposition 1 and is therefore omitted. At $t = 1$, we derive the equilibrium $n(\theta_1^*, x_1^*)$ similar to Proposition 1 with x_1^* as signal threshold and with $n(\theta_0^*, x_1^*) = 1$. Now to characterize the equilibrium at $t = 2$, we first compute $n(\theta_{t-1}^*, x_t^*)$ in terms of $n(\theta_{t-1}^*, x_{t-1}^*) = \frac{x_{t-1}^* - \theta_{t-1}^* + \bar{\varepsilon}_{t-1}}{2\bar{\varepsilon}_{t-1}}$, thus we obtain:

$$n(\theta_{t-1}^*, x_t^*) = \frac{(x_t^* - x_{t-1}^*)}{2\bar{\varepsilon}_t} + \left[2n(\theta_{t-1}^*, x_{t-1}^*) - 1\right] \frac{\bar{\varepsilon}_{t-1}}{2\bar{\varepsilon}_t} + \frac{1}{2} \tag{13}$$

By plugging $n(\theta_1^*, x_1^*)$ into (13) as $n(\theta_{t-1}^*, x_{t-1}^*)$ and by using (10), x_2^* can be solved along with $n(\theta_1^*, x_2^*)$. Then, $n(\theta_1^*, x_2^*)$ is entered into (11) to obtain $n(\theta_2^*, x_2^*)$. Repeating this for $t \geq 3$, we obtain a sequence of $n(\theta_t^*, x_t^*)$ along with its associated x_t^*. Finally, since $n(\theta_t^*, x_t^*)$ and θ_t^* can be used interchangeably as bank-run threshold (from Lemma 2), any sequence $\{n(\theta_t^*, x_t^*), x_t^*\}_{t=1}^T$ characterizes a monotone equilibrium (from Lemma 1).

To show uniqueness, notice that (12) has the same form as (5) except for P_t and the intercept $\frac{R}{n(\theta_{t-1}^*, x_t^*)}$. Thus, $\Pi\left(n_t^*, n_{t-1}^*, x_t^*\right)$ also monotonically decreases in $n(\theta_t^*, x_t^*)$ at any $t \geq 1$. ■

Proposition 2 is in fact a generalization of Proposition 1 that identifies a unique equilibrium at every time t. In period 1, the result coincides with the static model that derives $n(\theta_1^*, x_1^*)$ and x_1^*, given that $P_1 = P$, $n(\theta_0^*, x_1^*) = 1$, and $x_0^* = \underline{\theta} - \varepsilon_0$ (a

case where nobody runs). The values of $n(\theta_0^*, x_1^*)$ and x_0^* show the trivial fact that there was no bank-run at $t = 0$. For any time $t \geq 2$, the threshold equilibrium is continuously depicted by $n(\theta_t^*, x_t^*)$ by using the derived values of x_t^*, $n(\theta_{t-1}^*, x_{t-1}^*)$ and $n(\theta_{t-1}^*, x_t^*)$.

This is made clearer through the help of Figure 2 which presents $g_2(n(\theta_2^*, x_2^*), n(\theta_1^*, x_2^*))$ as the function of the expected payoff for waiting at $t = 2$, given the past threshold $n(\theta_1^*, x_1^*)$ and function $g_1(n(\theta_1^*, x_1^*), n(\theta_0^*, x_1^*))$ which was introduced in Figure 1[6]. The figure also presents A' and B' (the regions bounded by dashed lines) as the agent's expected payoffs for waiting and withdrawing, respectively. To summarize the mechanics of equilibrium characterization, start at time 1. In this period, set $n(\theta_1^*, x_1^*)$ to be the unique solution to $\Pi(n_1^*, n_0^*, x_1^*) = 0$, where $n(\theta_0^*, x_1^*) = 1$. Next, at $t = 2$, use the derived $n(\theta_1^*, x_1^*)$ to solve $\Pi(n_2^*, n_1^*, x_2^*) = 0$, naming its solution as $n(\theta_2^*, x_2^*)$. Graphically, the use of $n(\theta_1^*, x_1^*)$ increases the vertical intercept of the function $g_2(n(\theta_2^*, x_2^*), n(\theta_1^*, x_2^*))$ from R to $\frac{R}{n(\theta_1^*, x_2^*)}$; and the function's intersection with P_2 derives the value for $n(\theta_2^*, x_2^*)$. By continuing the same process for every $t \geq 3$ until $t = T$, we generate the complete sequence $\{n(\theta_t^*, x_t^*), x_t^*\}_{t=1}^T$.

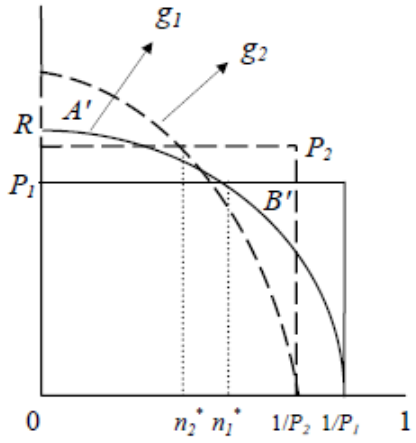

Figure 2: Unique equilibrium path in a dynamic framework

[6]Note that $g_t(n(\theta_t^*, x_t^*), n(\theta_{t-1}^*, x_t^*))$ is concave for all $n(\theta_t^*, x_t^*), n(\theta_{t-1}^*, x_t^*) \in (0, 1)$ and $t \geq 1$ since $\frac{\partial}{\partial n(\theta_t^*, x_t^*)} \left(\frac{R}{n(\theta_{t-1}^*, x_t^*)} \frac{1-n(\theta_t^*, x_t^*)P_t}{1-n(\theta_t^*, x_t^*)} \right) < 0$ and $\frac{\partial^2}{\partial n^2}(\cdot) < 0$.

3.2. Probability Bank-run

The main message of Proposition 2 is that a unique threshold equilibrium continues to exist even in a dynamic global games framework and for as long as bank-run has not happened in the past, there remains a possibility that it can occur at any time $t \geq 1$. This possibility however declines over time as is shown in our next proposition.

Proposition 3: *Given that bank-run has not occurred at any time in the past, at the threshold equilibrium* $\{n(\theta_t^*, x_t^*), x_t^*\}_{t=1}^T$,

(i) x_t^* *can increase or decrease over time,*

(ii) $n(\theta_t^*, x_t^*)$ *is decreasing over time, and*

(iii) the probability of bank-run at t*, decreases over time.*

Proof.

(i) From (10), we have $\frac{\partial n(\theta_{t-1}^*, x_t^*)}{\partial P_t} = R\left(\frac{1}{P_t} - \ln\left(\frac{P_t}{P_t - 1}\right)\right) < 0$ for any $P_t > 1$ and so, since P_t increases in t, $n(\theta_{t-1}^*, x_t^*)$ must decrease in t no matter what the value of x_t^* is. Assume that information becomes precise over time, *i.e.* $\bar{\varepsilon}_t \to 0$. Hence from (13), we see that if $n(\theta_{t-1}^*, x_{t-1}^*) \geq \frac{1}{2}$, then x_t^* must decrease as $n(\theta_{t-1}^*, x_t^*)$ decreases; whereas if $n(\theta_{t-1}^*, x_{t-1}^*) < \frac{1}{2}$, x_t^* can go up or down, depending on the level of the decrease in $n(\theta_{t-1}^*, x_t^*)$.

(ii) From (11), we have $\frac{\partial n(\theta_t^*, x_t^*)}{\partial P_t} = \left(\ln\left(\frac{P_t}{P_t - 1}\right) - \frac{1}{P_t - 1}\right) / \left(P_t^2 \ln^2\left(\frac{P_t}{P_t - 1}\right)\right) < 0$, for all $P_t > 1$, and so $n(\theta_t^*, x_t^*)$ must decrease over time as P_t increases in t.

(iii) From (i) and (ii), we have $\partial\left(\frac{n(\theta_t^*, x_t^*)}{n(\theta_{t-1}^*, x_t^*)}\right) / \partial P_t > 0$, for all $P_t > 1$. Since the probability of bank-run at t, given that it has not occurred at $t-1$ is given by $1 - \frac{n(\theta_t^*, x_t^*)}{n(\theta_{t-1}^*, x_t^*)}$ (from Lemma 2), we therefore have this probability decreasing over time. ■

The fact that a bank can fail in one period after having survived in the past only shows that the threshold $n(\theta_t^*, x_t^*)$ is decreasing over time. When $n(\theta_t^*, x_t^*) \geq n(\theta_{t-1}^*, x_{t-1}^*)$, and given that bank-run has not occurred at $t-1$ (i.e. $n(\theta) < n(\theta_{t-1}^*, x_{t-1}^*)$), then it is with probability 1 that bank-run will also not occur at t. On the other hand, if $n(\theta_t^*, x_t^*) < n(\theta_{t-1}^*, x_{t-1}^*)$, there is a positive probability for bank-run to occur at t and the threshold $n(\theta_t^*, x_t^*)$ is determined by solving $\Pi\left(n_t^*, n_{t-1}^*, x_t^*\right) = 0$. If in the end, bank-run did not occur at t, then an agent will have to update the bank-run probability at $t+1$ based on her private information at $t+1$ and the past threshold

$n(\theta_t^*, x_t^*)$. If again $n(\theta_{t+1}^*, x_{t+1}^*) \geq n(\theta_t^*, x_t^*)$, then bank-run will not happen at $t+1$; otherwise, there is again a positive probability of bank-run at $t+1$ at the threshold $n(\theta_{t+1}^*, x_{t+1}^*)$. This mechanics is made clearer in Figure 3, which shows the derivation of threshold $n(\theta_t^*, x_t^*)$ over time.

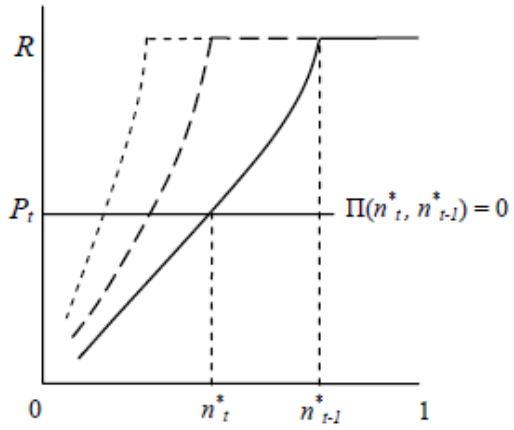

Figure 3: Decreasing threshold equilibrium over time

4. Learning with Shocks

We analyze in this section how the introduction of shocks in the learning process affects the probability of bank-run incidence. The dynamic model we discussed before shall be modified in the following manner. Suppose initially that everyone follows a monotone strategy at all $t \in \{1, 2, ...T\}$ such that any one withdraws if and only if

$$\bar{x}_t \leq \tilde{x}_t + \upsilon\sigma_t$$

where σ_t parameterizes an exogenous shock at time t, independent of $\tilde{x}_t$ and with a uniform support over $[-\bar{\varepsilon}_t, \bar{\varepsilon}_t]$; while υ represents the volatility of σ_t. As before, $\bar{x}_t$ is the agent's summary of signals received until t while $\tilde{x}_t$ is the threshold signal when there is no shock. Notice that as agents are aware of the presence of unobservable shocks, they tend to be cautious and decide to withdraw even when receiving a higher signal about the fundamentals.

Given the above strategy, the measure of agents who will withdraw in period t at a certain level of fundamentals θ is:

$$\zeta\left(\theta,\tilde{x}_t\right)=\frac{1}{2\bar{\varepsilon}_t}\int_{\theta-\bar{\varepsilon}_t}^{\tilde{x}_t+\upsilon\sigma_t}d\bar{x}_t=\frac{\tilde{x}_t+\upsilon\sigma_t-\theta+\bar{\varepsilon}_t}{2\bar{\varepsilon}_t}=n(\theta,\tilde{x}_t)+\frac{\upsilon\sigma_t}{2\bar{\varepsilon}_t}$$

Note here that when $\upsilon=0$, this setup coincides with the dynamic game without shock as $\zeta\left(\theta,\tilde{x}_t\right)=n(\theta,\tilde{x}_t)$. Now since $\zeta\left(\theta,\tilde{x}_t\right)$ is continuous and decreasing in θ, there is a θ^* that solves $\zeta\left(\theta^*,\tilde{x}_t\right)=\theta^*$ such that bank-run occurs if and only if $\theta\leq\theta^*=\frac{\tilde{x}_t+\upsilon\sigma_t+\bar{\varepsilon}_t}{1+2\bar{\varepsilon}_t}$. This condition is equivalent $\sigma_t\geq\hat{\sigma}_t^{\upsilon}\left(\cdot\right)$ where

$$\hat{\sigma}_t^{\upsilon}\left(\cdot\right)=\frac{1}{\upsilon}\left(\theta\left(1+2\bar{\varepsilon}_t\right)-\tilde{x}_t-\bar{\varepsilon}_t\right)=\frac{2\bar{\varepsilon}_t}{\upsilon}\left(\theta-\zeta(\theta,\tilde{x}_t)\right)+\sigma_t. \tag{14}$$

This shows that while σ_t summarizes the disturbances caused by any macroeconomic variable that can affect the sentiments of agents in retaining their assets, this should not cause bank-run unless the level reaches $\hat{\sigma}_t^{\upsilon}\left(\cdot\right)$. It follows then from (14) that the probability of bank-run to occur at t, conditional on θ and using threshold $\tilde{x}_t$ is

$$\rho_t^{\upsilon}\left(\zeta(\theta,\tilde{x}_t)\right)=1-F\left(\hat{\sigma}_t^{\upsilon}(\cdot)\right)=\rho_t^{\upsilon}\left(\theta;\tilde{x}_t\right) \tag{15}$$

Now at any time t, the difference between expected payoff from attacking and the payoff from withdrawing, for an agent with a summarized private information of $\bar{x}_t$, is given by

$$\Pi_t^{\upsilon}\left(\tilde{x}^t\right)=RP_t\int_0^{\frac{1}{P_t}}\left(\frac{1-\zeta(\theta,\tilde{x}_t)P_t}{1-\zeta(\theta,\tilde{x}_t)}\right)f_t^{\upsilon}\left(\zeta(\theta,\tilde{x}_t);\,\tilde{x}^{t-1}\right)d\zeta(\theta,\tilde{x}_t)-P_t \tag{16}$$

where $f_t^{\upsilon}\left(\zeta(\theta,\tilde{x}_t);\,\tilde{x}^{t-1}\right)$ is the density function of the private posterior at time t and is computed using Baye's rule, *i.e.*

$$f_t^{\upsilon}\left(\zeta(\theta,\tilde{x}_t);\,\tilde{x}^{t-1}\right)=\frac{\zeta(\theta,\tilde{x}_t)\;f_t^{\upsilon}\left(\zeta(\theta);\,\tilde{x}^{t-1}\right)}{\int_0^1\acute{\zeta}(\theta,\tilde{x}_t)\;f_t^{\upsilon}\left(\acute{\zeta}(\theta);\,\tilde{x}^{t-1}\right)d\acute{\zeta}(\theta)} \tag{17}$$

where $f_t^{\upsilon}\left(\zeta(\theta);\,\tilde{x}^{t-1}\right)$ is the density function of the common posterior about $\zeta(\theta)$, when agents in previous periods follow the strategy thresholds $\tilde{x}^{t-1}=\{\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_{t-1}\}$ and is computed also by Baye's rule.

From this setup, we show that monotone equilibrium strategy can still be achieved in $\{x_t^*\}_{t=1}^T$ although this does not necessarily be unique. We proceed in showing this by presenting first the following lemmas.

Lemma 3. *Given thresholds θ_t^* and $\tilde{x}_t$ at time t and past thresholds $\tilde{x}^{t-1}$, $F_t^v\left(\zeta(\theta_t^*, \tilde{x}_t);\, \tilde{x}^{t-1}\right) = 1 - F_t^v\left(\theta_t^* \,|(\tilde{x}_t, \sigma_t);\, \tilde{x}^{t-1}\right)$.*

Proof.

First, we show that the common posteriors on $\zeta(\theta)$ and θ are equal given the past thresholds $\tilde{x}^{t-1}$. Thus, we have

$$\begin{aligned} f_t^v\left(\zeta(\theta);\, \tilde{x}^{t-1}\right) &= \frac{\left(1-\rho_{t-1}^v\left(\zeta(\theta, \tilde{x}_{t-1})\right)\right) f_{t-1}^v\left(\zeta(\theta);\, \tilde{x}^{t-2}\right)}{\int_0^1 \left(1-\rho_{t-1}^v\left(\acute{\zeta}(\theta, \tilde{x}_{t-1})\right)\right) f_{t-1}^v\left(\acute{\zeta}(\theta);\, \tilde{x}^{t-2}\right) d\acute{\zeta}(\theta)} \\ &= \frac{\sqcap_{r=1}^{t-1}\left(1-\rho_r^v\left(\zeta(\theta, \tilde{x}_r)\right)\right) f_1^v\left(\zeta(\theta)\right)}{\int_0^1 \sqcap_{r=1}^{t-1}\left(1-\rho_r^v\left(\acute{\zeta}(\theta, \tilde{x}_r)\right)\right) f_1^v\left(\acute{\zeta}(\theta)\right) d\acute{\zeta}(\theta)} \end{aligned}$$

where the density of the initial prior $f_1^v\left(\zeta(\theta)\right) = f_1^v(\theta) = \frac{1}{2\varepsilon_1}$. From (15), we have

$$\begin{aligned} &= \frac{\sqcap_{r=1}^{t-1}\left(1-\rho_r^v\left(\theta; \tilde{x}_r\right)\right) f_1^v\left(\theta\right)}{\int_{\underline{\theta}}^{\overline{\theta}} \sqcap_{r=1}^{t-1}\left(1-\rho_r^v\left(\acute{\theta}; \tilde{x}_r\right)\right) f_1^v\left(\acute{\theta}\right) d\acute{\theta}} \\ &= f_t^v\left(\theta;\, \tilde{x}^{t-1}\right) \end{aligned}$$

Then, from (17) and recalling that $\zeta(\theta, \tilde{x}_t) = \frac{\tilde{x}_t + \upsilon\sigma_t - \theta + \bar{\varepsilon}_t}{2\bar{\varepsilon}_t}$, we obtain

$$f_t^v\left(\zeta(\theta, \tilde{x}_t);\, \tilde{x}^{t-1}\right) = \frac{\sqcap_{r=1}^{t-1}\left(1-\rho_r^v\left(\theta; \tilde{x}_r\right)\right)\left(\tilde{x}_t + \upsilon\sigma_t - \theta + \bar{\varepsilon}_t\right)}{\int_{\underline{\theta}}^{\overline{\theta}} \sqcap_{r=1}^{t-1}\left(1-\rho_r^v\left(\acute{\theta}; \tilde{x}_r\right)\right)\left(\tilde{x}_t + \upsilon\sigma_t - \acute{\theta} + \bar{\varepsilon}_t\right) d\acute{\theta}} \tag{18}$$

At threshold θ_t^*, we have the c.d.f. of $f_t^{\upsilon}(\cdot)$ as

$$\begin{aligned}
F_t^{\upsilon}\left(\zeta(\theta_t^*,\tilde{x}_t);\,\tilde{x}^{t-1}\right) &= \int_0^{\zeta(\theta_t^*,\tilde{x}_t)} f_t^{\upsilon}\left(\zeta(\theta,\tilde{x}_t);\,\tilde{x}^{t-1}\right)d\zeta(\theta,\tilde{x}_t)\\
&= \frac{\sqcap_{r=1}^{t-1}\left(1-\rho_r^{\upsilon}\left(\theta_t^*;\tilde{x}_r\right)\right)\left(\tilde{x}_t+\upsilon\sigma_t-\theta_t^*+\bar{\varepsilon}_t\right)}{\int_{\underline{\theta}}^{\overline{\theta}}\sqcap_{r=1}^{t-1}\left(1-\rho_r^{\upsilon}\left(\acute{\theta}_t^*;\tilde{x}_r\right)\right)\left(\tilde{x}_t+\upsilon\sigma_t-\acute{\theta}_t^*+\bar{\varepsilon}_t\right)d\acute{\theta}_t^*}\\
&= \frac{\left(\tilde{x}_t+\upsilon\sigma_t-\theta_t^*+\bar{\varepsilon}_t\right)f_t^{\upsilon}\left(\theta_t^*;\,\tilde{x}^{t-1}\right)}{\int_{\underline{\theta}}^{\overline{\theta}}\left(\tilde{x}_t+\upsilon\sigma_t-\acute{\theta}_t^*+\bar{\varepsilon}_t\right)f_t^{\upsilon}\left(\acute{\theta}_t^*;\,\tilde{x}^{t-1}\right)d\acute{\theta}_t^*}\\
&= \frac{\Pr\left(\theta\geq\theta_t^*\,|(\tilde{x}_t,\sigma_t)\right)f_t^{\upsilon}\left(\theta_t^*;\,\tilde{x}^{t-1}\right)}{\int_{\underline{\theta}}^{\overline{\theta}}\Pr\left(\acute{\theta}\geq\theta_t^*\,|(\tilde{x}_t,\sigma_t)\right)f_t^{\upsilon}\left(\acute{\theta}_t^*;\,\tilde{x}^{t-1}\right)d\acute{\theta}_t^*}\\
&= 1-F_t^{\upsilon}\left(\theta_t^*\left|(\tilde{x}_t,\sigma_t);\,\tilde{x}^{t-1}\right.\right)
\end{aligned}$$

■

Lemma 4. *For $t=1$, $\Pi_1^{\upsilon}(\tilde{x}_1)$ is continuous in $\tilde{x}_1$ for any $\tilde{x}_1\in\left(\underline{\theta}-\varepsilon,\overline{\theta}+\varepsilon\right)$ while for $t\geq 2$, $\Pi_t^{\upsilon}(\tilde{x}^t)$ is continuous in $\tilde{x}^t$ for any $\tilde{x}^t\in\left(\underline{\theta}-\varepsilon,\overline{\theta}+\varepsilon\right)^t$.*

Proof.

Case 1: $\upsilon=0$.

Since this case is equivalent to a dynamic case without shock, $\Pi_1^0(\tilde{x}_1)=\Pi\left(n(\theta_1^*,x_1^*)\right)$ while $\Pi_t^0(\tilde{x}^t)=\Pi\left(n(\theta_t^*,x_t^*),n(\theta_{t-1}^*,x^{*t-1})\right)$ for $t\geq 2$. Note that for all t, $n(\theta_t^*,x^{*t})=\max\left\{n(\theta):n(\theta)=\frac{x_r^*-\theta+\bar{\varepsilon}_r}{2\bar{\varepsilon}_r}\leq\theta,\text{ for all }r\leq t\right\}$ is continuous in x^{*t} and is bounded in $[0,1]$. And so, since $\Pi\left(n(\theta_1^*,x_1^*)\right)$ is continuous in $n(\theta_1^*,x_1^*)$ and $\Pi\left(n(\theta_t^*,x_t^*),n(\theta_{t-1}^*,x^{*t-1})\right)$ is continuous in $(n(\theta_t^*,x_t^*),n(\theta_{t-1}^*,x^{*t-1}))\in[0,1]^2$, it implies that for all $t\geq 1$, $\Pi_t^0(\tilde{x}^t)$ is continuous in $\tilde{x}^t$.

Case 2: $\upsilon>0$.

In this case, note first that the function $\rho_t^{\upsilon}(\theta;\tilde{x}_t)=1-F\left(\frac{1}{\upsilon}\left(\theta\left(1+2\bar{\varepsilon}_t\right)-\tilde{x}_t-\bar{\varepsilon}_t\right)\right)$ is increasing in $\tilde{x}_t$ and decreasing in θ. If $\tilde{x}_t$ increases such that $\frac{1}{\upsilon}\left(\theta\left(1+2\bar{\varepsilon}_t\right)-\tilde{x}_t-\bar{\varepsilon}_t\right)<-\bar{\varepsilon}_t$ then $\rho_t^{\upsilon}(\cdot)=1$; while if $\tilde{x}_t$ decreases such that $\frac{1}{\upsilon}\left(\theta\left(1+2\bar{\varepsilon}_t\right)-\tilde{x}_t-\bar{\varepsilon}_t\right)>\bar{\varepsilon}_t$ then $\rho_t^{\upsilon}(\cdot)=0$; thus, $\rho_t^{\upsilon}(\cdot)$ is bounded in $[0,1]$. For $t=1$, the c.d.f. of $\zeta(\theta)$ given the threshold $\tilde{x}_1$, $F_1^{\upsilon}\left(\zeta(\theta)\,|\tilde{x}_1\right)=\zeta\left(\theta,\tilde{x}_1\right)=\frac{\tilde{x}_1+\upsilon\sigma_t-\theta+\bar{\varepsilon}_1}{2\bar{\varepsilon}_1}$, is continuous in $\tilde{x}_1$ and θ. This implies that $\Pi_1^{\upsilon}(\tilde{x}_1)=RP_1\int\frac{1-\zeta(\cdot)P_1}{1-\zeta(\cdot)}dF_1^{\upsilon}(\cdot)$ is continuous in $\tilde{x}_1$.

For $t\geq 2$, observe from the proof of Lemma 3 that $F_t^{\upsilon}\left(\zeta(\theta_t^*,\tilde{x}_t);\,\tilde{x}^{t-1}\right)$ is continuous in θ, $\tilde{x}_t$ and $\tilde{x}^{t-1}$, which implies that $\Pi_t^{\upsilon}(\tilde{x}^t)$ is continuous in $\tilde{x}^t$. ■

Proposition 4. *For $\upsilon > 0$, any monotone equilibrium strategy $\{s_t\}_{t=1}^T$ is characterized by a sequence $\{x_t^*\}_{t=1}^T$ if and only if*

(i) at any $t \geq 1$, an agent withdraws $(s_{it} = 0)$ if $\bar{x}_t \leq x_t^*$ and waits $(s_{it} = 1)$ if $\bar{x}_t > x_t^*$.

(ii) for $t = 1$, $x_1^* \in \left(\underline{\theta} - \varepsilon_1, \overline{\theta} + \varepsilon_1\right)$ solves $\Pi_1^{\upsilon}(x_1^*) = 0$; while for $t \geq 2$, $x_t^* \in \left(\underline{\theta} - \bar{\varepsilon}_t, \overline{\theta} + \bar{\varepsilon}_t\right)$ solves $\Pi_t^{\upsilon}(x^{*t}) = 0$.

Multiple equilibria can exist for any $\upsilon > 0$.

Proof.

Necessity. Set $\{s_t(\cdot)\}_{t=1}^T$ as the monotone equilibrium strategy such that one withdraws if $\bar{x}_t \leq x_t^* = \tilde{x}_t + \upsilon\sigma_t$. Since the proportion of agents withdrawing at any t is decreasing in θ in such a strategy, the probability of bank-run is also decreasing in θ. Solving for the fixed point θ_t^* that solves $\theta_t^* = \zeta(\theta_t^*, x_t^*)$, we obtain $\theta_t^* = \frac{x_t^* + \bar{\varepsilon}_t}{1+2\bar{\varepsilon}_t}$. Now consider the payoffs. Since $\zeta(\theta_t^*, \tilde{x}_t) \equiv \zeta(\theta_t^*, x_t^*)$ and by applying this on (16) along with the value of θ_t^*, we solve the sequence $\{x_t^*\}_{t=1}^T$. Since the strategy is an equilibrium that gives maximal payoff at every t, an agent surely withdraws if $\bar{x}_t \leq x_t^*$ since $\Pi_t^{\upsilon}(x^{*t}) \leq 0$ and waits if $\bar{x}_t > x_t^*$ since $\Pi_t^{\upsilon}(x^{*t}) > 0$.

Sufficiency. The monotonicity of $\Pi_t^{\upsilon}(x^{*t})$ in x^{*t} follows directly from Lemma 4 such that by considering a sequence $\{x_t^*\}_{t=1}^T$ that satisfies conditions (i) and (ii) of the proposition, monotone equilibrium is obtained where $\Pi_t^{\upsilon}(x^{*t}) \leq 0$ if $\bar{x}_t \leq x_t^*$ and $\Pi_t^{\upsilon}(x^{*t}) > 0$ if $\bar{x}_t > x_t^*$.

Multiplicity. While $\Pi_t^{\upsilon}(x^{*t})$ is monotonic in x^{*t} at any t and when $\upsilon > 0$, it is not so in θ_t^* nor in $\zeta(\theta_t^*, x_t^*)$. To show this, apply the fact that $\theta_t^* = \zeta(\theta_t^*, \tilde{x}_t) \equiv \zeta(\theta_t^*, x_t^*)$ on (16) so that $\Pi_t^{\upsilon}(\tilde{x}^t)$ is depicted in terms of $\zeta(\theta_t^*, x_t^*)$. Then, since $f_t^{\upsilon}(\zeta(\theta_t^*, x_t^*); x^{*t-1}) = \frac{\sqcap_{r=1}^{t-1}(1-\rho_r^{\upsilon}(\zeta(\theta_r^*, x_r^*)))\zeta(\theta_t^*, x_t^*)}{\int_0^1 \sqcap_{r=1}^{t-1}\left(1-\rho_r^{\upsilon}(\zeta(\theta_r^*, x_r^*))\right)\acute{\zeta}(\theta_t^*, x_t^*)d\acute{\zeta}(\theta_t^*, x_t^*)}$, the equation

$$RP_t\left(\frac{1-\zeta(\theta_t^*, x_t^*)P_t}{1-\zeta(\theta_t^*, x_t^*)}\right) f_t^{\upsilon}\left(\zeta(\theta_t^*, x_t^*); x^{*t-1}\right) - P_t = 0$$

is a polynomial of degree t in $\zeta(\theta_t^*, x_t^*)$. By the fundamental theorem of algebra there is at least one solution to this equation which therefore applies also for $\Pi_t^{\upsilon}(x^{*t}) = 0$. Hence, there exists a possible multiple equilibria in this game. ■

The main difference in the dynamics of equilibrium between the basic dynamic setup (without unobservable shocks) and in that of Proposition 4 is that in the latter, the complete sequence of past thresholds x^{*t-1} is monitored at every stage t, while

in the former the entire influence of x^{*t-1} on deriving the current threshold x_t^* is captured by the fact that θ_{t-1}^* is not breached.

While there exists a monotone equilibrium strategy with respect to x_t^* in this current setting, the payoff $\Pi_t^{\upsilon}(x^{*t})$ is no longer monotonic in $\zeta(\theta_t^*, x_t^*)$ which gives way to multiple equilibria. To understand this better, Figure 4 presents an example that admits multiple equilibria when $T = 2$. As before, the solid line represents an agents payoff in period 2 that is truncated by period 1's threshold $\zeta(\theta_1^*, x_1^*) = n(\theta_1^*, x_1^*)$. Note that there is only a positive probability bank-run in period 2 whenever $\zeta(\theta_2^*, x_2^*) < \zeta(\theta_1^*, x_1^*)$ since above $\zeta(\theta_1^*, x_1^*)$ everyone remains and gets a payoff R. On the other hand, the dashed line represents the payoff $\Pi_2^{\upsilon}(x^{*2})$ of a game with noisy shocks ($\upsilon > 0$) and the two values of $\zeta(\theta_2^*, x_2^*)$ are the solutions to $\Pi_2^{\upsilon}(x^{*2}) = 0$. Notice that in the presence of noisy shocks, the truncation at $\zeta(\theta_1^*, x_1^*)$ is already lost and the posterior on $\zeta(\theta)$ has retained full support over $[0, 1]$. Thus, eventhough bank-run has survived in the past, an agent at every stage does not anymore see the threshold $\zeta(\theta_1^*, x_1^*)$ as clearly as in the game without shocks but rather draws it from a range of probability. Thus, even if an agent receives a high signal $\bar{x}_2$ which means that the probability of bank-run at time 2 is low (or that $\zeta(\theta_2^*, x_2^*) > \zeta(\theta_1^*, x_1^*)$), bank-run could still persist due to the uncertainty on the past period's threshold.

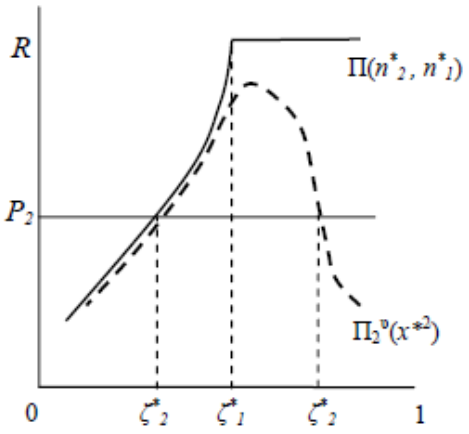

Figure 4: Multiple equilibria in a dynamic setup with unobservable shocks

From the same figure above, it is worthwhile also to note that as $\upsilon \to 0$, the "noisy" private signal threshold in period 1 converges to x_1^* and the probability of

bank-run approaches 0 for $\zeta(\theta_2^*, x_2^*) \geq \zeta(\theta_1^*, x_1^*)$. This simply means that $\Pi_2^v(x^{*2})$ converges to the payoff function $\Pi(n_2^*, n_1^*, x_2^*)$ of the original dynamic game.

5. Conclusion

This study which main objective is to set a dynamic bank-run model has established the following results. First, by applying the static global games framework on bank-runs, we have characterized a unique monotone equilibrium in terms of the threshold measure of early withdrawals. In particular, this equilibrium threshold n^* can now be defined by the level of interest rates which represents the payoff derived at the time of withdrawal. Second, the use of threshold n^* in the payoff analysis, instead of following the typical use of θ^*, is instrumental in extending the uniqueness result to dynamic game framework. Through a simple recursive mechanism, a unique equilibrium path n_t^* is generated which maps out the crisis threshold point at every time t. This also allows comparative statics to show that for as long as bank-run has not occurred in the past, its probability incidence decreases over time. Finally, we demonstrate that in the presence of some unobservable macroeconomic shocks, this equilibrium uniqueness result no longer hold since the perturbed learning process fails to certainly identify the thresholds in the previous periods.

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Appendix:

Alternative Proof of Proposition 1:

This proof obtains the result of Proposition 1 without using the bank-run threshold θ^*. Given from equation (2) that $n(\theta, x^*) = \frac{x^*-\theta+\varepsilon}{2\varepsilon}$, the expected utility from waiting, $R\left(\frac{1-Pn(\theta,x^*)}{1-n(\theta,x^*)}\right)$, can be rewritten as $R\left(\frac{2\varepsilon-P(x^*-\theta+\varepsilon)}{2\varepsilon-(x^*-\theta+\varepsilon)}\right)$. Note that as $R(\cdot) = 0$ whenever $\theta = x^*-\varepsilon\left(\frac{2}{P}-1\right)$, an equilibrium is characterized by any $\theta \in \left[x^*+\varepsilon, x^*-\varepsilon\left(\frac{2}{P}-1\right)\right]$. Thus, the threshold equilibrium payoff is computed as follows

$$\frac{P}{2\varepsilon}\int_{x^*-\varepsilon\left(\frac{2}{P}-1\right)}^{x^*+\varepsilon} R\frac{2\varepsilon - P(x^*-\theta+\varepsilon)}{2\varepsilon-(x^*-\theta+\varepsilon)}d\theta - P = 0$$

$$\int_{x^*-\varepsilon\left(\frac{2}{P}-1\right)}^{x^*+\varepsilon} R\frac{2\varepsilon - P(\varepsilon+x^*)+P\theta}{\varepsilon-x^*+\theta}d\theta = 2\varepsilon$$

Let $a = 2\varepsilon - P(\varepsilon + x^*)$ and $b = \varepsilon - x^*$ such that we have

$$R\int_{x^*-\varepsilon\left(\frac{2}{P}-1\right)}^{x^*+\varepsilon} \frac{a+P\theta}{b+\theta}d\theta = 2\varepsilon$$

By integrating by parts, we obtain

$$(a+\theta)\ln(b+\theta) - \int \ln(b+\theta)d\theta \;\Big|_{\theta=x^*-\varepsilon\left(\frac{2}{P}-1\right)}^{\theta=x^*+\varepsilon} = \frac{2\varepsilon}{R}$$

$$(a-Pb)\ln(b+\theta) + P\theta \;\Big|_{\theta=x^*-\varepsilon\left(\frac{2}{P}-1\right)}^{\theta=x^*+\varepsilon} = \frac{2\varepsilon}{R}$$

Substituting back the values of a and b and by some algebra,

$$(1-P)\ln\left(\frac{P}{P-1}\right) + 1 = \frac{1}{R}$$

By substituting R on the equation $R\frac{1-n(\cdot)P}{1-n(\cdot)} = P$, we have

$$n(\theta, x^*) = 1 - \frac{1}{P\ln\left(\frac{P}{P-1}\right)}$$

■