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Fiscal Sustainability, Macroeconomic Stability, and Welfare under Fiscal Discipline in a Small Open Economy*

Keiichi Morimoto† Takeo Hori‡ Noritaka Maebayashi§
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Abstract

We construct a small open economy model of endogenous growth with public capital accumulation and examine how a debt policy rule under which the government gradually reduces its debt-GDP ratio to a target level affects macroeconomic stability, fiscal sustainability, and welfare. We obtain the following implications for fiscal policy design in small countries. First, to ensure fiscal sustainability, the government should adjust public spending rather than the income tax rate to finance public debt. In addition, it has to set the target level of the debt-GDP ratio sufficiently low to avoid expectations-driven economic instabilities. Under sustainability and stability, a tighter (looser) debt rule brings welfare gains when the world interest rate is relatively high (low).

JEL classification: E32; E62; H63
Keywords: Fiscal policy, Public debt, Welfare, Fiscal sustainability, Equiribrium indeterminacy

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1 Introduction

The discretionary fiscal policies implemented during the 2008-2009 world crisis resulted in serious increases in government debt in the Euro area. In 2011, the average debt-GDP ratio in the Euro area reached 88 percent of GDP, some 20 percentage points higher than at the start of the crisis in 2007. Public debt as a share of GDP in Greece equaled 166.1 percent in 2012. Debt-GDP ratios in Italy, Ireland, and Portugal also exceeded 100 percent. This weak fiscal condition raised doubts about these countries' ability to finance their increased debt. As a response to the crisis, the EU has introduced strong fiscal consolidations under the surveillance of the European Commission. Overall public deficits were reduced thanks to this act. In the Euro area, the average general government deficit fell from 6.2 percent of GDP in 2010 to 4.1 percent of GDP in 2011. However, this may not be enough. The Maastrichit Treaty asks the EU countries to keep their deficit and debt levels below 3 and 60 percent, respectively, to ensure compliance with budgetary discipline. As for the corrective arm, a debt-reduction benchmark has been established against the insufficiently diminishing debt-GDP ratio. Under an assessment of the relevant factors, the European Commission can now launch the Excessive Deficit Procedure for Member States whose debt exceeds 60 percent of GDP and does not comply with the debt benchmark even if they show a deficit below 3 percent of GDP.[1] According to plans set out in the Stability and Convergence Programmes, which were submitted to the Commission and Council in Spring 2012, Member States project to base further fiscal consolidation on expenditure cuts including reductions in public investment.

This paper studies the debt policy rule under which government debt must be reduced if its level is beyond a criterion level as in the Maastrichit Treaty and examines the effects of the rule on macroeconomic stability, fiscal sustainability, and welfare. In an endogenous growth model whose growth engine is the flow of public service as in Barro (1990), Futagami et al. (2008) investigate the effects of a government bond-issuance rule under which the government gradually reduces its debt to a target level if its debt is beyond the criterion level, whereas Maebayashi et al. (2013) uses an endogenous growth model whose engine of growth is public capital accumulation to study the same issue. These authors provide interesting results. However, their investigations are confined to closed economies and accordingly transactions in foreign capital markets are removed. In reality, both the government and private sector can borrow and lend their assets in the foreign capital market. Countries holding large levels of debt such as Greece, Italy, Ireland, and Portugal hold large external debt as well. It is important to study the debt policy rule described above in a model of an open economy.

For our purpose, we extend the model of Maebayashi et al. (2013) into a small open economy model. As in Futagami et al. (1993) and Turnovsky (1996), public capital accumulated through public investment has positive effects on private goods production. The government finances its spending on public investment by imposing a tax on income and by issuing bonds. Public bond-issuance is under the restriction of the same debt policy rule as that in Futagami et al. (2008) and Maebayashi et al. (2013). Public debt must be decreased if the debt-GDP ratio exceeds the target level. We consider two types of public finance budget regimes. In budgetary regime (I), the government adjusts its expenditure with the tax rate fixed to rule the debt-GDP ratio in the long run: if the

[1] It sets out schedules and deadlines for the Council, following reports from and on the basis of opinions by the Commission and the Economic and Financial Committee, to reach a decision that an excessive deficit exists in a Member State.

debt-GDP ratio exceeds the target level, it adjusts public investment so as to decrease its debt. In budgetary regime (II), the government controls the tax rate with its expenditure ratio fixed to rule the debt-GDP ratio in the long run: if the debt-GDP ratio exceeds the target level, it adjusts the tax rate so as to decrease its debt. We show that the difference between the two budgetary regimes generates different effects on economic stability and fiscal sustainability.

In budgetary regime (I), there exists a unique steady-state equilibrium. The target (long-run) debt-GDP ratio is a crucial determinant of steady-state stability and equilibrium (in)determinacy. When the target (long-run) debt-GDP ratio is sufficiently low, the steady state is saddle stable and hence exhibits equilibrium determinacy. The economy does not fluctuate endogenously. However, if the government sets a high target debt-GDP ratio, equilibrium indeterminacy arises because the steady state is a sink or there exists a limit cycle around the steady state. The economy may exhibit expectations-driven endogenous business cycles. Then, to stabilize the economy under budgetary regime (I), the government in a small open economy should keep its debt-GDP ratio sufficiently low in the long-run. It should be noted that as Maebayashi et al. (2013) show, in a closed economy version of the model, equilibrium indeterminacy does not arise under budgetary regime (I). These endogenous fluctuations in our model are derived from the feature of a small open economy such that a small open economy can import resources from abroad without affecting the world resource constraint.[2]

Assuming that the target debt-GDP ratio is low enough to ensure equilibrium determinacy, we examine the welfare effects of fiscal consolidation (debt reduction) under budgetary regime (I). The welfare effects of debt reduction depend on the rates of returns from private savings and public investment. In our small open economy, the rate of return from private savings is equal to the world interest rate. When the world interest rate is high (low) relative to the rate of return from public investment, reductions in government debt improve (deteriorate) welfare. Furthermore, we show that the pace of debt reduction is an important determinant of the magnitude of welfare gains (losses). When the world interest rate is higher than the rate of return from public savings, the government can further improve welfare by reducing the debt at a faster pace. In contrast, if the world interest rate is relatively low, the government can mitigate welfare losses by reducing the debt at a slower pace.

Further, in budgetary regime (II), there exists a unique steady-state equilibrium. However, the steady state is always unstable under budgetary regime (II). Because the economy cannot reach the unstable steady state, it makes little sense to study the characteristics of the unstable steady state and hence we do not examine the welfare effects of debt reduction in budgetary regime (II). Nevertheless, our model provides the following important implication. Suppose that the initial asset holdings of domestic households are smaller than the initial outstanding debt of the government. This implies that private domestic savings cannot afford to absorb outstanding government debt and the government borrows from foreign investors. In such a situation, under regime (II), there exists no equilibrium such that the government can keep its debt-GDP ratio at a finite level in the long run. Hence, the government cannot sustain its debt. It should be emphasized that as Maebayashi et al. (2013) show, in a closed economy version of the model, the economy converges to a unique steady state where the long-run debt-GDP ratio is finite even under regime (II). This result in our model is derived from the feature of a small open economy that households in

[2]We provide a detailed discussion on this in Section 4.1.

a small open economy can borrow from abroad without affecting the world asset market.[3]

To sum up, the results of this paper provide the following implications for fiscal policy design with budgetary discipline in small countries.

(i) For fiscal sustainability, the government should adjust public spending rather than the income tax rate to finance public debt.

(ii) In addition, it has to set the target level of debt-GDP ratio sufficiently low to avoid expectations-driven economic instabilities.

(iii) Under sustainability and stability, a tighter (looser) debt rule brings welfare gains when the world interest rate is relatively high (low).[4]

There exist researches on public debt finance in endogenous growth models where government service through public investment is an input for private goods production (see, for example, Bruce and Turnovsky (1999), Greiner and Semmler (2000), Ghosh and Mourmouras (2004), Greiner (2007, 2012), and Yakita (2008)). These studies do not investigate the rule that targets government debt in the long run as in the Maastricht Treaty. Furthermore, they do not consider public debt finance in open economies. Our study contributes to these studies by examining the effects of the debt policy rule on economic growth, equilibrium (in)determinacy, fiscal sustainability, and welfare in a small open economy.

The existing studies on fiscal sustainability are mostly based on closed economy models. It is well known that permanent budget deficits cannot be sustainable unless the interest rate is less than the growth rate in closed economies (see Diamond (1965) and Chalk (2000)). Bräuninger (2005) and Yakita (2008) examine the sustainabilities of public debt assuming that the government adjusts the income tax rate and keeps the government spending-GDP ratio and the budget deficit-GDP ratio constant. They show that public debt is not sustainable when the initial debt is too large. However, these papers do not explore whether different budgetary regimes generate differences in macroeconomic consequences. In our model, a choice between the two regimes significantly affects fiscal sustainability. Under budgetary regime (I) where public investment is endogenously adjusted, public debt can be sustained. However, under regime (II) where the tax rate is endogenously adjusted, growing indebtedness from abroad can be a source of government failure in a small open economy.

In the literature studying relations between fiscal policies and equilibrium (in)determinacy or endogenous business cycles, Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004, 2008) discuss the occurrence of equilibrium indeterminacy in closed economies, focusing on income taxation. These authors show that conditions for (in)determinacy depend on whether the tax rate or government expenditure is endogenized and whether government expenditure is wasteful or productive. These studies mainly consider balanced budget rules because they bear in mind a traditional argument that balanced budget rules amplify business cycles.[5] Maebayashi et al. (2013) show that in a closed economy model with productive government expenditure where public debt is allowed but constrained by the same debt policy rule as ours, equilibrium indeterminacy never arises, irrespective of whether

[3]We provide a detailed discussion on this in Section 5.

[4]Here, a tight (loose) debt rule means a rapid (slow) adjustment in public debt given a low (high) target level of public debt.

[5]It is thought that under a balanced budget, aggregate demand is stimulated during booms through tax cuts while aggregate demand is reduced during recessions by a tax increase. In this manner, a balanced budget rule may amplify business cycles.

the tax rate or government expenditure is endogenized. Extending a model of Maebayashi et al. (2013) into a small open economy, we show that even when debt finance is allowed for the government, equilibrium indeterminacy arises if public expenditure is endogenously adjusted and the long-run debt-GDP ratio is sufficiently high.[6] In the literature of limit cycles, Farmer (1986) and Greiner (2007) show that in closed economies, limit cycles emerge when the government controls the value of its *deficit* rather than the value of its *debt*.[7] In contrast, we show that the target rule of debt in a small open economy will be a cause of limit cycles.

Focusing on production externalities and the curvature of the utility function, some authors show that in small open economies, indeterminacy occurs under less restrictive conditions than in closed economies.[8] Most of these studies do not consider fiscal policies. We show that the fiscal policy under which equilibrium is always determinate in a closed economy will be a source of indeterminacy in a small open economy. Our study complements the literature on indeterminacy and open economies.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 derives the dynamic systems of the economy. Section 4 investigates the stability effects of public finance under budgetary regime (I) and its effects on welfare. Section 5 examines budgetary regime (II). Section 6 concludes.

2 Model

We consider a small open economy. A single final good is produced using labor, private capital, and public capital (infrastructure). It is assumed that the final good and private capital are freely traded beyond the country's borders. However, individuals cannot migrate, and public capital (social infrastructure) too cannot cross borders. To construct social infrastructure, the government must make public investments.

2.1 Households

We consider a representative household. The size of population is normalized to one. Let C_t be consumption at time t. The utility of the representative household is given by

$$U_0 = \int_0^{+\infty} [\ln C_t] e^{-\rho t} dt, \tag{1}$$

where $\rho > 0$ is the subjective discount rate. The household inelastically supplies one unit of labor at each moment of time. The government imposes a tax on the household's income, $I_t \equiv rA_t + w_t$, where r is the (constant and exogenously given) world interest rate, w_t is the wage rate at time t, and A_t is the asset holdings of the household at time t. We assume that the tax takes the residence base form. According to residence-based tax, residences' income is taxed at a uniform rate regardless of its source country while non-residences'

[6] Recently, using a two-sector, two-country model with production externalities, Hu and Mino (2013) show that trade structures affect conditions for equilibrium indeterminacy although they do not consider fiscal policies.

[7] Starting with Benhabib and Nishimura (1979), there exist many works on limit cycles. As more recent examples, see, Benhabib and Rustichini (1990), Dockner and Feichtinger (1993), Greiner and Hanusch (1994), Greiner and Semmler (1996) and Wirl (2002).

[8] See, for example, Meng (2003), Meng and Velasco (2003, 2004), and Weder (2001).

income is not taxed. Thus, the flow budget constraint of the representative household is:

$$\dot{A}_t = (1-\tau_t)I_t - C_t, \tag{2}$$

where τ_t denotes the income tax rate at time t. In contrast to Futagami et al. (2008) who assume a constant tax rate, we allow τ_t to vary over time, as we discuss later. Because we assume an open economy, the household can borrow from foreign countries and hence $A(t)$ can be negative. However, A_t must satisfy the no-Ponzi game (NPG) condition, $\lim_{T\to\infty} A_T e^{-\int_t^T (1-\tau_v)rdv} \geq 0$. The household maximizes (1) subject to (2), which yields

$$\dot{C}_t = \{(1-\tau_t)r - \rho\}C_t, \tag{3}$$

and the transversality condition (TVC), $\lim_{T\to\infty} A_T e^{-\int_t^T (1-\tau_v)rdv} = 0$.

2.2 Firms

As in Futagami, Morita and Shibata (1993), The production function of the representative firm is given by $Y_t = F(K_t, K_{g,t}L_t)$, where Y_t, K_t, $K_{g,t}$, and L_t are output, private capital, stock of infrastructure, and labor input at time t, respectively.[9] The presence of $K_{g,t}$ in the production function reflects the external effects. As we will see in the next subsection, stock of infrastructure, $K_{g,t}$, accumulates through government investments. The production function satisfies the standard neoclassical characteristics, especially, the constant returns to scale with respect to K_t and $K_{g,t}L_t$. Accordingly, we can transform this into the following intensive form:

$$Y_t = F(x_t, 1)K_{g,t}L_t = f(x_t)K_{g,t}L_t, \tag{4}$$

where $x_t \equiv K_t/(K_{g,t}L_t)$ and $f(x_t) \equiv F(x_t, 1)$. Given perfect competition and profit maximization, we obtain

$$r = f'(x_t), \tag{5a}$$
$$w_t = \{f(x_t) - f'(x_t)x_t\}K_{g,t} \equiv \omega_t K_{g,t}. \tag{5b}$$

The world interest rate, r, is constant because of the assumption of a small country. Thus, x_t and ω_t become constant over time. We omit time index t from x_t and ω_t in the following discussion. In equilibrium, we have $L_t = 1$. Then, Y_t grows at the same rate as $K_{g,t}$ (see (4)). This is because infrastructure constructions by the government stimulate private investments and attract foreign capital. For later use, we define $K_{g,t}/Y_t = 1/f(x) \equiv k_g$.

2.3 Government

To construct infrastructure, the government makes public investments. The amount of public investments in time t is denoted by G_t. Then, a stock of infrastructure accumulates:[10]

$$\dot{K}_{g,t} = G_t. \tag{6}$$

[9] In contrast to us, Futagami et al. (2008) assume that the flow of public service enters the production function as an input.

[10] As in Futagami, Morita and Shibata (1993), we ignore the depreciation of public capital to simplify the analysis.

For later use, we define $g_t \equiv G_t/Y_t$. The government finances its expenditure by two means. One is by levying an income tax, and the other is by issuing bonds. Thus, the government's budget constraint is

$$rB_t + G_t = \dot{B}_t + \tau_t(rA_t + w_t), \tag{7}$$

where B_t stands for outstanding government debt and $\dot{B}_t$ denotes newly-issued government bonds. The government must satisfy the NPG condition, $\lim_{t\to\infty} B_t e^{-rt} \leq 0$.

We assume that the government has a target level for its outstanding debt. The government attempts to maintain its debt-GDP ratio at a constant level in the long run. Following Futagami et al. (2008), we assume that the government adjusts its debt-GDP ratio, $b_t \equiv B_t/Y_t$, gradually so that it equals the target level of the debt-GDP ratio in the long run, as follows:

$$\dot{b_t} = -\phi(b_t - \bar{b}), \quad \phi > 0 \tag{8}$$

where $\bar{b} > 0$ and $\phi > 0$ represent the target level of the government's debt-GDP ratio and the adjustment coefficient of the rule, respectively. If $b_t > \bar{b}$ holds, the government reduces its debt according to the difference between the current and target levels of b: $b_t - \bar{b}$. If the adjustment coefficient, ϕ, takes a large (small) value, the government adjusts b_t to the target level at a fast (slow) pace. Futagami et al. (2008) and Maebayashi et al. (2013) gauge the size of the economy by the level of private capital, K_t, and hence they define $b_t \equiv B_t/K_t$. Our approach is more direct and realistic than theirs and hence the interpretation of (8) is easier. However, the following point deserves to be mentioned. From (4) and the definition of x, we have $Y_t = (f(x)/x)K_t$, and hence $b_t = B_t/Y_t = (x/f(x))(B_t/K_t)$ holds. Because $x/f(x)$ is constant in our small open economy model, the last equation shows that whether we define $b_t = B_t/Y_t$ or $b_t = B_t/K_t$ does not affect our main results. Because of this reality, we assume that $b_t = B_t/Y_t$.

Given this adjustment rule, the government chooses either τ_t or g_t so as to satisfy the budget constraint, (7). The present study considers two types of budgetary regimes: (I) the government sets a constant τ and then adjusts g_t so that it satisfies (7) and (8), or (II) the government chooses a constant g and then τ_t is endogenously determined so as to satisfy (7) and (8). It should be noted that Futagami et al. (2008) consider only the first case where τ is constant and g_t varies so as to satisfy (7) and (8).

3 Dynamic Systems

We derive the equilibrium paths of the economy. In order to derive the dynamic system, we define the following variables: $c_t \equiv C_t/Y_t$, $a_t \equiv A_t/Y_t$, and $\gamma_t \equiv \dot{Y}_t/Y_t$. Note that Y_t is given by (4), x remains constant, $K_{g,t}$ is a state variable, and $L_t = 1$ holds in equilibrium. We have to treat Y_t as a predetermined variable. Then, a_t and b_t are state variables and c_t is a jump variable. From (4) and (6), we have $g_t = k_g\gamma_t$. From (2), (3), (5b), (7), and the definitions of I_t and γ_t, we obtain

$$\dot{b}_t = (r - \gamma_t)b_t + g_t - \tau_t(ra_t + \omega k_g), \tag{9a}$$

$$\dot{a}_t = \{(1-\tau_t)r - \gamma_t\}a_t - c_t + (1-\tau_t)\omega k_g, \tag{9b}$$

$$\dot{c}_t = \{(1-\tau_t)r - \rho - \gamma_t\}c_t. \tag{9c}$$

We first consider budgetary regime (I) where the government sets a constant $\tau \in (0,1)$ and adjusts g_t so as to satisfy (8) and (9a). Eliminating $\dot{b}_t$ from (8) and (9a) and solving

for γ_t using $g_t = k_g\gamma_t$, we obtain[11]

$$\gamma_t \left(= \frac{g_t}{k_g}\right) = \frac{\tau(ra_t + \omega k_g) - rb_t - \phi(b_t - \bar{b})}{k_g - b_t} \equiv \gamma(a_t, b_t). \tag{10}$$

The above equation shows that when the government decreases $\bar{b}$, γ_t becomes smaller (larger) in the short run if $b_t < (>)k_g$. To understand this result, we differentiate $b_t \equiv B_t/Y_t$ with respect to t:

$$\dot{b}_t = \frac{\dot{B}_t}{Y_t} - b_t\frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left(\dot{B}_t - \frac{b_t}{k_g}G_t\right). \tag{11}$$

In the second equality, we use $\gamma_t = g_t/k_g = G_t/(k_gY_t)$. The above equation shows that the government can reduce its debt-GDP ratio by two means. One is to reduce its debt, B_t, and the other is to enhance the output growth through public investments. In addition, as b_t is smaller (larger) so as to satisfy $b_t < (>)k_g$, increases in G_t have smaller (larger) reducing effects on b_t.[12] Then, when $b_t > k_g$ holds, the government increases public investments, $g_t(= k_g\gamma_t)$, in order to stimulate output growth and then reduce b_t. However, when b_t is small relative to k_g, the government must cut public investments so as to decrease B_t. The following two points should be mentioned as well: these effects of reductions in $\bar{b}$ on public investments, $g_t(= k_g\gamma_t)$, become stronger as ϕ is larger because the government must decrease b_t at a faster pace, and then omit it entirely as b_t comes close to the target level, $\bar{b}$.

Substituting (10) into (9b) and (9c) yields

$$\dot{a}_t = \{(1-\tau)r - \gamma(a_t, b_t)\}a_t - c_t + (1-\tau)\omega k_g, \tag{12a}$$
$$\dot{c}_t = \{(1-\tau)r - \rho - \gamma(a_t, b_t)\}c_t. \tag{12b}$$

The dynamic system is then given by (8), (12a), and (12b). We know from (10) that γ_t can be negative, which means that the government uninstalls some of the social infrastructure. To focus on positive-growth cases, we restrict our attention to equilibria where growth rates are positive in the steady state.[13]

We now focus on budgetary regime (II) where the government sets a constant $\gamma(= g/k_g)$ and adjusts τ_t so as to satisfy (8) and (9a). Eliminating $\dot{b}_t$ from (8) and (9a) and solving for τ_t, we obtain

$$\tau_t = \frac{(r-\gamma)b_t + k_g\gamma + \phi(b_t - \bar{b})}{ra_t + \omega k_g} \equiv \tau(a_t, b_t). \tag{13}$$

When the government reduces $\bar{b}$, τ_t becomes larger in the short run because the government must set a high tax rate and earn a larger primary surplus to reduce b_t. As in budgetary regime (I), the effects of reductions in $\bar{b}$ on τ_t become stronger as ϕ increases, and gradually disappear as b_t becomes close to $\bar{b}$. The following two points also deserve to be mentioned. First, when $a_t = -\omega k_g/r$, τ_t cannot be defined and hence the optimization problem of the

[11]Because $g_t = k_g\gamma_t$, the choice of g_t is equivalent to that of γ_t. When we discuss regime (I) in the following discussion, we mainly focus on γ_t, rather than g_t.

[12]For example, suppose that $B_t = 100$ and $Y_t = 1$. Then, we have $b_t = 100$. If Y_t increases by one from $Y_t = 1$ to $Y_t = 2$, b_t reduces to 50 by 50%. Next suppose that $B_t = 100$ and $Y_t = 100$. Then, we have $b_t = 1$ in this case, and a one-unit increase in Y_t has only a relatively small effect on b_t.

[13]The steady state is defined in Subsection 4.1.

household is not well-defined. Then, the transitional paths along which a_t moves across $-\omega k_g/r$ must not be equilibrium. Second, when a_t is so small that household income $(ra_t + \omega k_g)Y_t$ is negative, τ_t tends to be negative, and hence household tax payments $\tau_t(ra_t + \omega k_g)Y_t$ tend to be positive even though household income is negative. Then, the representative household must borrow from abroad to make tax payments beyond its income. In another respect, the government forces the household to borrow from abroad in order to meet the budget constraint of the government. Although the case makes little sense in practice, it is theoretically possible and provides important implications as we will see in Section 5. Substituting (13) into (9b) and (9c) yields

$$\dot{a}_t = (r-\gamma)(a_t - b_t) - \phi(b_t - \bar{b}) - c_t + (\omega - \gamma)k_g, \tag{14a}$$

$$\dot{c}_t = \{(1-\tau(a_t, b_t))r - \rho - \gamma\}c_t. \tag{14b}$$

The dynamic system is given by (8), (14a), and (14b).

4 Adjustments in Public Investments: Regime (I)

We first consider the economy under budgetary regime (I) where the government sets a constant $\tau \in (0,1)$ and adjusts public investments so as to satisfy (7) and (8). The steady-state equilibrium is derived and then its stability is examined. The end of this section examines the welfare effects of reductions in b_t.

4.1 Steady-State Equilibrium

We first derive the steady-state equilibrium where $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ holds. The dynamic system consists of (8), (12a), and (12b). We set $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ in (8), (12a), and (12b), and solve for b_t, a_t, c_t, and γ_t using (10). The steady-state values of b_t, a_t, c_t, and γ_t are, respectively, given by $b_{\rm I}^* = \bar{b}(>0)$,

$$a_{\rm I}^* = \frac{(\gamma_{\rm I}^* - \tau\omega)k_g + (\rho + \tau r)\bar{b}}{\tau r}, \tag{15a}$$

$$c_{\rm I}^* = \frac{\gamma_{\rm I}^*(\rho + \tau\omega)k_g + \rho(\rho + \tau r)\bar{b}}{\tau r} > 0, \tag{15b}$$

$$\gamma_{\rm I}^* = (1-\tau)r - \rho. \tag{15c}$$

To ensure positive growth, we assume that $\gamma_{\rm I}^* = (1-\tau)r - \rho > 0$. From (15a), household income at the steady state is given by $(ra_{\rm I}^* + \omega k_g)Y_t = \frac{(\tau r+\rho)\bar{b}+\gamma_{\rm I}^* k_g}{\tau} Y_t$ that is apparently positive. Because $\tau \in (0,1)$ and $\rho > 0$, we have $\gamma_{\rm I}^* < (1-\tau)r$, which ensures TVC because $\dot{A}_t/A_t = \gamma_{\rm I}^*$ holds at the steady state. The NPG condition of the government is also satisfied because we have $\dot{B}_t/B_t = \gamma_t < (1-\tau)r < r$ in the steady state. Apparently, the policy parameters, $\bar{b}$ and ϕ, have no effects on the long-run growth rate, γ_I^*. We obtain the following proposition.

Proposition 1 *Consider budgetary regime (I). Suppose that* $\gamma_{\rm I}^* = (1-\tau)r - \rho > 0$. *There exists a unique steady-state equilibrium where* $b_{\rm I}^* = \bar{b}$ *and* (15a)–(15c) *hold.* $\bar{b}$ *and* ϕ *have no effects on the long-run growth rate,* γ_I^*.

We now examine the stability of the steady state characterized by $b_{\rm I}^* = \bar{b}$ and (15a)–(15c). We observe that the long-run debt-GDP ratio, $\bar{b}$, plays crucial roles in stability and

equilibrium indeterminacy, although it does not affect the steady-state values of the endogenous variables. We linearize the dynamic system around the steady state characterized by $b_{\rm I}^* = \bar{b}$ and (15a)–(15c), and then obtain

$$\begin{pmatrix} \dot{b}_t \\ \dot{a}_t \\ \dot{c}_t \end{pmatrix} = \begin{pmatrix} -\phi & 0 & 0 \\ -\frac{\partial\gamma(a_I^*,\bar{b})}{\partial b_t}a_{\rm I}^* & \rho - \frac{\partial\gamma(a_I^*,\bar{b})}{\partial a_t}a_{\rm I}^* & -1 \\ -\frac{\partial\gamma(a_I^*,\bar{b})}{\partial b_t}c_{\rm I}^* & -\frac{\partial\gamma(a_I^*,\bar{b})}{\partial a_t}c_{\rm I}^* & 0 \end{pmatrix} \begin{pmatrix} b_t - \bar{b} \\ a_t - a_{\rm I}^* \\ c_t - c_{\rm I}^* \end{pmatrix}, \tag{16}$$

where

$$\frac{\partial\gamma(a_I^*,\bar{b})}{\partial b_t} \equiv \left.\frac{\partial\gamma(a_t,b_t)}{\partial b_t}\right|_{(a_t,b_t)=(a_{\rm I}^*,\bar{b})} = -\frac{\phi+\rho+\tau r}{k_g-\bar{b}}, \tag{17}$$

$$\frac{\partial\gamma(a_I^*,\bar{b})}{\partial a_t} \equiv \left.\frac{\partial\gamma(a_t,b_t)}{\partial a_t}\right|_{(a_t,b_t)=(a_{\rm I}^*,\bar{b})} = \frac{\tau r}{k_g-\bar{b}}. \tag{18}$$

We denote the Jacobian matrix of (16) by M. Let us denote the (i,j)-element of M by m_{ij} $(i,j=1,2,3)$. One of the eigenvalues of M is equal to $-\phi<0$. The other two, λ_1 and λ_2, are the solutions of the characteristic equation $\Omega(z) \equiv z^2 - m_{22}z + m_{32} = 0$, and thus satisfy

$$\lambda_1+\lambda_2 = m_{22} = \rho - \frac{\partial\gamma(a_I^*,\bar{b})}{\partial a_t}a_{\rm I}^* \quad \text{and} \quad \lambda_1\cdot\lambda_2 = m_{32} = -\frac{\partial\gamma(a_I^*,\bar{b})}{\partial a_t}c_{\rm I}^*. \tag{19}$$

Checking the signs of m_{22} and m_{32}, we can know about the stability of the steady-state equilibrium and can prove the next proposition.

Proposition 2 *Consider budgetary regime (I). Suppose that $\gamma_{\rm I}^* = (1-\tau)r - \rho > 0$. Then, we have the following.*

1. *If $\bar{b} < k_g$, the steady state is locally saddle stable and exhibits local determinacy.*
2. *If $\bar{b} > k_g$, the following hold:*
   - *(a) Let $\tilde{b} \equiv \frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r}k_g$. When $\tau > r/\omega$,*
     - *i. if $k_g < \bar{b} < \tilde{b}$, the steady state is locally stable and exhibits local indeterminacy.*
     - *ii. if $\bar{b} > \tilde{b}$, the steady state is locally unstable.*[14]
   - *(b) When $\tau < r/\omega$, the steady state is locally unstable.*

(Proof) See Appendix A.

Proposition 2 shows that when $\bar{b}$ is large ($\bar{b} > k_g$), the steady state becomes unstable under some conditions. However, this does not mean that there is no equilibrium. In fact, Proposition 3 shows the possibility of an equilibrium that exhibits cyclical fluctuations.

Proposition 3 *Consider budgetary regime (I). Suppose that $\gamma_{\rm I}^* = (1-\tau)r - \rho > 0$ and $\bar{b} > k_g$. In addition, suppose that $r/\omega < 1$ and $\hat{b}_- < \tilde{b} < \hat{b}_+$ and that $\bar{b}$ is sufficiently close to $\tilde{b}$, where $\hat{b}_-$ and $\hat{b}_+$ are as defined in Appendix B. Then, there exists at least a limit cycle around the steady state, and the steady state exhibits equilibrium indeterminacy and the possibility of cyclical fluctuations.*

[14] Note that $\tilde{b}$ is larger than k_g if $\tau > r/\omega$ holds. See Appendix A.

(Proof) See Appendix B.

The properties of the equilibrium path heavily depend on $\bar{b}$. When $\bar{b}$ is small enough to satisfy $\bar{b} < k_g$, the equilibrium path is uniquely determined. Therefore, the economy does not fluctuate endogenously. However, when $\bar{b}$ takes a moderate value ($k_g < \bar{b} < \tilde{b}$) and $\tau > r/\omega$ holds, the steady state is locally stable and exhibits indeterminacy. There are multiple equilibrium paths that converge to the steady state. Depending on households' expectations, any of these equilibrium paths can be realized and then the economy can exhibit endogenous business cycles. In other cases, the steady state is unstable. However, as proved in Proposition 3, there exists a limit cycle around the steady state. In Figures 1 and 2, we illustrate a numerical example of a stable limit cycle.[15] The economy converges to a limit cycle and then cyclically fluctuates around the steady state. There are multiple equilibrium paths that converge to the limit cycle. Depending on households' expectations, any of these equilibrium paths can be realized and then the business cycles endogenously arise. These results provide the following implications. Consider a small open economy where the current debt-GDP ratio is high. If the government leaves its debt-GDP ratio high, the economy is destabilized. However, if it attempts to decrease its debt-GDP ratio to a low level at a steady pace by adjusting its spending, the government can stabilize the economy. Conversely, even when the current debt-GDP ratio is low, the economy is destabilized if the government steadily increases its debt-GDP ratio by adjusting its spending. To stabilize the economy under budgetary regime (I), the government should choose a sufficiently small $\bar{b}$ so as to satisfy $\bar{b} < k_g$.

[Figures 1 and 2]

To interpret the intuition behind equilibrium indeterminacy, we differentiate (10) with respect to a_t and then evaluate the derivative in the steady state, which yields (18). We have $\partial\gamma_t/\partial a_t < (>)0$ if and only if $\bar{b} > (<)k_g$. Decreases in a_t reduce the tax revenue of the government, which tightens the budget constraint of the government. To reduce $b_t \equiv B_t/Y_t$ according to (8), the government—that is constrained with tight budgets—has two options: one is to pay back the debt, B_t, which requires reductions in public investments, G_t, and the other is to increase public investments, G_t, in order to stimulate growth of Y_t. As discussed below (11), when b_t is larger (smaller) than k_g, increasing public investments (increases in G_t) has larger (smaller) reducing effects on b_t. Then, if $\bar{b} > (<)k_g$ holds, in reaction to decreases in a_t, the government increases (decreases) public investment, $g_t(= k_g\gamma_t)$, around the steady state. Using this result, we provide an intuitive explanation of equilibrium indeterminacy. Suppose that households expect future increases in G_t. Because increases in G_t stimulate output growth, this expectation implies that households also expect future increases in γ_t. In addition, because w_t increases at rate γ_t (see (5b)), future increases in labor income are also expected. Households then have a lesser incentive to save, and a_t thus decreases. As a result, the government's tax revenue decreases. When $\bar{b}$ is large enough to satisfy $\bar{b} > k_g$, the government increases public investments, G_t, as discussed above. Households' expectations are self-fulfilling and then equilibrium indeterminacy arises. Note that in the numerical example of the limit cycle in Figure 2, government spending and private asset holdings exhibit counter-cyclical

[15] In this example, we assume the Cobb-Douglas production function: $Y_t = {K_t}^{\alpha}(K_{g,t}L_t)^{1-\alpha}$. We use the following parameters: $\alpha = 0.36$, $r = 0.03$, $\rho = 0.01$, $\phi = 0.1$, and $\tau = 0.3$. Under these parameters, we have $\omega \approx 2.59$, $k_g \approx 0.25$, $\hat{b}_- \approx 2.01$, $\hat{b}_+ \approx 238.61$, and $\tilde{b} \approx 6.612$. We then assume that $b_0 \approx 6.613$. These values satisfy $1 > \tau > r/\omega$, $\bar{b} > k_g$, and $\hat{b}_- < \tilde{b} < \hat{b}_+$.

movements in equilibrium. This is consistent with the above interpretation for equilibrium indeterminacy. When $\bar{b} < k_g$ holds, expectations of future increases in G_t are not self-fulfilling because the government—faced with reductions in tax revenue—decreases G_t. Hence, the steady-state exhibits equilibrium determinacy.

Maebayashi et al. (2013) show that in a closed economy where the government follows (8) and adopts budgetary regime (I), equilibrium indeterminacy does not arise. Consider a closed economy where public capital is productive. As in our open economy, increases in public investments have positive growth effects. At the same time, increases in public investments crowd out private investments, which counteracts the positive growth effects. Then, the growth effects of increases in public investments become weaker, as compared to in our small open economy. The government—constrained with tight budgets—has only one choice: to cut public investments in order to pay back its debt. Households' expectations of increases in G_t are not self-fulfilling, and thus equilibrium indeterminacy does not arise in a closed economy.

In contrast, because an *open* economy can import capital from abroad and because economic activities in a *small* economy do not affect the world resource constraint, increases in public investments do not necessarily crowd out private investments in a *small open* economy and hence can have stronger growth effects. As such, in our small open economy, equilibrium indeterminacy can arise under some conditions.

Before we move on to the welfare effects of reductions in $\bar{b}$ in the next subsection, the following comment is appropriate here. Suppose that the economy is initially on a steady state and the government suddenly reduces $\bar{b}$. After this policy change, if the new steady state is stable, the economy begins to move toward it. If the transitional path is not unique, welfare effects of this policy change cannot be examined. The next subsection assumes that $\bar{b} < k_g$ so that the equilibrium path is uniquely determined.

4.2 Welfare Analysis

Focusing on budgetary regime (I) and the case of $\bar{b} < k_g$ where the steady state is locally determinate, this subsection examines the welfare effects of reductions in $\bar{b}$. We consider the following scenario. The economy is initially in the steady state equilibrium characterized by $b_{\mathrm{I}}^* = \bar{b}$ and (15a)–(15c). At time 0, the government unexpectedly reduces $\bar{b}$ marginally. After marginal reductions in $\bar{b}$, the economy begins to move toward the new steady state along the saddle path, and the transitional dynamics is generated. Although the value of ϕ has no influence on the steady-state equilibrium as shown in (15a)–(15c), it affects the transitional dynamics as is clear from (8). Then, ϕ plays important roles in examining the welfare effects of decreases in $\bar{b}$.

Our welfare measure is (1). Because $C_t = C_0 e^{\{(1-\tau)r-\rho\}t} = c_0 f(x) K_{g,0} e^{\{(1-\tau)r-\rho\}t}$, we can rewrite (1) as

$$U_0 = \frac{1}{\rho} \ln c_0 + \frac{1}{\rho} \left\{ \ln f(x) K_{g,0} + \frac{(1-\tau)r - \rho}{\rho} \right\}. \tag{20}$$

When the government reduces $\bar{b}$ at time 0 unexpectedly, c_0 jumps to its new value just at time 0; however, other variables in (20) do not change. Then, marginal reductions in $\bar{b}$ affect households' welfare only through c_0. Because $\bar{b} < k_g$ holds and the steady state is saddle stable, we can consider $\lambda_1 < 0$ and $\lambda_2 > 0$ that are solutions of the characteristic equation $\Omega(z) = 0$ as we discussed in Subsection 4.1. Appendix C shows that using $\lambda_2 > 0$,

c_0 can be expressed as

$$c_0 = c_{\rm I}^* + (b_0 - \bar{b})v_3 + \{a_0 - a_{\rm I}^* - (b_0 - \bar{b})v_2\}\lambda_2, \tag{21}$$

where v_2 and v_3 are defined in (C.3a) and (C.3b).

We now explain our scenario in more detail and examine the welfare effects of marginal reductions in $\bar{b}$. Initially, $\bar{b}$ is set to be equal to $\bar{b}^{init}$. We denote the initial steady-state value of a_t as $a_{\rm I}^{init}$. Because the economy is initially in the steady-state equilibrium, we have $b_0 = \bar{b}^{init}$ and $a_0 = a_{\rm I}^{init}$ in (21). Let us denote the target level of b_t after the policy change as $\bar{b}^{new}$. Because we consider marginal reductions in $\bar{b}$, $\bar{b}^{new}$ is approximately equal to $\bar{b}^{init}$ and then we have $b_0 = \bar{b}^{init} \approx \bar{b}^{new}$. We denote the steady-state value of a_t after the policy change as $a_{\rm I}^{new}$. Because we consider marginal changes, $a_{\rm I}^{new}$ is approximately equal to $a_{\rm I}^{init}$ and then we have $a_0 = a_{\rm I}^{init} \approx a_{\rm I}^{new}$. Note that after the policy change, we have $\bar{b} = \bar{b}^{new}$ and $a_{\rm I}^* = a_{\rm I}^{new}$ in (21), which implies that $b_0 - \bar{b} = \bar{b}^{init} - \bar{b}^{new} \approx 0$ and $a_0 - a_{\rm I}^* = a_{\rm I}^{init} - a_{\rm I}^{new} \approx 0$. Then, the effect of marginal changes in $\bar{b}$ can be expressed as

$$\begin{aligned}
\left.\frac{\partial c_0}{\partial \bar{b}}\right|_{(b_0,a_0)\approx(\bar{b}^{new},a_{\rm I}^{new})} &= \frac{\partial c_{\rm I}^*}{\partial \bar{b}} - \frac{\partial a_{\rm I}^*}{\partial \bar{b}}\lambda_2 + v_2\lambda_2 - v_3, \\
&= \frac{\lambda_2 - \rho}{\tau r}\left\{\frac{\lambda_2(\phi + \rho + \tau r)}{\phi + \lambda_2} - (\rho + \tau r)\right\} \equiv \Psi(\phi).
\end{aligned} \tag{22}$$

The calculation procedure from the first to the second lines in the above equation is presented in Appendix D. We can easily show that $\lambda_2 - \rho > 0$ and $\Psi(0) = 0$.[16] Differentiating $\Psi(\phi)$, we obtain

$$\Psi'(\phi) = \frac{\lambda_2(\lambda_2 - \rho)}{\tau r(\phi + \lambda_2)}\{\lambda_2 - (\rho + \tau r)\}.$$

We can show that if $r < (>)\omega$ holds, we have $\rho + \tau r < (>)\lambda_2$.[17] Then, if $r < (>)\omega$ holds, we have $\Psi(\phi) > (<)0$ and $\Psi'(\phi) > (<)0$ for $\phi > 0$. From (20), we obtain

$$\left.\frac{\partial U_0}{\partial \bar{b}}\right|_{(b_0,a_0)\approx(\bar{b}^{new},a_{\rm I}^{new})} = \frac{1}{\rho c_{\rm I}^*}\Psi(\phi).$$

Because ϕ has no effect on $c_{\rm I}^*$, the discussion this far yields the following proposition.

Proposition 4 *Consider budgetary regime (I). Suppose that $\gamma_{\rm I}^* = (1-\tau)r - \rho > 0$ and $\bar{b} < k_g$ hold and that the economy is initially in the steady-state equilibrium.*
(a) If $r < \omega$ holds, marginal reductions in $\bar{b}$ reduce households' welfare. As $\phi > 0$ becomes smaller, decreases in households' welfare too become smaller.
(b) If $r > \omega$ holds, marginal reductions in $\bar{b}$ raise households' welfare. As $\phi > 0$ becomes larger, increases in households' welfare too become larger.

The welfare effects of reductions in $\bar{b}$ depend on the relationship between r and ω. When $r > \omega$ holds, reductions in $\bar{b}$ improve welfare. Furthermore, the value of ϕ provides an important implication. When $r > \omega$ holds, the government should choose a large ϕ.

[16] λ_2 is a solution of $\Omega(z) = 0$. We have $0 < z < \lambda_2$ $(z > \lambda_2)$ if $\Omega(z) < (>)0$ holds for $z > 0$. Because $\Omega(\rho) = \rho^2 - m_{22}\rho + m_{32} = -(1-\tau)\omega k_g \tau r/(k_g - \bar{b}) < 0$, we have $\lambda_2 - \rho > 0$.

[17] As in footnote 16, $0 < z < \lambda_2$ $(z > \lambda_2)$ holds if $\Omega(z) < (>)0$ holds for $z > 0$. We have $\Omega(\rho + \tau r) = \tau r k_g(r - \omega)/(k_g - \bar{b})$. Because $\bar{b} < k_g$ is assumed, we have $\rho + \tau r < (>)\lambda_2$ if $r < (>)\omega$ holds.

More concretely, the government can improve welfare further by reducing b_t at a faster pace. In contrast, when $r < \omega$ holds, the government can mitigate welfare losses by reducing b_t at a slower pace.

To interpret Proposition 4 intuitively, we rewrite households' income, $I_t = rA_t + w_t$, using (5b):

$$I_t = rA_t + \omega K_{g,t}.$$

Households' income is generated from their assets, A_t, and public capital, $K_{g,t}$. Needless to say, A_t accumulates through savings of the private sector (households), whereas $K_{g,t}$ accumulates through public investments by the government. We can consider public investments as savings of the public sector. Consequently, r and ω can be considered as the rates of return on the private and public savings, respectively. If $r < \omega$ holds, accumulating assets through *public* savings is socially more efficient. Because reductions in $\bar{b}$ have negative effects on public investments (savings) when $\bar{b} < k_g$ holds as discussed below (10), reductions in $\bar{b}$ reduce households' welfare. When ϕ is small, the initial decline in γ_t is also small, which mitigates the negative welfare effects. When $r > \omega$ holds, accumulating assets through public savings is not socially efficient. Then, reductions in $\bar{b}$ depress public investments (savings) and improve welfare. When ϕ is large, the initial decline in γ_t is also large. Then, public investments are depressed further and households' welfare improves even further.

5 Tax Adjustments: Regime(II)

We now move onto budgetary regime (II). The dynamic system is given by (8), (14a), and (14b). We first derive the steady-state equilibrium where $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ holds. We set $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ in (8), (13), (14a), and (14b), and solve for b_t, a_t, c_t, and τ_t. The steady-state values of b_t, a_t, c_t, and τ_t are, respectively, given by $b^*_{\mathrm{II}} = \bar{b}(> 0)$,

$$a^*_{\mathrm{II}} = \frac{(r-\gamma)\bar{b} + \gamma k_g}{r-\rho-\gamma} - \frac{\omega k_g}{r}, \tag{23a}$$

$$c^*_{\mathrm{II}} = \frac{(r-\gamma)\bar{b} + \gamma k_g}{r-\rho-\gamma}\rho + \frac{\gamma\omega k_g}{r}, \tag{23b}$$

$$\tau^*_{\mathrm{II}} = \frac{r-(\rho+\gamma)}{r}. \tag{23c}$$

From (23a), households' income is given by $(ra^*_{\mathrm{II}} + \omega k_g)Y_t = \frac{(r-\gamma)\bar{b}+\gamma k_g}{r-\rho-\gamma} rY_t$ at the steady state. If $\gamma < r - \rho$ holds, the steady-state income level becomes strictly positive. This inequality ensures that $\tau^*_{\mathrm{II}} \in (0,1)$ and $c^*_{\mathrm{II}} > 0$. Accordingly, the remainder of this section assumes that $\gamma < r - \rho$. Because $\dot{A}_t/A_t = \gamma$ holds in the steady state and (23c) implies $(1-\tau^*_{II})r - \gamma = \rho > 0$, TVC is satisfied. The NPG condition of the government is also satisfied because we have $\dot{B}_t/B_t = \gamma < r - \rho < r$ in the steady state.

As shown in Appendix E, the Jacobian matrix of the linearized dynamic system has only one stable root, $-\phi$. Because there are two state variables, b_t and a_t, the economy cannot approach the steady-state equilibrium unless b_0 and a_0 are equal to $\bar{b}$ and a^*_{II} by chance. We thus obtain the next proposition.

Proposition 5 *Consider budgetary regime (II). Suppose that $\gamma < r - \rho$. There exists a unique steady-state equilibrium where $b^*_{\mathrm{II}} = \bar{b}$ and* (23a)–(23c) *hold. Unless b_0 and a_0 are equal to $\bar{b}$ and a^*_{II} by chance, the economy cannot approach the steady-state equilibrium.*

Because the economy cannot reach the steady-state equilibrium, it makes little sense to study the characteristics of the steady state any more. However, our model provides other important implications. Using (8), we solve (14a) and then rearrange the solution using inequality $c_t > 0$:[18]

$$a_t < b_t + \left\{(a_0 - b_0) + \frac{\omega - \gamma}{r - \gamma} k_g\right\} e^{(r-\gamma)t} - \frac{\omega - \gamma}{r - \gamma} k_g. \tag{24}$$

Note that we have $\lim_{t\to+\infty} b_t = \bar{b}$ because of (8) and that $r - \gamma > 0$ is assumed. Suppose that $a_0 - b_0$ is sufficiently small so that $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$ holds. Then, (24) implies $\lim_{t\to\infty} a_t = -\infty$. If $a_0 > -\omega k_g/r$ holds, a_t necessarily moves across $-\omega k_g/r$. As discussed below (13), such transitional paths cannot be the equilibrium because τ_t is not defined at $a_t = -\omega k_g/r$ and households' optimization problem is not well-defined.

It is also shown that even if $a_0 < -\omega k_g/r$ holds and a_t does not move across $-\omega k_g/r$ along the transition, there exists no equilibrium. As shown in (13), τ_t tends to zero as a_t tends to $-\infty$. Then, for a large t, the discounted sum of households' labor income can be written as

$$\int_t^{+\infty} (1 - \tau_v) w_v e^{-\int_t^v (1-\tau_s) r ds} dv = \int_t^{+\infty} \omega k_g Y_v e^{-r(v-t)} dv = \frac{\omega k_g}{r - \gamma} Y_t > 0. \tag{25}$$

In the first equality, we use $\tau_t = 0$ for a large t, (5b), and the definition of k_g. The second equality uses $\dot{Y}_t/Y_t = \gamma$ and $r > \gamma$. The assumption $\gamma < r - \rho (< r)$ ensures the last inequality. Because we have $a_t = -\infty$ for a large t, inequality $-a_t > \omega k_g/(r - \gamma)$ holds for a large t. From (25) and the definition of a_t, the following inequality is obtained for a large t:

$$-A_t > \int_t^{+\infty} (1 - \tau_v) w_v e^{-\int_t^v (1-\tau_s) r ds} dv.$$

The above inequality implies that for a large t, households' borrowing eventually exceeds the discounted sum of their labor income, and households would thus be unable to pay off their borrowing. In our model, rational expectations and perfect foresight ensure that foreign economies do not lend to such households.

We come to the following conclusion. Under budgetary regime (II), if $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$ holds, there exists no equilibrium such that the government can keep its debt-GDP ratio at a finite level, $\bar{b}$, in the long run.[19] In other words, under budgetary regime (II), if $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$ holds, the government cannot sustain its debt. The next remark summarizes the discussion thus far.

Remark

Consider budgetary regime (II). Suppose that $\gamma < r - \rho$ and $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$.

[18] Appendix F provides the derivation of (24).

[19] If $r < \gamma$ holds, (24) does not necessarily imply $\lim_{t\to\infty} a_t = -\infty$ even when $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$ holds. The discussion in this paragraph does not apply to the case of $r < \gamma$. As shown in Appendix E, when $r < \gamma$ holds, the steady state characterized by $b^*_{\mathrm{II}} = \bar{b}$ and (23a)–(23c) is saddle stable and hence the economy can reach the steady state in the long-run. However, in this steady state, the government cannot satisfy the NPG condition because $\dot{B}_t/B_t = \gamma > r$. Furthermore, TVC is also not satisfied because $\dot{A}_t/A_t = \gamma > r$. The steady state cannot be an equilibrium. Furthermore, both the income tax rate, τ^*_{II}, and households' income $(ra^*_{\mathrm{II}} + \omega k_g)Y_t = \frac{(r-\gamma)\bar{b} + \gamma k_g}{r - \rho - \gamma} r Y_t$ are negative. Then, the steady state characterized by $b^*_{\mathrm{II}} = \bar{b}$ and (23a)–(23c) makes little sense when $r < \gamma$ holds.

Then, there exists no equilibrium such that the government can keep its debt-GDP ratio at a finite level in the long run.

Inequality $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$ indicates that households' initial asset holdings tend to be smaller than the government's initial outstanding debt, which means that private domestic savings cannot absorb the outstanding government debt and the government borrows from foreign investors. Our result implies that in such a case, if the government in a small open economy attempts to reduce its debt-GDP ratio by adjusting the income tax rate without cutting public investments, the economy cannot sustain its debt and eventually loses the ability to pay back its debt.

Maebayashi et al. (2013) show that in a closed economy where the government follows (8) and adopts budgetary regime (II), the unique steady state is always saddle stable and the economy converges to the steady state where the debt-GDP ratio remains constant over time. In other words, the government can always sustain its debt if it is a closed economy. To reduce its debt-GDP ratio, the government imposes a high tax rate on the interest income under regime (II) (see (13)), which discourages household savings and negatively affects a_t. However, households in a closed economy cannot borrow from abroad and hence a_t cannot be smaller than b_t. Therefore, households in a closed economy do not lose the ability to pay back debt and then there exists an equilibrium where the government can keep its debt-GDP ratio at a finite level. In contrast, because households in an *open* economy can borrow from abroad and because households in a *small* economy do not affect the world asset market, households in a *small open* economy are at risk of getting overextended with debt (although this does not happen in equilibrium).

6 Concluding Remarks

We examine how a debt policy rule under which the debt ratio is gradually adjusted to the targets level affects macroeconomic stabilities and welfare. We construct a small open economy model where the government can borrow from abroad to finance its debt. Following Futagami et al (1993), Turnovsky (1996), and Fisher and Turnovsky (1998), public capital accumulated by public investments has positive externalities on goods production.

In budgetary regime (I), where the government controls public spending so as to follow a debt policy rule, there exists a unique steady state equilibrium. When the target (long-run) debt-GDP ratio is sufficiently low (high), the steady state exhibits equilibrium (in)determinacy. Our results suggest that to stabilize the economy under budgetary regime (I), the government in a small open economy should keep its debt-GDP ratio sufficiently low in the long run. With regard to the welfare analysis of fiscal consolidation when the steady state exhibits equilibrium determinacy, we obtain the following results. If the rate of return from private savings is larger (smaller) than that from public savings, reductions in government debt improve (reduce) welfare and the government can realize (may suffer) larger welfare gains (losses) by reducing its debt at a faster pace. In budgetary regime (II), where the government controls the income tax rate so as to follow a debt policy rule, the steady state is always unstable. If the initial asset holdings of households are smaller than the initial outstanding government debt, there exists no equilibrium such that the government can keep its debt-GDP ratio at a finite level in the long run. Hence, the government cannot sustain its debt.

Appendix

A Proof of Proposition 2

1. If $\bar{b} < k_g$ holds, we have $m_{32} < 0$ because $c_{\mathrm{I}}^* > 0$ and $\partial\gamma(a_I^*, \bar{b})/\partial a_t = \tau r/(k_g - \bar{b}) > 0$ (see (18)). From the second equation of (19), one of the eigenvalues of the Jacobian matrix in (16) is negative while the other is positive. Then, the steady state is locally saddle stable and exhibits local determinacy.
2. If $\bar{b} > k_g$ holds, $m_{32} > 0$ (see (19) and (18)). Using (15a) and (18), we rewrite m_{22} as

$$m_{22} = \frac{\{(1-\tau)r - 2\rho - \tau\omega\}k_g + (2\rho + \tau r)\bar{b}}{\bar{b} - k_g}. \tag{A.1}$$

(a) If $\tau > r/\omega$ holds, $(1-\tau)r - 2\rho - \tau\omega$ takes a negative value. Let $\tilde{b} \equiv \frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r}k_g$. Then, if $k_g < \bar{b} < \tilde{b}$ holds, $m_{22} < 0$. Inequality $\tau > r/\omega$ ensures that $\frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r} = 1 + \frac{\tau\omega - r}{2\rho+\tau r} > 1$, and thus ensures the existence of a $\bar{b}$ that satisfies $k_g < \bar{b} < \tilde{b}$. From $m_{22} < 0$, $m_{32} > 0$, and (19), we know that the real parts of both λ_1 and λ_2 are negative. The steady state is locally stable and exhibits local indeterminacy.

If $\bar{b} > \tilde{b}$ holds, $\bar{b} > k_g$ because inequality $\tau > r/\omega$ ensures that $\frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r} = 1 + \frac{\tau\omega - r}{2\rho+\tau r} > 1$. Then, we have $m_{22} > 0$ and $m_{32} > 0$, which implies that the real parts of the two eigenvalues are positive.
(b) We now examine the signs of m_{22} and m_{32} in the case of $\tau < r/\omega$ and $\bar{b} > k_g$. Apparently, $m_{32} > 0$ holds. Because $\bar{b} > k_g$, we have

$$\begin{aligned} m_{22} &> \frac{\{(1-\tau)r - 2\rho - \tau\omega\}k_g + (2\rho + \tau r)k_g}{\bar{b} - k_g} \\ &= \frac{r - \tau\omega}{\bar{b} - k_g}k_g \\ &> 0. \end{aligned}$$

Then, we have $m_{22} > 0$ and $m_{32} > 0$, which implies that the real parts of the two eigenvalues are positive.

B Proof of Proposition 3

We show the existence of limit cycles by applying the Hopf bifurcation theorem.[20] Since Jacobian matrix M (defined in 16) has one negative eigenvalue, $-\phi$, the dynamic system can exhibit Hopf bifurcation if its submatrix

$$M' \equiv \begin{pmatrix} \rho - \frac{\partial\gamma(a_I^*, \bar{b})}{\partial a_t}a_{\mathrm{I}}^* & -1 \\ -\frac{\partial\gamma(a_I^*, \bar{b})}{\partial a_t}c_{\mathrm{I}}^* & 0 \end{pmatrix}$$

has a couple of complex eigenvalues. A necessary and sufficient condition for complex eigenvalues is $(trM')^2 - 4detM' < 0$, that is,[21]

$$\left(\rho - \frac{\tau r}{k_g - \bar{b}}a_{\mathrm{I}}^*\right)^2 < -4\frac{\tau r}{k_g - \bar{b}}c_{\mathrm{I}}^*. \tag{B.1}$$

[20] To see the Hopf bifurcation theorem in detail, refer to Theorem 3.4.2 (pp. 151) of Guckenheimer and Holmes (1983).

[21] This inequality requires $\bar{b} > k_g$ and as mentioned earlier, holds even in case 2.(a).ii of Proposition 2 where $\tau > \frac{r}{\omega}$ and $\bar{b} > \tilde{b}$.

Substituting (15a) and (15b) into (B.1), we obtain

$$\left[\rho - \frac{(\gamma_{\mathrm{I}}^* - \tau\omega)k_g + (\rho + \tau r)\bar{b}}{k_g - \bar{b}}\right]^2 < -4\frac{\gamma_{\mathrm{I}}^*(\rho + \tau\omega)k_g + \rho(\rho + \tau r)\bar{b}}{k_g - \bar{b}}.$$

Multiplying both sides by $(k_g - \bar{b})^2$ and arranging it as a polynomial of $\bar{b}$, we have

$$(\tau r)^2\bar{b}^2 - 2\tau r\big[(1-\tau)(\omega - r) + \omega\big]k_g\bar{b} + \big[(1-\tau)r + \tau\omega\big]^2 k_g^2 < 0. \tag{B.2}$$

The critical values of quadratic inequality (B.2) are

$$\hat{b}_- \equiv \frac{(1-\tau)(\omega - r) + \omega - 2\sqrt{(1-\tau)(\omega - r)\omega}}{\tau r},$$
$$\hat{b}_+ \equiv \frac{(1-\tau)(\omega - r) + \omega + 2\sqrt{(1-\tau)(\omega - r)\omega}}{\tau r},$$

where these real numbers are well-defined under the assumption that $r/\omega < 1$. As such, a necessary and sufficient condition for complex eigenvalues is

$$\hat{b}_- < \bar{b} < \hat{b}_+.$$

Note that M' has complex eigenvalues by the assumption that $\hat{b}_- < \tilde{b} < \hat{b}_+$. The real part of these eigenvalues, $\frac{detM'}{2} = \frac{m_{22}}{2}$, equals zero only when $\bar{b} = \tilde{b}$; it is also differentiable and increasing in $\bar{b}$. Thus, $\tilde{b}$ is the unique bifurcation value of parameter $\bar{b}$, and submatrix M' has a couple of complex eigenvalues in any sufficiently small neighborhood of $\tilde{b}$. Therefore, by the Hopf bifurcation theorem, there exists a sufficiently small $\varepsilon > 0$ such that either of the following cases is true: (i) there exists a stable limit cycle for every $\bar{b}$ such that $\tilde{b} < \bar{b} < \tilde{b} + \varepsilon$, or (ii) there exists an unstable limit cycle for every $\bar{b}$ such that $\tilde{b} - \varepsilon < \bar{b} < \tilde{b}$. Since any solution emerging from the neighborhoods of the cycle winds around it in case (i) and remains in the cycle (asymptotically on the (a_t, c_t) plane) in case (ii), there are innumerable initial values of c_t consistent with TVC in each case. This implies the multiplicity of equilibria. The proof of Proposition 3 is thus complete.

C Derivation of c_0

We derive the saddle path that converges to the steady state we consider. Because $\bar{b} < k_g$ holds, we can consider that $\lambda_1 < 0$ and $\lambda_2 > 0$. Because $m_{32} < 0$ holds in the case of $\bar{b} < k_g$ (see Appendix A), $\sqrt{{m_{22}}^2 - 4m_{32}} > |m_{22}|$. Then, λ_1 and λ_2 are given by

$$\lambda_1 = \frac{m_{22} - \sqrt{{m_{22}}^2 - 4m_{32}}}{2} < 0 \quad \text{and} \quad \lambda_2 = \frac{m_{22} + \sqrt{{m_{22}}^2 - 4m_{32}}}{2} > 0. \tag{C.1}$$

Given b_0 and a_0, we solve (16) and obtain

$$\begin{pmatrix} b_t \\ a_t \\ c_t \end{pmatrix} = \begin{pmatrix} b_{\mathrm{I}}^* \\ a_{\mathrm{I}}^* \\ c_{\mathrm{I}}^* \end{pmatrix} + \theta_1 \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} e^{-\phi t} + \theta_2 \begin{pmatrix} 0 \\ 1 \\ \lambda_2 \end{pmatrix} e^{\lambda_1 t}. \tag{C.2}$$

In the above equation, $\theta_1 \equiv b_0 - \bar{b}$ and $\theta_2 \equiv a_0 - a_{\mathrm{I}}^* - (b_0 - \bar{b})v_2$ are determined by the initial condition. Vectors $(1\ v_2\ v_3)^T$ and $(0\ 1\ \lambda_2)^T$ are the eigenvectors associated with

eigenvalues $-\phi$ and λ_1, respectively.[22] v_2 and v_3 are expressed as

$$v_2 = \frac{m_{11}m_{21} - m_{31}}{m_{11}(m_{11} - m_{22}) + m_{32}} = \frac{\partial\gamma(a_I^*, \bar{b})}{\partial b_t}\frac{\phi a_{\rm I}^* + c_{\rm I}^*}{\phi^2 + m_{22}\phi + m_{32}} = \frac{\partial\gamma(a_I^*, \bar{b})}{\partial b_t}\frac{\phi a_{\rm I}^* + c_{\rm I}^*}{(\phi + \lambda_1)(\phi + \lambda_2)}, \tag{C.3a}$$

$$v_3 = \frac{(m_{11} - m_{22})m_{31} + m_{21}m_{32}}{m_{11}(m_{11} - m_{22}) + m_{32}} = \frac{\partial\gamma(a_I^*, \bar{b})}{\partial b_t}\frac{(\rho + \phi)c_{\rm I}^*}{\phi^2 + m_{22}\phi + m_{32}} = \frac{\partial\gamma(a_I^*, \bar{b})}{\partial b_t}\frac{(\rho + \phi)c_{\rm I}^*}{(\phi + \lambda_1)(\phi + \lambda_2)}. \tag{C.3b}$$

In equations (C.3a) and (C.3b), we use (19) that implies that $\phi^2 + m_{22}\phi + m_{32} = (\phi + \lambda_1)(\phi + \lambda_2)$. From (C.2), c_0 is given by (21).

D Derivation of (22)

From (15a) and (15b), we have $\partial a_{\rm I}^*/\partial\bar{b} = (\rho + \tau r)/(\tau r)$ and $\partial c_{\rm I}^*/\partial\bar{b} = \rho\partial a_{\rm I}^*/\partial\bar{b}$. Using the definition of v_2 and v_3, we rearrange $v_2\lambda - v_3$ as

$$v_2\lambda_2 - v_3 = \frac{\phi + \rho + \tau r}{(k_g - \bar{b})(\phi + \lambda_1)(\phi + \lambda_2)}\{(\rho - \lambda_2)c_{\rm I}^* + \phi(c_{\rm I}^* - \lambda_2 a_{\rm I}^*)\}. \tag{D.1}$$

Because λ_2 is a solution of $\Omega(z) = 0$, $\Omega(\lambda_2) = {\lambda_2}^2 - m_{22}\lambda_2 + m_{32} = 0$, which implies that

$$c_{\rm I}^* - \lambda_2 a_{\rm I}^* = \left(\frac{\partial\gamma(a_I^*, \bar{b})}{\partial a_t}\right)^{-1}\lambda_2(\lambda_2 - \rho),$$

where $\partial\gamma(a_I^*, \bar{b})/\partial a_t$ is given by (18). In deriving the above equation, we use the definitions of m_{22} and m_{32}. Substituting the above equation into (D.1) yields

$$\begin{aligned} v_2\lambda_2 - v_3 &= \frac{(\lambda_2 - \rho)(\phi + \rho + \tau r)}{\tau r(\phi + \lambda_1)(\phi + \lambda_2)}\left(\phi\lambda_2 - \frac{\partial\gamma(a_I^*, \bar{b})}{\partial a_t}c_{\rm I}^*\right), \\ &= \frac{(\lambda_2 - \rho)(\phi + \rho + \tau r)}{\tau r(\phi + \lambda_1)(\phi + \lambda_2)}\left(\phi\lambda_2 + m_{32}\right), \\ &= \frac{(\lambda_2 - \rho)\lambda_2(\phi + \rho + \tau r)}{\tau r(\phi + \lambda_2)}. \end{aligned}$$

On the second line, we use the definition of m_{32}. The second equation of (19) is used on the last line. Substituting the above expression, $\partial a_{\rm I}^*/\partial\bar{b} = (\rho+\tau r)/(\tau r)$, and $\partial c_{\rm I}^*/\partial\bar{b} = \rho\partial a_{\rm I}^*/\partial\bar{b}$ into the first line of (22) yields the second line of (22).

E Stability of the Steady State: Regime (II)

To examine the stability of the steady-state equilibrium, we linearize the dynamic system around the steady state characterized by $b_{\rm II}^* = \bar{b}$ and (23a)–(23c):

$$\begin{pmatrix} \dot{b}_t \\ \dot{a}_t \\ \dot{c}_t \end{pmatrix} = \begin{pmatrix} -\phi & 0 & 0 \\ -(r - \gamma + \phi) & r - \gamma & -1 \\ -\frac{r-\gamma+\phi}{ra_{\rm II}^* + \omega k_g}rc_{\rm II}^* & \frac{r-\rho-\gamma}{ra_{\rm II}^* + \omega k_g}rc_{\rm II}^* & 0 \end{pmatrix}\begin{pmatrix} b_t - \bar{b} \\ a_t - a_{\rm II}^* \\ c_t - c_{\rm II}^* \end{pmatrix}.$$

[22] To derive the eigenvector associated with λ_1, we use (19). Let us denote the eigenvector as $\hat{v}^T \equiv (\hat{v}_1\ 1\ \hat{v}_3)^T$. From $M\hat{v} = \lambda\hat{v}$, we obtain $\hat{v}_1 = 0$ and $\hat{v}_3 = m_{22} - \lambda_1 = m_{32}/\lambda_1$. Then, the two equations of (19) imply that $\hat{v}_3 = \lambda_2$.

One of the eigenvalues of the Jacobian matrix is given by $-\phi < 0$. The other two, μ_1 and μ_2, satisfy

$$\mu_1 + \mu_2 = r - \gamma > 0 \quad \text{and} \quad \mu_1 \cdot \mu_2 = \frac{r - \rho - \gamma}{r a^*_{\rm II} + \omega k_g} r c^*_{\rm II} > 0.$$

The above inequalities hold because $\gamma < r - \rho$ is assumed. The real parts of both μ_1 and μ_2 are positive. Remember that there are two state variables, b_t and a_t. Unless b_0 and a_0 are equal to $\bar{b}$ and a^*_{II} by chance, the economy cannot achieve the steady-state equilibrium.

If $r < \gamma$ holds as discussed in footnote 19, we have $\mu_1 + \mu_2 < 0$ and $\mu_1 \cdot \mu_2 < 0$. Either of μ_1 or μ_2 has a positive real part while the other has a negative one. Then, the steady state is saddle stable.

F Derivation of (24)

Let us define $z_t \equiv a_t - b_t$. Using (8) and the definition of z_t, we rewrite (14a) as

$$\dot{z}_t = (r - \gamma) z_t - c_t + (\omega - \gamma) k_g.$$

Given $z_0 \equiv a_0 - b_0$, the solution for the above is

$$z_t = \left(z_0 + \frac{\omega - \gamma}{r - \gamma} k_g \right) e^{(r-\gamma)t} - \frac{\omega - \gamma}{r - \gamma} k_g - \int_0^t c_v e^{(r-\gamma)(t-v)} dv.$$

Because $c_t > 0$ holds for all $t \geq 0$, the above equation implies the following inequality:

$$z_t < \left(z_0 + \frac{\omega - \gamma}{r - \gamma} k_g \right) e^{(r-\gamma)t} - \frac{\omega - \gamma}{r - \gamma} k_g.$$

Substituting $z_t = a_t - b_t$ into the above equation yields (24).

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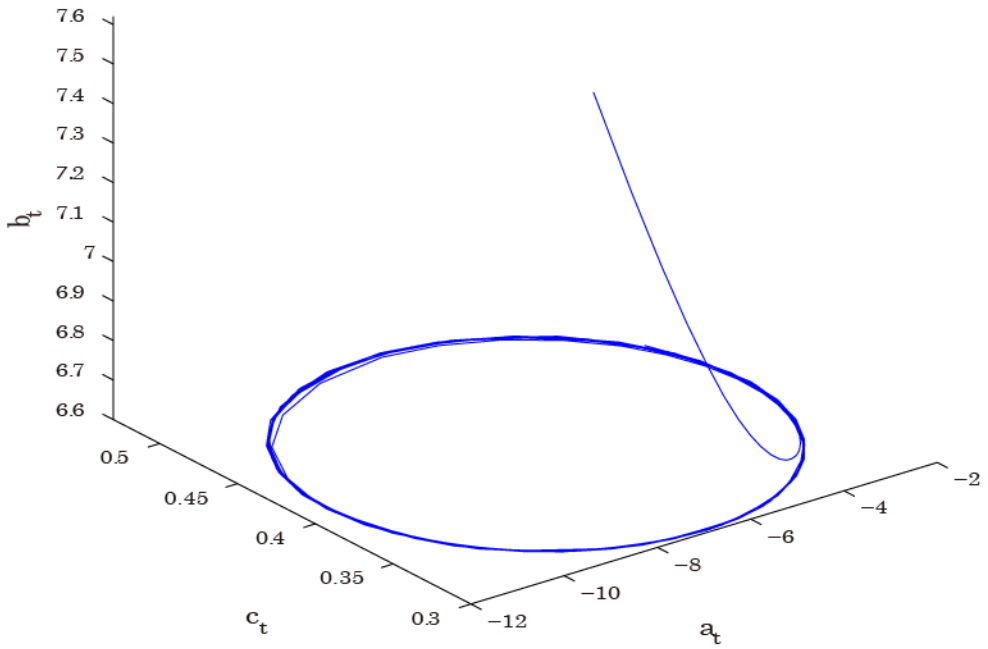

Figure 1: An Example of Limit Cycle

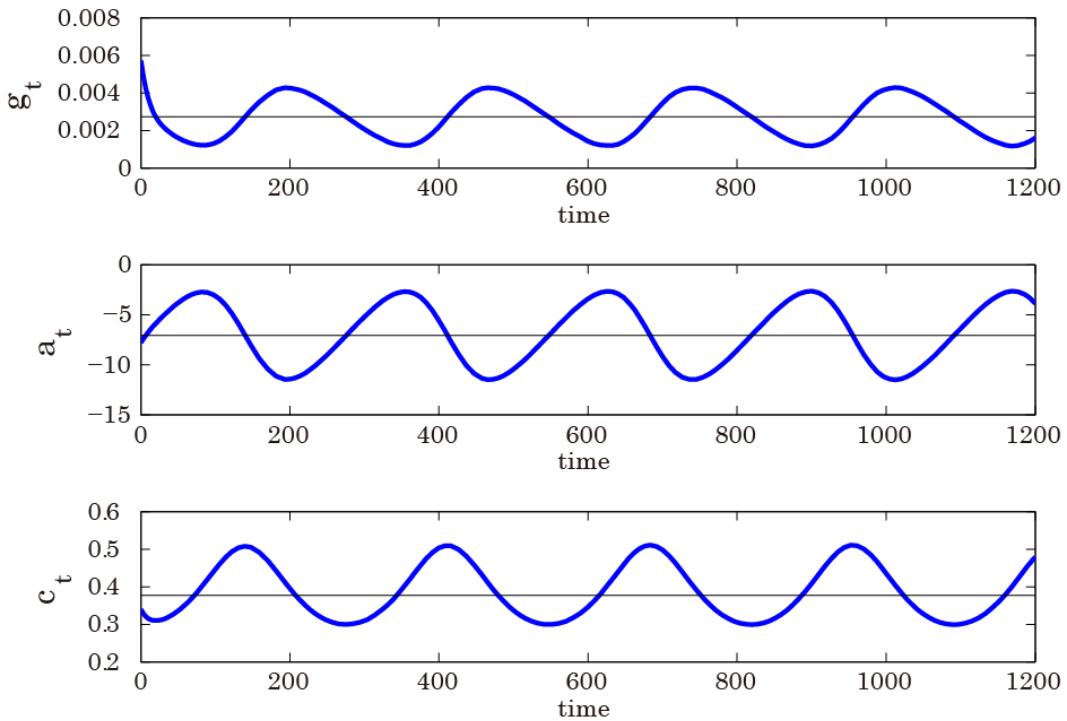

Figure 2: Fluctuations of (g_t, a_t, c_t)