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Volatility and realized quadratic variation of differenced returns. A wavelet method approach.

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Abstract

This paper analyzes some asymptotic results for an alternative estimator of integrated volatility in a continuous-time diffusion process of high frequency data (used in asset pricing finance).

The estimator, which is computationally efficient, is based on the quadratic variation of the second order log-price differences. This is contrary to the well known realized quadratic variation of intra daily returns (which is based on first order log-price differences). This latter is known as realized volatility.

Analytically, the asymptotics of the proposed estimator is compared to the usual realized volatility estimators. Lastly, we provide some simulation experiments to illustrate the results.

JEL Classification: C1, G1.

Keywords: continuous-time methods; quadratic variation; realized volatility; second order quadratic variation

*I thank Asger Lunde for suggesting to me the idea of estimating volatility by wavelets.

1. Introduction

We consider a stochastic volatility model, which we write in a generic form. The (univariate) log-price is assumed to follow a diffusion process of the following kind:

$$dp(t) = \mu(t)dt + \sigma(t)dw(t), \tag{1}$$

where $\mu(t)$ and $\sigma(t)$ are time dependent functions and $w(t)$ is a standard Brownian motion.

It was suggested in Andersen & Bollerslev (1998*a*), that in this setting the relevant notion of volatility over a particular time span, normalized to $[0, 1]$, is

$$\int_0^1 \sigma^2(u)du.$$

This is called integrated volatility or integrated variance (by e.g. Barndorff-Nielsen & Shephard (2002)), and is the focal point in the pricing of derivative securities under stochastic volatility (see e.g. Hull & White (1987)). The method to estimate the integrated volatility analyzed in the present paper was suggested by Høg & Lunde (2003).

The key tool is the simple Haar wavelet $\psi(t) = \phi(2t) - \phi(2t-1)$, where $\phi(t) = 1_{[0,1)}(t)$ is the indicator function on the unit interval. Then

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

This wavelet is orthogonal. For any scale $a > 0$ and dilation[1] b the continuous transform of the incremental price process dp is

$$\begin{aligned}(\mathcal{W}_\psi dp)(a,b) &= a^{-1/2}\int \psi(\frac{t-b}{a})dp(t) \\ &= a^{-1/2}\int \psi(\frac{t-b}{a})\sigma(t)dw(t) + a^{-1/2}\int \psi(\frac{t-b}{a})\mu(t)dt \end{aligned} \tag{2}$$

An important property which is used in the development of this analysis is

Lemma 1 *The second order moment of the Wavelet transform in (2) is given by*

$$\begin{aligned}E[(\mathcal{W}_\psi dp)(a,b)]^2 &= a^{-1}\int \psi^2(\frac{t-b}{a})\sigma^2(t)dt + [(\mathcal{W}_\psi \mu)(a,b)]^2 \\ &= a^{-1}\int_b^{a+b} \sigma^2(t)dt + [(\mathcal{W}_\psi \mu)(a,b)]^2. \end{aligned} \tag{3}$$

Proof. This is easily shown by using the standard rules of Ito calculus. ■

2. Method

Without loss of generality we normalize the time window of observations $[0,\tau]$ to $[0,1]$. Let T be the number of observations of the process in (1). For $k = 0, \cdots, T-1$ we have the disjoint partition

$$\bigcup_{k=0}^{T-1} [\frac{k}{T}, \frac{k+1}{T}[\tag{4}$$

[1]For example $a = 2^{-j}$ and $b = 2^{-j}k$

of the interval $[0,1]$ and the following so-called wavelet coefficients of $dp(t)$ on the interval $[0,1]$,

$$\begin{aligned}\omega_{Tk}(\mathrm{d}p) &\equiv (\mathcal{W}_\psi \mathrm{d}p)(\frac{1}{T},\frac{k}{T}) \\ &= \sqrt{T}\left[\int_{k/T}^{(k+\frac{1}{2})/T} \mathrm{d}p[t] - \int_{(k+\frac{1}{2})/T}^{(k+1)/T} \mathrm{d}p[t]\right] \\ &= \sqrt{T}\left[2p\left[(k+1/2)/T\right] - p\left[k/T\right] - p\left[(k+1)/T\right]\right]. \qquad (5)\end{aligned}$$

From (3) it follows that the second order moments of the wavelet coefficients of dp are

$$\mathrm{E}[\omega_{Tk}^2(\mathrm{d}p))] = 2^j \int_{k/T}^{(k+1)/T} \sigma^2(t)dt + [(\mathcal{W}_\psi \mu)(1/T, k/T)]^2. \qquad (6)$$

We assume that $p[\cdot]$ is observed at times $t_1, \cdots, t_T$, where without loss of generality we have $0 \le t_1 \le \cdots \le t_T \le 1$. By previous tick–interpolation we put $p[\frac{x}{T}] = p[j']$ whenever $t_{j'} \le \frac{x}{T} \le t_{j'+1}$. Then (5) is observed for every $k = 0, \cdots, T-1$.

It is convenient to define the following

$$x_{1k} = \textstyle\int_{k/T}^{(k+1/2)/T} \mathrm{d}p(t), \qquad \mu_{1k} = \sqrt{T}\int_{k/T}^{(k+1/2)/T} \mu(t)\mathrm{d}t, \quad \sigma_{1k}^2 = T\int_{k/T}^{(k+1/2)/T} \sigma^2(t)\mathrm{d}t,$$

$$x_{2k} = \sqrt{T}\textstyle\int_{(k+1/2)/T}^{(k+1)/T} \mathrm{d}p(t), \quad \mu_{2k} = \sqrt{T}\int_{(k+1/2)/T}^{(k+1)/T} \mu(t)\mathrm{d}t, \quad \sigma_{2k}^2 = T\int_{(k+1/2)/T}^{(k+1)/T} \sigma^2(t)\mathrm{d}t.$$

Furthermore we use the following notation

$$\mu_k = \mu_{1k} - \mu_{2k}, \qquad V_k = \sigma_{1k}^2 + \sigma_{2k}^2 = T\int_{k/T}^{(k+1)/T} \sigma^2(t)\mathrm{d}t.$$

Then it is easily seen that x_{1k} and x_{2k} are independent and

$$x_{1k} \sim N(\mu_{1k}, \sigma_{1k}^2), \qquad x_{2k} \sim N(\mu_{2k}, \sigma_{2k}^2),$$

hence

$$\omega_{Tk}(\mathrm{d}p) = x_{1k} - x_{2k} \sim N(\mu_{1k} - \mu_{2k}, \sigma_{1k}^2 + \sigma_{2k}^2).$$

Therefore

$$\frac{\omega_{Tk}(\mathrm{d}p)}{\sqrt{\sigma_{1k}^2 + \sigma_{2k}^2}} \sim N(\frac{\mu_{1k} - \mu_{2k}}{\sqrt{\sigma_{1k}^2 + \sigma_{2k}^2}}, 1).$$

The following lemma holds:

Lemma 2 *Consider the sequence of squared values from (5)*

$$\omega_{Tk}^2(dp) = T\left[2p\left[\frac{k+1/2}{T}\right] - p\left[\frac{k}{T}\right] - p\left[\frac{k+1}{T}\right]\right]^2.$$

The mean and variance of $\omega_{Tk}^2(dp)$ *are*

$$\begin{aligned} E\left[\omega_{Tk}^2(dp)\right] &= V_k + \mu_k^2, \\ Var\left[\omega_{Tk}^2(dp)\right] &= 2V_k^2 + 4\mu_k^2 V_k. \end{aligned} \tag{7}$$

Proof. It is a well known fact that for a normal variate X with mean μ and variance σ^2 the second and fourth non-central moments are

$$\mathrm{E}(X^2) = \sigma^2 + \mu^2, \text{ and } \mathrm{E}(X^4) = 3\sigma^4 + \mu^4 + 6\mu^2\sigma^2.$$

Hence

$$\begin{aligned} \mathrm{E}\left[\frac{\omega_{Tk}^2(\mathrm{d}p)}{\sigma_{1k}^2 + \sigma_{2k}^2}\right] &= 1 + \frac{\mu_k^2}{V_k}, \\ \mathrm{Var}\left[\frac{\omega_{Tk}^2(\mathrm{d}p)}{\sigma_{1k}^2 + \sigma_{2k}^2}\right] &= 2 + \frac{4\mu_k^2}{V_k}, \end{aligned}$$

which proves the lemma.

Remark: $\frac{\omega_{Tk}^2(\mathrm{d}p)}{\sigma_{1k}^2+\sigma_{2k}^2}$ is a non-central χ^2 variate with one degree of freedom and non-central parameter $\frac{\mu_k^2}{V_k}$. ■

We are now able to establish the following asymptotic result

Theorem 3 *Given a sample of T observations from(1): $p[t_1], \cdots, p[t_T]$ and the relation between these and the partition of the unit interval given in (4), we have the following result*

$$E[\frac{1}{T}\sum_{k=0}^{T-1}\omega_{Tk}^2(dp)] \longrightarrow \int_0^1 \sigma^2(t)dt \text{ for } T \to \infty$$

Remark: *Note that the statement in the theorem suggests that we estimate the integrated volatility by the sample mean of the squared wavelet coefficients at the scale corresponding to $1/T$.*

Proof. Sice $V_k = T\int_{k/T}^{(k+1)/T}\sigma^2(t)\mathrm{d}t$, we have that $\sum_{k=0}^{T-1}V_k = T\int_0^1\sigma^2(t)\mathrm{d}t$. Furthermore we note that $\omega_{Tk}^2(\mu) = \mu_k^2$. Since ψ is an orthogonal wavelet, the so-called "Parseval" identity (see for example Chui (1992)) holds for *any* integer $M > 0$

$$\int_0^1 \mu^2(t)\mathrm{d}t = \left[\int_0^1 \mu(t)\mathrm{d}t\right]^2 + \sum_{j=0}^{M}\sum_{k=0}^{2^j-1} w_{jk}^2(\mu).$$

Therefore, for $j = M$ we have

$$\sum_{k=0}^{2^M-1} w_{Mk}^2(\mu) < \int_0^1 \mu^2(t)\mathrm{d}t - \left[\int_0^1 \mu(t)\mathrm{d}t\right]^2,$$

and from (7) we get

$$\mathrm{E}\left[\sum_{k=0}^{T-1}\omega_{Tk}^2(\mathrm{d}p)\right] = \int_0^1\sigma^2(t)\mathrm{d}t + \sum_{k=0}^{T-1}\omega_{Tk}^2(\mu)). \tag{8}$$

Now let $M = [\log_2 T] + 1$, (i.e. $2^{M-1} \leq T < 2^M$). Then

$$\sum_{k=0}^{T-1} \omega_{Tk}^2(\mu)) < \sum_{k=0}^{2^M-1} w_{Mk}^2(\mu) < \int_0^1 \mu^2(t)\mathrm{d}t - \left[\int_0^1 \mu(t)\mathrm{d}t\right]^2.$$

Hence (7) implies

$$\mathrm{E}[\frac{1}{T}\sum_{k=0}^{T-1} \omega_{Tk}^2(\mathrm{d}p)] \;\longrightarrow\; \int_0^1 \sigma^2(t)\mathrm{d}t \text{ for } T \to \infty$$

■

An asymptotic distributional result is obtained under simplified assumptions on the drift process $\mu(t)$. First we have the following theoretical result, which is not, however, immediately practically useful.

Theorem 4 *The weighted sum of the squared wavelet coefficients is exactly distributed as a non central χ^2 with T degrees of freedom and non central parameter $\sum_{k=0}^{T-1} \frac{\mu_k^2}{V_k}$:*

$$\sum_{k=0}^{T-1} \frac{\omega_{Tk}^2(dp)}{V_k} \quad \overset{exact}{\sim} \quad \chi'^2\left(T, \sum_{k=0}^{T-1} \frac{\mu_k^2}{V_k}\right).$$

Furthermore we have the following approximation to a central χ^2

$$\frac{T + \sum_{k=0}^{T-1} \frac{\mu_k^2}{V_k}}{T + 2\sum_{k=0}^{T-1} \frac{\mu_k^2}{V_k}} \sum_{k=0}^{T-1} \frac{\omega_{Tk}^2(dp)}{V_k} \quad \overset{approx.}{\sim} \quad \chi^2\left(\frac{T + \sum_{k=0}^{T-1} \frac{\mu_k^2}{V_k}}{T + 2\sum_{k=0}^{T-1} \frac{\mu_k^2}{V_k}}\right).$$

Proof. These results follow from definitions and basic properties of the non central χ^2 distributions, see Kendall & Stuart (1961). ■

3. The second order realized variance

We have the following asymptotic and main result

Theorem 5 *Consider the sequence of squared values*

$$\tilde{r}_j^2 = \frac{1}{T}\omega_{Tk}^2(dp) = \left[2p\left[\frac{k+1/2}{T}\right] - p\left[\frac{k}{T}\right] - p\left[\frac{k+1}{T}\right]\right]^2,$$

$k = 0, \cdots, T-1$.

Define the second order realized variance as SORV=$\sum_{j=0}^{T-1}\tilde{r}_j^2$.

Then

$$\frac{\sqrt{T}\left[SORV - \int_0^1 \sigma^2(t)\,dt\right]}{\sqrt{2\int_0^1 \sigma^4(t)\,dt}} \quad \xrightarrow{d} \quad N(0,1)$$

Proof. The statement of the theorem follows from Lemma 2 and the Lindeberg (or Liapunov) Central Limit Theorem (see for example Spanos (1986)) for non-i.i.d. independent variates. ■

Remark:

Note the analogy to the similar result for RV (Barndorff-Nielsen & Shephard 2002). The analogy is based on the following, which is easily verified

Since

$$\tilde{r}_j = \frac{1}{\sqrt{T}}\omega_{Tj}(\mathrm{d}p) = \left[2p\left[\frac{j+1/2}{T}\right] - p\left[\frac{j}{T}\right] - p\left[\frac{j+1}{T}\right]\right],$$

we can write

$$\tilde{r}_j = r_{2j} - r_{2j+1},$$

where

$$r_{2j} = p\left[\frac{j+1/2}{T}\right] - p\left[\frac{j}{T}\right],$$

and so

$$r_{2j+1} = p\left[\frac{j+1}{T}\right] - p\left[\frac{j+1/2}{T}\right].$$

r_{2j} is the jth return over a $\Delta = \frac{1}{2T}$ interval.

Note that $\sqrt{T}\, r_{2j}$ and $\sqrt{T}\, r_{2j+1}$ coincide with the definitions of x_{1j} and x_{2j} respectively, given on page 4.

The second order realized variance estimator SORV is then

$$\begin{aligned} SORV = \sum_{k=0}^{T-1} \tilde{r}_k^2 &= \sum_{k=0}^{T-1} \left[r_{2k} - r_{2k+1}\right]^2 \\ &= r_0^2 + r_{2T}^2 + 2\sum_{k=1}^{T-1} r_{2k}^2 - 2\sum_{k=0}^{T-1} r_{2k} r_{2k+1}. \end{aligned}$$

And the realized variance estimator, correspondingly is

$$RV = \sum_{k=0}^{T-1} \left[r_{2k} + r_{2k+1}\right]^2.$$

Also note the analogy to the Fourier analysis method of Barucci & Reno (2002). They estimate integrated variance by

$$\frac{1}{2\pi}\int_0^1 \sigma^2(t)\mathrm{d}t = \lim_{n\to\infty} \frac{\pi}{2n} \sum_{s=1}^{n} \left(a_s^2(dp) + b_s^2(dp)\right). \tag{9}$$

Originally the method in the present paper was thought as an alternative to the Fourier method as it is also based on some transform: In the present context we

actually estimate by the wavelet transform

$$\int_0^1 \sigma^2(t)\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \omega_{Tk}^2(\mathrm{d}p).$$

4. **Monte Carlo Simulations**

We simulate

$$\begin{aligned} \mathrm{d}p_t &= \sigma_t \mathrm{d}W_{1t} \\ \mathrm{d}\ln\sigma_t^2 &= \theta(\omega - \ln\sigma_t^2)\mathrm{d}t + \eta\mathrm{d}W_{2t}, \end{aligned}$$

where W_{1t} and W_{2t} are uncorrelated Wiener processes. Thus spot log-variance evolves as a mean reverting Ornstein-Uhlenbeck process with mean ω, mean reversion parameter θ, and volatility η.

4.1. *Simulation design*

We set the parameters as follows, see (Andersen, Benzoni, and Lund (2002)):

$(\theta, \omega, \eta) = (0.032, -0.103, 0.115)$, and $p_0 = 0$, $\ln\sigma_0^2 = \omega$.

$T = 5,000$ daily replications are generated from this model. We divide the day into 23,400 seconds (which corresponds to 6.5 hours of a working day). The results are reported for every sampling frequency M that divides 23,400 evenly. This corresponds to 72 different frequencies.

Figures 1-7 show smoothed densities of the centered RV and the centered SORV (both centered by the true integrated variance).

The graphs clearly support the theoretical findings that RV and SORV have similar asymptotic variances.

To assess the performances with regard to the root mean squared error (RMSE), the following graph shows the RMSE as a function of a selection of the 72 different frequencies. There is no difference, reflecting that also in that respect, RV and SORV perform equally well.

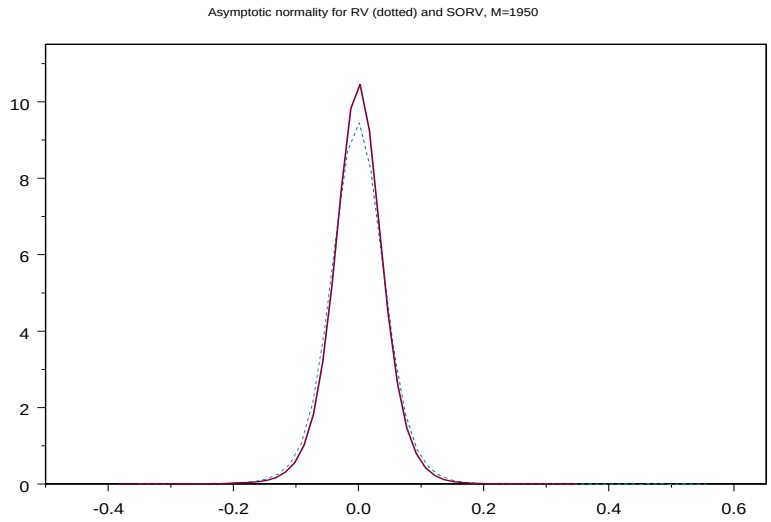

Figure 1: Comparison of distributions of RV and SORV for $M = 1950$

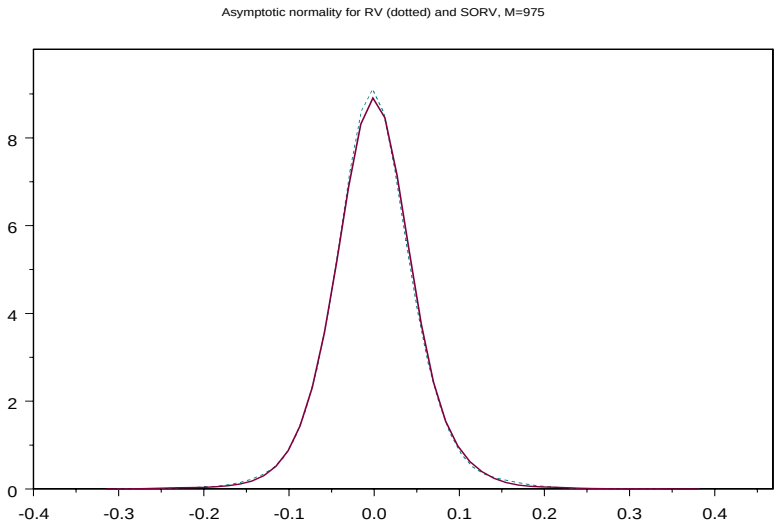

Figure 2: Comparison of distributions of RV and SORV for $M = 975$

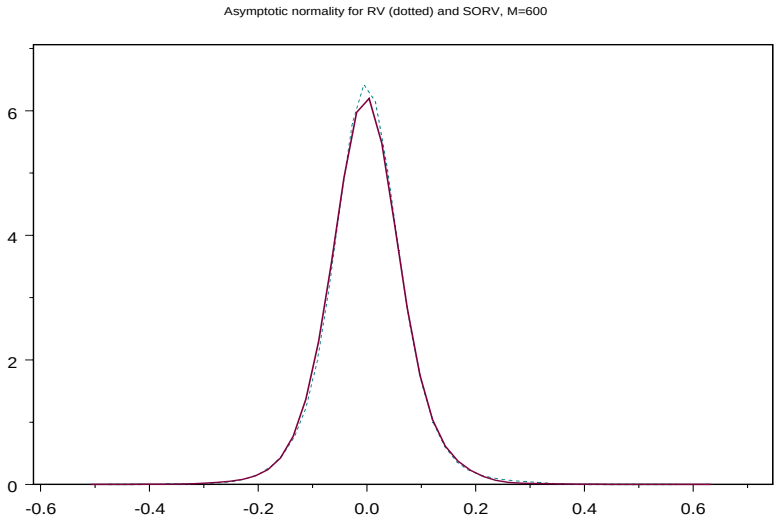

Figure 3: Comparison of distributions of RV and SORV for $M = 600$

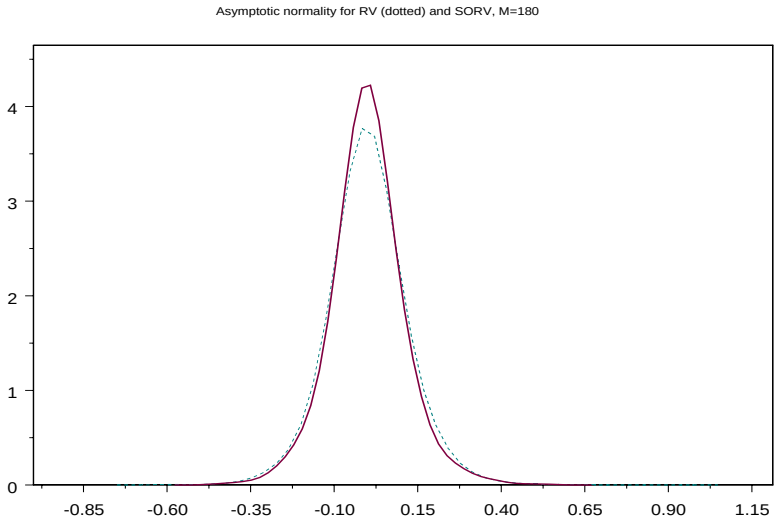

Figure 4: Comparison of distributions of RV and SORV for $M = 180$

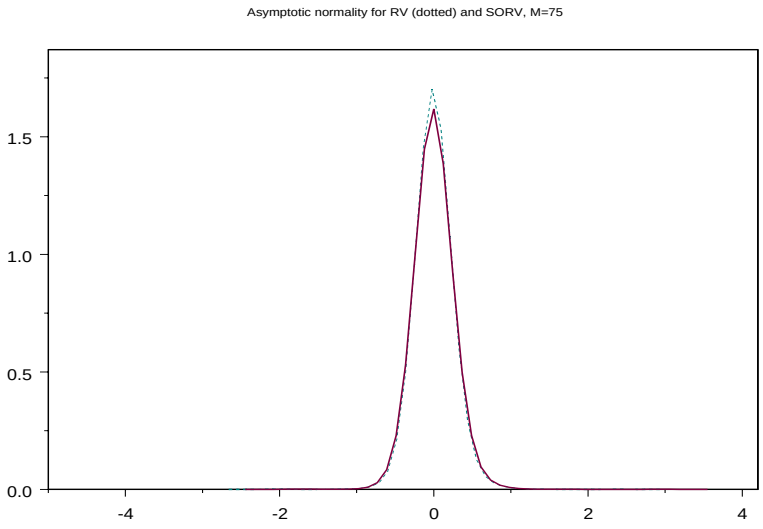

Figure 5: Comparison of distributions of RV and SORV for $M = 75$

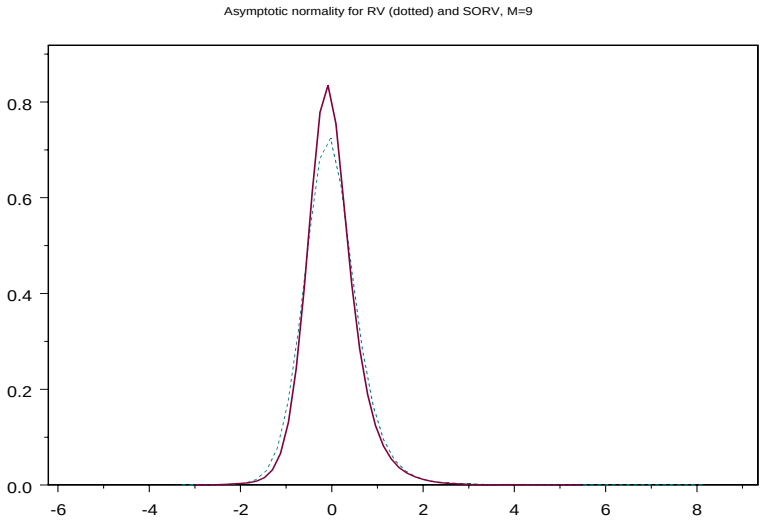

Figure 6: Comparison of distributions of RV and SORV for $M = 9$

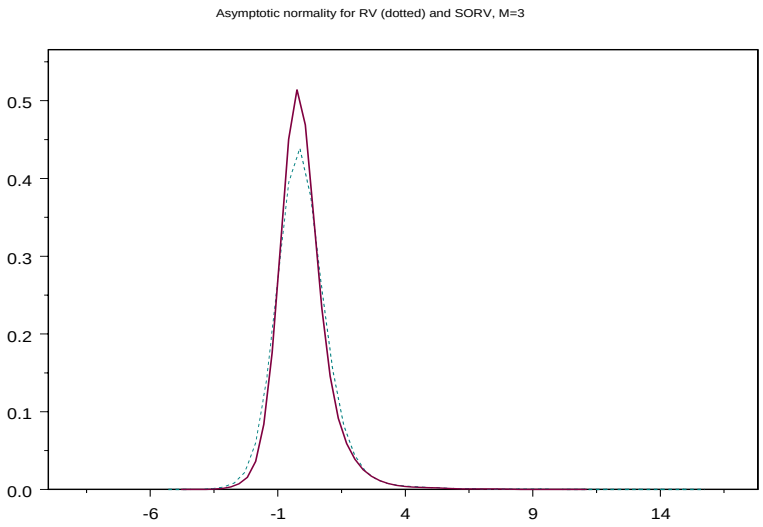

Figure 7: Comparison of distributions of RV and SORV for $M = 3$

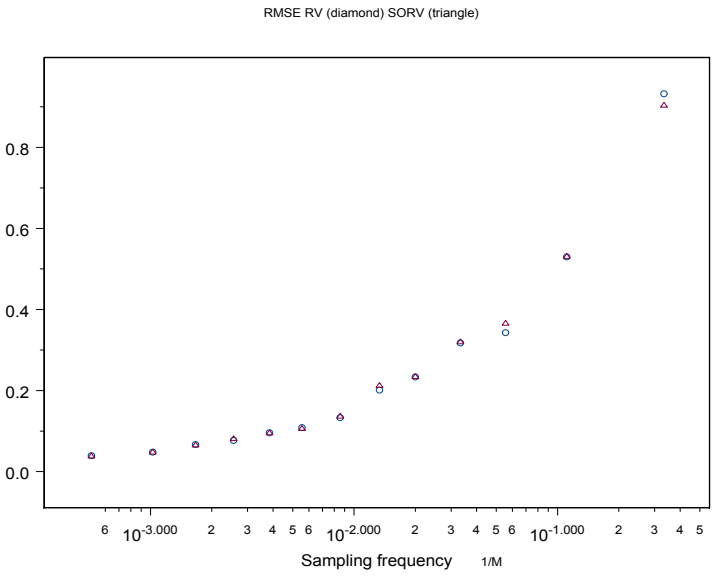

Figure 8: Comparison of RMSE of RV and SORV

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