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# Altruism in Networks

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*Abstract*: We provide the first theoretical analysis of altruism in networks. Agents are embedded in a fixed, weighted network and care about their direct friends. Given some initial distribution of incomes, they may decide to support their poorer friends. We study the resulting non-cooperative transfer game. Our analysis highlights the importance of indirect gifts, where an agent gives to a friend because his friend himself has a friend in need. We uncover four main features of this interdependence. First, we show that there is a unique profile of incomes after transfers, for any network and any utility functions. Uniqueness in transfers holds on trees, but not on arbitrary networks. Second, there is no waste in transfers in equilibrium. In particular, transfers flow through indirect paths of highest altruistic strength. Third, a negative shock on one agent cannot benefit others and tends to affect socially closer agents first. In addition, an income redistribution that decreases inequality ex-ante can increase inequality ex-post. Fourth, altruistic networks decrease income inequality. In contrast, more altruistic or more homophilous networks can increase inequality.

*Keywords*: private transfers, social networks, altruism, income redistribution, income inequality.

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# I. Introduction

Private transfers of money, time and other resources represent a large portion of our economies. For instance, aggregate private transfers received are estimated at 23% of GDP for Spain in 2000, 30% for the US in 2003, and 54% for the Philippines in 1999 (Lee & Donehover 2011). In developing countries, remittances alone often represent a significant portion of these transfers. Remittances received in 2009 are evaluated at 12% of GDP for the Philippines, 19% for Honduras and 25% for Lesotho (Worldbank 2011). Identifying the ultimate motives behind these transfers is not an easy empirical task. Still, introspection as well as an extensive empirical literature show that altruism is a main motivation (De Weerdt & Fafchamps 2011, Foster & Rosenzweig 2001, Leider et al. 2009). Individuals give to others they care about. Moreover, and as increasingly recognized by economists, individuals have different social neighborhoods (Jackson 2008). Thus, private transfers often flow through *altruistic networks*.

In this paper, we provide the first theoretical analysis of altruism in networks. We assume that agents are embedded in a fixed, weighted network and care about their direct friends. Given some initial distribution of incomes, agents may decide to support their poorer friends. Incentives to give are intricately linked to the network structure. Gifts made in one part of the network may depend on gifts received or made in other parts. We characterize the Nash equilibria of this non-cooperative transfer game.

Our analysis highlights the importance of indirect gifts. In equilibrium, an agent may give to a friend because his friend himself has a friend in need. Transfers, and incomes after transfers, ultimately depend on the whole network structure. We uncover four main features of this interdependence.

Our first main result establishes *the uniqueness of incomes after transfers*, for any network and any utility functions.[1] This extends the uniqueness result obtained by Arrow (1981) for groups to arbitrary weighted networks. In contrast, there are typically multiple

[1] Utility functions may differ between agents. They must be twice continuously differentiable, strictly increasing, strictly concave and satisfy an assumption guaranteeing that an agent never gives to a richer friend, see Section II.

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equilibria in transfers. We show that uniqueness in transfers holds on trees, which means that multiplicity is related to the presence of cycles in the network. We also characterize the mathematical structure of the equilibrium set.

Second, our analysis reveals *a principle of economy in transfers* at work in altruistic networks. The interplay of individual giving decisions somehow eliminates waste in transfers. We derive two formal results in support of this intuition. First, we show that transfers must flow through paths of highest altruistic strength. When all links have the same strength, it means that transfers flow through shortest paths of the network. Thus, transfers cannot take roundabout ways to connect two agents. Second, we find that, conditional on equilibrium incomes, equilibrium transfers must minimize a weighted sum of transfers. When all links have the same strength, equilibrium transfers simply minimize the aggregate transfer needed to reach equilibrium incomes.

Third, we study how incomes after transfers depend on initial incomes. We establish *a natural monotonicity property of altruistic networks.* If an agent suffers a loss in initial income, his income after transfers decreases while the income of every other agent either decreases or stays unchanged. In altruistic networks, one agent cannot gain from the loss of another. Moreover, the shock affects socially closer agents first. We then look at income redistributions. We find that *a Pigou-Dalton redistribution from richer to poorer can end up increasing inequality.* This happens when the initial redistribution removes resources from an agent playing an important supporting role in his local neighborhood. This result may have important implications for the design of public policies in altruistic societies.

Fourth, we study how income inequality depends on the structure of the network. We first show that *altruistic networks decrease inequality*, in the sense of second-order stochastic dominance. Furthermore, an increase in altruism tends to reduce the maximal income spread. However, an increase in altruism can also lead to an increase in income variance. New links can help relieve financial demands on one part of the network; this may improve the situation of rich agents and may increase variance. We then study the impact of homophily, the tendency of similar people to be connected and a key structural feature of social networks (McPherson, Smith-Lovin & Cook 2001). We find that homophily with

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respect to income tends to generate more inequality. Moreover, this relation is subject to "small-world" effects. The first few links connecting the poor and the rich can have a dramatic impact on inequality reduction. This is a consequence of indirect gifts, as the connected poor help out unconnected ones.

Finally, we obtain further results when preferences satisfy Constant Absolute Risk Aversion (CARA). We uncover the existence of a potential function in that case and provide a characterization of Nash equilibria as solutions of a convex minimization program.

Our analysis contributes to at least three literatures concerned, respectively, with altruism, private transfers, and social networks.

Our paper, first, introduces social networks to the economics of altruism initiated by Becker (1974) (Kolm & Mercier Ythier 2006). Most analysis in this literature consider altruistic individuals interacting in pairs (Alger & Weibull 2010, Bernheim & Stark 1988, Stark 1995). In a pioneering study, Arrow (1981) analyzes the transfer game for groups of any size and when every agent cares equally for everyone else. We essentially extend his analysis to arbitrary weighted networks. We show that income uniqueness holds in general. In contrast, many other features of the equilibria highlighted by Arrow are not robust to changes in the social structure. For instance an agent can both give and receive in an altruistic network in equilibrium, and income rankings can be reversed. In addition, the techniques developed by Arrow to solve his model do not extend to arbitrary networks. We discuss his results and the relationship between his analysis and ours in more detail in Section III. Overall, we find that the network has a first-order impact on outcomes in an altruistic society.

Second, our analysis contributes to the economic literature on private transfers. Following Townsend (1994), empirical studies typically find that risk sharing in rural communities in developing countries is good but imperfect (Mazzocco & Saini 2012). One mechanism explored by economists to explain this finding is that households trade informal insurance possibilities but that trade is constrained by a lack of commitment ability (Dubois, Jullien & Magnac 2008, Ligon, Thomas & Worrall 2002). However, altruism provides an alternative explanation (Cox & Fafchamps 2008). Distinguishing altruism from exchange

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is empirically challenging (Arrondel & Masson 2006). Still, exchange cannot explain pure redistribution when money always flow in the same direction.[2] In contrast, altruism can explain redistribution as well as good but imperfect risk-sharing. And indeed, altruism does seem to be a main motive behind transfers.[3]

In addition, detailed studies on gifts and loans find that transfers generally flow through social networks (De Weerdt & Dercon 2006, Fafchamps & Gubert 2007, Fafchamps & Lund 2003). People help their family, friends and neighbors in case of need. Building on these insights, a recent theoretical literature has studied risk sharing in networks where transfers, by assumption, must flow through social links.[4] The study most related to ours is Ambrus, Mobius & Szeidl (2013). Authors consider a fixed, weighted network and assume that links have monetary values which can be used as social collateral. These values limit the amount of money that can flow through links. They characterize the Pareto-constrained risk-sharing arrangements. In contrast, we characterize the Nash equilibria of a non-cooperative transfer game. Transfer schemes in their setup and in ours share some common features. In particular, they both satisfy monotonicity with respect to incomes and the fact that socially closer agents are more affected by a shock. As discussed in Ambrus, Mobius & Szeidl (2013), these two features are consistent with empirical findings of Angelucci & De Giorgi (2009) and Angelucci, De Giorgi & Rasul (2012). The two models also yield different predictions.[5] In Ambrus, Mobius & Szeidl (2013), small shocks are perfectly insured while large shocks are not. And small shocks can trigger arbitrarily long transfer chains. In contrast, in altruistic networks small shocks usually do not elicit network support while large shocks do. And transfer chains are of bounded length.

To sum up, we develop and study one of the first economic models able to explain good

[2]Exchange is consistent with redistribution in expectations, but not with redistribution in all states of the world.

[3]Empirical studies finding evidence of altruism include Agarwal & Horowitz (2002), De Weerdt & Fafchamps (2011), Foster & Rosenzweig (2001), Kazianga (2006), Leider et al. (2009), Ligon & Schechter (2012), Mitrut & Nordblom (2010).

[4]Bloch, Genicot & Ray (2008) and Bramoullé & Kranton (2007a,b) analyze network stability in various risk sharing contexts. In contrast, we take the network as given in our approach.

[5]A further distinction is that, in a network with $n$ agents and $L$ links, Pareto-constrained risk sharing arrangements depend on $L+n-1$ parameters: the link values and the Pareto weights. In contrast, incomes after transfers in altruistic networks only depend on $L$ parameters: the links' altruistic strengths.

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but imperfect risk-sharing, redistribution and transfer flows through social networks.

Applied researchers have recently explored yet another motive that could explain private transfers. People may give to others because they conform to redistributive social norms (Baland, Guirkinger & Mali 2011). We note here that our results ultimately depend on the best-replies rather than the preferences. Therefore, our whole analysis also applies to a society prescribing to act towards family and friends *as if* truly altruistic.

Finally, our paper contributes to the literature on games played on fixed networks (Ballester, Calvó-Armengol & Zenou 2006, Galeotti et al. 2008, Bramoullé, Kranton & D'amours 2013). This literature has, so far, focused on one-dimensional strategies. In contrast, an individual strategy here is a vector of transfers. This makes our analysis quite different from the existing studies. Our results for CARA utility functions confirm the interest of potential functions to study network games emphasized by Bramoullé, Kranton & D'amours (2013).

The remainder of the paper is organized as follows. We introduce the model and establish existence in Section II. We illustrate the effect of the network in Section III. We analyze equilibrium incomes and transfers in Section IV. We study comparative statics with respect to initial incomes and to the network in Section V and we conclude in Section VI.

## II. The model

We consider a community of $n \geq 2$ agents. Agents make private transfers to each other. Agent $i$ has initial income $y_i^0 \geq 0$ and may give $t_{ij} \geq 0$ to agent $j$. The collection of transfers defines a $n$ by $n$ matrix $\mathbf{T}$ with non-negative entries. By convention, $t_{ii} = 0$. Income after transfers, $y_i$, is equal to

$$y_i = y_i^0 - \sum_j t_{ij} + \sum_j t_{ji} \tag{1}$$

where $\sum_j t_{ij}$ represents total gifts made *by* $i$ and $\sum_j t_{ji}$ total gifts made *to* $i$. We assume that $y_i \geq 0$. Agents are budget-constrained: total gifts made must be lower than or equal

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to the sum of initial income and of total gifts received. Define $S$ as the set of admissible transfer profiles: $S = \left\{\mathbf{T} \in \mathbb{R}_+^{n^2} : \forall i, t_{ii} = 0 \text{ and } \sum_j t_{ij} \leq y_i^0 + \sum_j t_{ji}\right\}$. Since aggregate gifts made are equal to aggregate gifts received, $\sum_i y_i = \sum_i y_i^0$. Aggregate income is constant and redistributed through private transfers.

We assume that agents care about each other. Preferences have a private and a social component. Agent $i$'s private, or material, preferences are represented by utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$. We assume that $u_i$ is twice continuously differentiable and satisfies $u_i' > 0$ and $u_i'' < 0$. Agent $i$'s social, or altruistic, preferences are represented by utility function $v_i : \mathbb{R}_+^n \to \mathbb{R}$ such that

$$v_i(y_i, \mathbf{y}_{-i}) = u_i(y_i) + \sum_j \alpha_{ij} u_j(y_j) \tag{2}$$

and $\forall i, j, 0 \leq \alpha_{ij} < 1$. When $\alpha_{ij} > 0$, $i$ cares about $j$'s material well-being and the size of the coefficient measures the strength of the altruistic linkage.[6] By convention, $\alpha_{ii} = 0$. The collection of the bilateral coefficients $\alpha_{ij}$ defines a *directed and weighted altruistic network* $\boldsymbol{\alpha}$. We say that this network is *binary* when all links have the same strength and $\alpha_{ij} \in \{0, \alpha\}$.

We make the following joint assumption on private utilities and altruistic coefficients:

$$\forall i, j, \forall y, u_i'(y) > \alpha_{ij} u_j'(y) \tag{3}$$

This condition guarantees that an agent never gives to a richer friend: $t_{ij} > 0 \Rightarrow y_i > y_j$. This implies, in particular, that the budget constraint never binds and $y_i > 0$ except when $i$ has no income to give ($y_i^0 = \sum_j t_{ji} = 0$). Assumption (3) provides a counterpart in a network context to the assumption of "selfish preferences" in Arrow (1981, p. 203). Under common preferences when $\forall i$, $u_i = u$, Assumption (3) reduces to: $\forall i, j$, $\alpha_{ij} < 1$.

The collection of agents, utilities $v_i$, and transfers $t_{ij}$ define a non-cooperative game with joint strategies. Our main objective is to study the Nash equilibria of this transfer game. Agents simultaneously choose their transfers to maximize their altruistic utilities

[6] Alternatively, we could consider interdependence in altruistic utilities: $v_i = u_i + \sum a_{ij} v_j$. Assume that $\mathbf{I} - \mathbf{A}$ is invertible, where $\mathbf{I}$ is the identity matrix, and let $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$. Then, $\mathbf{v} = \mathbf{B}\mathbf{u}$ and setting $\alpha_{ij} = b_{ij}/b_{ii}$ leads us back to our formulation. Our analysis applies as long as $\forall i \neq j, 0 \leq b_{ij} < b_{ii}$.

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subject to their budget constraint. A Nash equilibrium is a matrix of transfers $\mathbf{T} \in S$ such that $\forall i, \forall \mathbf{T}'_i \in \mathbb{R}^n_+$ such that $(\mathbf{T}'_i, \mathbf{T}_{-i}) \in S$, $v_i(\mathbf{T}_i, \mathbf{T}_{-i}) \geq v_i(\mathbf{T}'_i, \mathbf{T}_{-i})$.

We next present a useful reformulation of equilibrium conditions. Observe that $v_i$ is concave in $\mathbf{T}_i$ for any $\mathbf{T}_{-i}$. This implies that the first-order conditions of $i$'s utility maximization problem are necessary and sufficient. Since the budget constraint never binds, we have:

**Lemma 1.** *A matrix of transfers* $\mathbf{T}$ *is a Nash equilibrium of the transfer game iff (1)* $\forall i, j$, $u'_i(y_i) \geq \alpha_{ij} u'_j(y_j)$ *and (2)* $\forall i, j$ *such that* $t_{ij} > 0$, $u'_i(y_i) = \alpha_{ij} u'_j(y_j)$.

In equilibrium an agent cannot be much richer than any of his friends. And a positive transfer must be such that the ratio of the giver's marginal utility on the receiver's is precisely equal to the altruistic coefficient. When $\alpha_{ij} = 0$, the first condition is trivially satisfied while the second condition is never satisfied. Thus, agents only give to others they care about: $t_{ij} > 0 \Rightarrow \alpha_{ij} > 0$. Still, as we will see below, agents may end up being affected by friends of friends and by others far away in the network.

To illustrate, consider common preferences satisfying Constant Absolute Risk Aversion, or CARA: $u_i(y) = -e^{-Ay}/A$ with $A > 0$. Lemma 1 becomes: (1) $\forall i, j$, $y_i \leq y_j - \ln(\alpha_{ij})/A$ and (2) $\forall i, j$ such that $t_{ij} > 0$, $y_i = y_j - \ln(\alpha_{ij})/A$. The difference between two friends' incomes cannot be greater than a threshold value, which is independent on the income levels. Alternatively, suppose that preferences satisfy Constant Relative Risk Aversion, or CRRA: $u_i(y) = y^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ or $u_i(y) = \ln(y)$ for $\gamma = 1$. Lemma 1 becomes: (1) $\forall i, j$, $y_i/y_j \leq \alpha_{ij}^{-1/\gamma}$ and (2) $\forall i, j$ such that $t_{ij} > 0$, $y_i/y_j = \alpha_{ij}^{-1/\gamma}$. Now, the ratio of friends' incomes cannot be greater than a threshold value.

We note that the transfer game exhibits a complex pattern of strategic interactions and externalities. The optimal $t_{ij}$ is weakly decreasing in $t_{kj}$ but weakly increasing in $t_{jk}$. An agent tends to reduce his gift to a friend if this friend receives more gifts from others. In contrast, he tends to increase his gift when his friend makes more gifts himself. Therefore, gifts to an agent from different givers are strategic substitutes while gifts to and from an agent are strategic complements. Next, suppose that $j$ is a friend of $i$ but $k$ is not: $\alpha_{ij} > 0$ and $\alpha_{ik} = 0$. Then, $v_i$ is decreasing in $t_{jk}$ but increasing in $t_{kj}$. Externalities are

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positive with respect to some transfers and negative with respect to others. Thus, strategic interactions and externalities depend on the network structure in complex ways.

In contrast, this game captures relatively simple rules of behavior. When preferences are common and the network is binary, each agent provides financial support to his poorest friends and, depending on his financial ability, brings them all up to a neighborhood-specific minimal income. More generally, an agent $i$ seeks to reduce the maximal level of altruistic marginal utility $\alpha_{ij}u'_j$ in his neighborhood. What happens when everyone is simultaneously behaving in this way, however, is not obvious. Our objective is to analyze the interplay of altruistic behavior on an arbitrary network.

To conclude this section, we establish existence. It follows from results due to Mercier Ythier (1993, 2006). In Appendix, we derive an alternative proof based on the theory of concave games (Rosen 1965).

**Proposition 1.** *For any altruistic network and any utility functions, a Nash equilibrium of the transfer game exists.*

One difficulty in showing existence is that the set of admissible strategies $S$ is unbounded. To address this issue, we show that transfer networks in equilibrium are acyclic (see Proposition 4 below) and that aggregate transfers in acyclic networks are bounded from above. Once this bound is established, existence follows from classical fixed point results.

## III. An illustration

In this section, we illustrate the impact of the altruistic network on transfers and incomes. We assume common preferences and contrast two benchmark structures: complete graphs and stars.

Consider complete graphs first. Everyone cares equally about everyone else. Formally, $\alpha_{ij} = \alpha < 1$, $\forall i \neq j$. This case is covered by Arrow (1981)'s analysis.[7] Our discussion

[7]More precisely, Arrow (1981) considers utility functions of the form $v_i(\mathbf{y}) = u_i(y_i) + \sum_{j\neq i} w(y_j)$. The two frameworks coincide when $\forall i, u_i = u$ and $w = \alpha u$. Our results also apply to complete networks

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follows his, and we refer to his paper for detailed explanations. On the complete graph, equilibrium conditions can be expressed through a unique endogenous variable: minimal income $y_{\min}$. To see why, denote by $\hat{y}_{\max}$ the income level such that $u'(\hat{y}_{\max}) = \alpha u'(y_{\min})$. Condition (1) of Lemma 1 implies that $\forall i, u'(y_i) \geq \alpha u'(y_{\min})$ and hence $y_i \leq \hat{y}_{\max}$. Condition (2) means that if $t_{ij} > 0$, then $u'(y_i) = \alpha u'(y_j) \leq \alpha u'(y_{\min}) = u'(\hat{y}_{\max})$ and hence $y_i = \hat{y}_{\max}$ and $y_j = y_{\min}$. There are two cases. If $\forall i, j, u'(y_i^0) \geq \alpha u'(y_j^0)$, there is no transfer in equilibrium. If, on the other hand, $u'(y_i^0) < \alpha u'(y_j^0)$ for some pair $i, j$, then an equilibrium is characterized by the following properties. If $y_i^0 > \hat{y}_{\max}$, then $y_i = \hat{y}_{\max}$, if $y_i^0 < y_{\min}$, then $y_i = y_{\min}$ and otherwise $y_i = y_i^0$. Minimal income $y_{\min}$ solves the following equation

$$\sum_{i:y_i^0 > \hat{y}_{\max}} (y_i^0 - \hat{y}_{\max}) = \sum_{i:y_i^0 < y_{\min}} (y_{\min} - y_i^0) \tag{4}$$

This accounting equality says that aggregate gifts made are equal to aggregate gifts received. Since the left hand side is decreasing in $y_{\min}$ while the right hand side is increasing in $y_{\min}$, this equation has a unique solution and equilibrium income $y_i$ is unique.

The equilibrium outcome can be described as a system of minimal income for the poor, paid for by the rich. Agents act as if any income above $\hat{y}_{\max}$ is contributed to a fund which serves to bring the income of the poorest agents up to $y_{\min}$. Who gives to whom precisely does not matter and there are usually many equilibria in transfers. Let us highlight two further properties of these equilibria. An agent never gives and receives at the same time. And rankings in the initial income distribution cannot be reversed: $y_i^0 \geq y_j^0 \Rightarrow y_i \geq y_j$. We see below that these two properties may not hold when agents interact through a network.

Consider star graphs next. Agent 1, the center, cares equally about all other agents and peripheral agents only care about the center. Formally, $\alpha_{1i} = \alpha_{i1} = \alpha < 1$, $\forall i \neq 1$ and $\alpha_{ij} = 0$ if $i, j \neq 1$. The equilibrium conditions on stars can also be expressed through a unique parameter, namely the center's equilibrium income $y_1$. Denote by $\hat{y}_R$ and $\hat{y}_P$ the income levels such that $u'(\hat{y}_R) = \alpha u'(y_1)$ and $u'(y_1) = \alpha u'(\hat{y}_P)$. Consider a peripheral agent $i \neq 1$ and rewrite the equilibrium conditions of Lemma 1. There are three cases. If

with heterogeneous preferences: $v_i(\mathbf{y}) = u_i(y_i) + \alpha \sum_{j \neq i} u_j(y_j)$, which are not covered by Arrow (1981)'s analysis.

$\hat{y}_P \leq y_i^0 \leq \hat{y}_R$, then $t_{i1} = t_{1i} = 0$ and $y_i = y_i^0$. If $y_i^0 > \hat{y}_R$, then $t_{i1} > 0$ and $y_i = \hat{y}_R$. If $y_i^0 < \hat{y}_P$, then $t_{1i} > 0$ and $y_i = \hat{y}_P$. The center's income $y_1$ solves the following equation

$$y_1 = y_1^0 + \sum_{i:y_i^0 > \hat{y}_R} (y_i^0 - \hat{y}_R) - \sum_{i:y_i^0 < \hat{y}_P} (\hat{y}_P - y_i^0) \tag{5}$$

Equation (5) represents a reformulation of accounting equality (1) for the center. It says that the center's final income must be equal to his initial income plus what he receives minus what he gives. Observe that the left hand side is now increasing in $y_1$ while the right hand side is decreasing in $y_1$. Therefore, this equation has a unique solution and there is now a unique equilibrium in transfers.

This characterization allows us to illustrate two key properties of altruism in networks: indirect giving and ranking reversal. Consider the numerical example presented in Figure 1 with common CARA preferences $u_i(y) = -e^{-Ay}/A$ and $-\ln(\alpha)/A = 1$. The upper panel depicts initial incomes. The center is richest with an income of 8. His left neighbor is the second richest with an income of 7. If society were composed of these two agents only, there would not be any transfer. However, the agent on the right is poor with an income of 0. The lower panel depicts equilibrium incomes and transfers. The left agent gives 1 to the center, who gives 4 to the right agent and equilibrium incomes are 6, 5 and 4. Here, the center receives and gives at the same time. The gift of the agent on the left is indirect. He does not give because his friend is initially in need but because his friend has a friend in need. In a complex structure, these indirect gifts will depend in intricate ways on the architecture of the network. In addition, the agent with the largest initial income does not have the largest income after transfers. Income rankings can be reversed and the position in the altruistic network is now an important determinant of equilibrium income. In the remainder of the paper, we analyze altruism in arbitrary networks.



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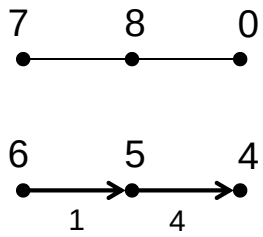


Figure 1: An example of altruistic network

## IV. Equilibrium incomes and transfers

### A. Income uniqueness

In section, we characterize equilibrium incomes and transfers. We start by presenting our first main result: the uniqueness of incomes after transfers. Our result extends Arrow (1981)'s finding to arbitrary weighted networks.

**Theorem 1.** *For any altruistic network and any utility functions, there is a unique profile of equilibrium incomes.*

Proof: Consider two equilibria $\mathbf{T}$ and $\mathbf{T}'$ with income distributions $\mathbf{y}$ and $\mathbf{y}'$. Assume that $\mathbf{y} \neq \mathbf{y}'$. Without loss of generality, assume that $y'_i > y_i$ for some $i$. Define $U = \{i : y'_i > y_i\}$. Our proof unfolds in two steps.

We show, first, that if $i \in U$ and $t_{ij} > 0$ or $t'_{ji} > 0$ then $j \in U$. Suppose that $y'_i > y_i$ and $t_{ij} > 0$. By Lemma 1, $u'_i(y_i) = \alpha_{ij} u'_j(y_j)$ and $u'_i(y'_i) \geq \alpha_{ij} u'_j(y'_j)$. Since $u'_i$ is strictly decreasing, $u'_i(y_i) > u'_i(y'_i)$ and hence $\alpha_{ij} u'_j(y_j) > \alpha_{ij} u'_j(y'_j)$, which means that $y'_j > y_j$. Suppose next that $t'_{ji} > 0$. Then, $u'_j(y'_j) = \alpha_{ji} u'_i(y'_i)$ and $u'_j(y_j) \geq \alpha_{ji} u'_i(y_i)$. Since $\alpha_{ji} > 0$, $\alpha_{ji} u'_i(y_i) > \alpha_{ji} u'_i(y'_i)$ and hence $u'_j(y_j) > u'_j(y'_j)$, and $y'_j > \hat{y}_j$.

Next, given two sets $U$ and $V$, define $t_{U,V} = \sum_{i \in U, j \in V} t_{ij}$ as aggregate transfers from agents in $U$ to agents in $V$. Let $N$ denote the set of all agents. Observe that $\sum_{i \in U} y_i = \sum_{i \in U} (y_i^0 - t_{i,U} - t_{i,N-U} + t_{U,i} + t_{N-U,i})$. Since $\sum_{i \in U} (t_{U,i} - t_{i,U}) = 0$, this yields

$$\sum_{i \in U} y_i = \sum_{i \in U} y_i^0 - t_{U,N-U} + t_{N-U,U}$$

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This accounting equation says that aggregate final income within is equal to aggregate initial income within minus gifts given outside plus gifts received within.

Finally, observe that statements in our first step mean that $t_{U,N-U} = 0$ while $t'_{N-U,U} = 0$. This implies that $\sum_{i\in U} y_i = \sum_{i\in U} y_i^0 + t_{N-U,U} \geq \sum_{i\in U} y_i^0$ while $\sum_{i\in U} y'_i = \sum_{i\in U} y_i^0 - t'_{U,N-U} \leq \sum_{i\in U} y_i^0$. Therefore, $\sum_{i\in U} y'_i \leq \sum_{i\in U} y_i$. But $i \in U \Leftrightarrow y'_i > y_i$ which means that $\sum_{i\in U} y'_i > \sum_{i\in U} y_i$, which establishes a contradiction. □

To show this result, we assume that two different income profiles exist. We combine Lemma 1 with elementary flow techniques to generate a contradiction. Our proof is fundamentally distinct from Arrow (1981)'s constructive reasoning. Arrow builds the equilibria from the bottom up and then checks the uniqueness of income profiles. Whether we can find a general constructive proof remains an open question. We note that Arrow (1981)'s arguments for the complete graph do not extend to arbitrary networks because equilibrium conditions can generally not be expressed through a single parameter.[8] We also note that classical uniqueness properties, such as contraction or Rosen (1965)'s conditions, do not apply here due to equilibrium multiplicity. We do provide a constructive argument for CARA preferences, see Section IV.C below. In any case, Theorem 1 means that there is no need to determine all Nash equilibria to study incomes after transfers. Finding one equilibrium is enough.

### B. Transfers

We next characterize equilibrium transfers. This section has three parts. First, we analyze the structure of the equilibrium set. We show that uniqueness holds if the altruistic network is a tree and that, in general, the equilibrium set is compact and convex. Second, we look at the shape of the network of transfers. We show that this network is acyclic and that transfers must flow through paths of highest altruistic strength. Third, we show that, conditional on reaching equilibrium incomes, a Nash equilibrium must minimize a weighted sum of transfers. The last two results illustrate a general principle of economy in transfers at work in altruistic networks.

[8] In fact, stars and complete networks seem to be the only two structures for which this property holds.

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Equilibrium multiplicity on the complete graph is noted by Arrow (1981, p. 221): "The actual flows, who gives to whom, are not unique (...), but this nonuniqueness is not very interesting". However, the star's example shows that actual flows may be unique on incomplete networks. In addition, multiplicity does not mean that anything goes and understanding the structure of the equilibrium set is of some interest.

We show, first, that equilibrium uniqueness extends to trees. We say that an altruistic network is a tree if the undirected graph where $i$ and $j$ are linked when $\alpha_{ij} > 0$ or $\alpha_{ji} > 0$ has no cycles.

**Proposition 2.** *The transfer game has a unique Nash equilibrium if the altruistic network is a tree.*

Proof: Proceed by induction on the number of agents. Define $H_n$ as the proposition's statement when the number of agents is lower than or equal to $n$. Direct computations show that $H_2$ is true. Suppose that $H_{n-1}$ is true and consider a tree with $n$ nodes. There are two cases. Either the tree has an isolated node, and $H_n$ is true. Or it has a node who has a unique neighbor. In this case, let $i$ have $j$ as his only neighbor. Let $\mathbf{T}$ be an equilibrium and $\mathbf{T}_{-i}$ the submatrix obtained by removing the $i^{\text{th}}$ row and column. By Theorem 1, there is a unique profile of equilibrium income $\mathbf{y}$. By Lemma 1, there are three cases. If $y_i = y_j$, then $t_{ij} = t_{ji} = 0$. If $y_i > y_j$, then $t_{ji} = 0$ and $t_{ij} = y_i^0 - y_i \geq 0$. If $y_i < y_j$, then $t_{ij} = 0$ and $t_{ji} = y_i - y_i^0 \geq 0$. In all cases, $t_{ij}$ and $t_{ji}$ are uniquely determined. Then, note that $\mathbf{T}_{-i}$ is an equilibrium for initial incomes $\hat{\mathbf{y}}^0$ and network $\boldsymbol{\alpha}_{-i}$ where $\hat{y}_j^0 = y_j^0 - t_{ji} + t_{ij}$ and $\hat{y}_k^0 = y_k^0$, $\forall k \neq j$. By $H_{n-1}$, $\mathbf{T}_{-i}$ is uniquely determined and hence $H_n$ is true. □

Proposition 2 means that multiplicity of equilibria is related to the presence of cycles in the altruistic network. Our next result characterizes the mathematical structure of the equilibrium set for arbitrary networks.

**Proposition 3.** *For any altruistic network and any utility functions, the set of Nash equilibria is compact and convex .*

Proof: Consider a sequence of equilibria $\mathbf{T}_n$ converging towards $\mathbf{T}$. By condition (1) of Lemma 1, $\forall i, j$, $u_i'(y_i^n) \geq \alpha_{ij} u_j'(y_j^n)$. Taking the limit and by continuity of $u'$, $u_i'(y_i) \geq$

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$\alpha_{ij}u'_j(y_j)$. Next consider $i, j$ such that $t_{ij} > 0$. There exists $N$ such that $n > N \Rightarrow t^n_{ij} > 0$. By condition (2), $u'_i(y^n_i) = \alpha_{ij}u'_j(y^n_j)$ and hence $u'_i(y_i) = \alpha_{ij}u'_j(y_j)$. Thus, $\mathbf{T}$ is an equilibrium and the set of Nash equilibria is closed. By the proof of Proposition 1, it is also bounded and hence compact.

Consider $\mathbf{T}$ and $\mathbf{T}'$ two equilibria with income distributions $\mathbf{y}$ and $\mathbf{y}'$, and $\lambda \in [0, 1]$. Let $\mathbf{T}_\lambda = \lambda\mathbf{T} + (1-\lambda)\mathbf{T}'$, which defines a final income distribution $\mathbf{y}_\lambda$. Note that $\mathbf{y} = \mathbf{y}^0 - \mathbf{T}\mathbf{1} + \mathbf{T}^T\mathbf{1}$. Therefore, $\mathbf{y}_\lambda = \mathbf{y}^0 - (\boldsymbol{\lambda}\mathbf{T} + (\mathbf{1}-\boldsymbol{\lambda})\mathbf{T}')\mathbf{1} + (\lambda\mathbf{T} + (1-\lambda)\mathbf{T}')^T\mathbf{1} = \lambda\mathbf{y} + (1-\lambda)\mathbf{y}'$. By Theorem 1, $\mathbf{y} = \mathbf{y}'$ and hence $\mathbf{y}_\lambda = \mathbf{y}$. This means that condition (1) of Lemma 1 is satisfied. As for condition (2), suppose that $\lambda t_{ij} + (1-\lambda)t'_{ij} > 0$. This implies that $t_{ij} > 0$ or $t'_{ij} > 0$. In either case, it means that $u'_i(y_i) = \alpha_{ij}u'_j(y_j)$ and condition (2) is satisfied. □

An implication of Proposition 3 is that when uniqueness fails to hold, the number of equilibria is infinite. Let us next look at the shape of the transfer networks. A Nash equilibrium $\mathbf{T}$ can be viewed as a weighted directed network where $i$ and $j$ are linked if $i$ gives to $j$. Since $\alpha_{ij} = 0 \Rightarrow t_{ij} = 0$, the transfer graph is a subgraph of the original altruistic network. Consider a chain $C$ of agents of length $l$: $i_1$,$i_2$,...,$i_l$. Define the *altruistic strength* $\alpha_C$ of chain $C$ as the product of the bilateral altruistic coefficients: $\alpha_C = \alpha_{i_1 i_2}\alpha_{i_2 i_3}...\alpha_{i_{l-1} i_l}$. This strength is positive if and only if the chain is a path of the altruistic network and any agent in the chain cares about his successor.

**Proposition 4.** *There are no cycles in transfers, and transfers flow through paths of highest altruistic strength.*

Proof: Suppose that $i$ and $j$ are connected through a path $P$ in the transfer network: $t_{ii_2} > 0$, $t_{i_2 i_3} > 0$, ..., $t_{i_{l-1} j} > 0$. We have: $y_i > y_{i_2}$, ...,$y_{i_{l-1}} > y_j$ and hence $y_i > y_j$ and there is no cycle. Condition (2) of Lemma 1 means that $u'_i(y_i) = \alpha_{ii_2}u'_{i_2}(y_{i_2})$, $u'_{i_2}(y_{i_2}) = \alpha_{i_2 i_3}u'_{i_3}(y_{i_3})$, ..., $u'_{i_{l-1}}(y_{i_{l-1}}) = \alpha_{i_{l-1} j}u'_j(y_j)$. Successive substitutions yield $u'_i(y_i) = \alpha_{ii_2}\alpha_{i_2 i_3}...\alpha_{i_{l-1} j}u'_j(y_j)$ and hence $u'_i(y_i)/u'_j(y_j) = \alpha_P$. Next, consider an arbitrary chain $C$ connecting $i$ and $j$: $i_1 = i$, $i_2$, ..., $i_{l-1}$, $i_l = j$. Condition (1) of Lemma 1 means that $u'_i(y_i) \geq \alpha_{ii_2}u'_{i_2}(y_{i_2})$, ..., $u'_{i_{l-1}}(y_{i_{l-1}}) \geq \alpha_{i_{l-1} j}u'_j(y_j)$. Successive substitutions yield: $u'_i(y_i) \geq \alpha_{ii_2}\alpha_{i_2 i_3}...\alpha_{i_{l-1} j}u'_j(y_j)$ and hence $\alpha_P \geq \alpha_C$. Therefore, $P$ is a chain of highest altruistic strength. □

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In a binary network, all links have strength $\alpha$ and $\alpha_C = \alpha^l$ when $C$ is a path of length $l$ in the altruistic network. Proposition 4 yields:

**Corollary 1.** *When all links have the same altruistic strength, transfers flow through shortest paths of the altruistic network.*

Money flows in this context share some characteristics of water flows across a rugged landscape (Viessman & Lewis 2002). Water always follows paths of least resistance. Similarly, gifts always follow paths of highest strength. Chains of transfers never take roundabout ways to connect two agents. Informally, we could say that there is no waste in transfers. Our last result provides further support to this intuition. Suppose that equilibrium incomes $\mathbf{y}$ are known. Observe that any collection of transfers $\mathbf{T}$ such that $\mathbf{T1} - \mathbf{T}^t\mathbf{1} = \mathbf{y}^0 - \mathbf{y}$ allows agents to reach $\mathbf{y}$ from $\mathbf{y}^0$. Our next result clarifies the properties satisfied by equilibrium transfers within this set.

**Proposition 5.** *Take equilibrium incomes* $\mathbf{y}$ *as given. Then, equilibrium transfers* $\mathbf{T}$ *minimize* $\sum_{i,j} -\ln(\alpha_{ij})t_{ij}$ *subject to:* $\forall i,j, t_{ij} \geq 0$, $\forall i,j$*:* $\alpha_{ij} = 0$, $t_{ij} = 0$ *and* $\forall i, \sum_j t_{ij} - \sum_j t_{ji} = y_i^0 - y_i$.

Proof: The objective function and the constrained set are convex, therefore first-order conditions are necessary and sufficient. The lagrangian of this problem is $L = \sum_{i,j} -\ln(\alpha_{ij})t_{ij} + \sum_i \lambda_i(\sum_j t_{ij} - \sum_j t_{ji} - y_i^0 + y_i)$ where $\lambda_i$ is the lagrange multiplier associated with $i$'s accounting equality. Kuhn-Tucker first-order conditions are: $\forall i,j, \lambda_i - \lambda_j \geq \ln(\alpha_{ij})$ and $t_{ij} > 0 \Rightarrow \lambda_i - \lambda_j = \ln(\alpha_{ij})$. Next, take the logarithm in the conditions of Lemma 1. A transfer profile $\mathbf{T}$ is a Nash equilibrium iff (1) $\forall i,j, \ln(u_i'(y_i)) - \ln(u_j'(y_j)) \geq \ln(\alpha_{ij})$ and (2) $t_{ij} > 0 \Rightarrow \ln(u_i'(y_i)) - \ln(u_j'(y_j)) = \ln(\alpha_{ij})$. Setting $\lambda_i = \ln(u_i'(y_i))$ yields the correspondence. □

Among all the transfer profiles leading to equilibrium incomes, equilibrium transfers minimize a specific weighted sum of transfers. Links with stronger altruistic coefficients have less weight in the sum. When all links have the same strength, equilibrium transfers simply minimize the aggregate transfer needed to reach equilibrium incomes. A direct implication of Proposition 5 is that if $\mathbf{T}$ and $\mathbf{T}'$ are two distinct equilibria, then

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$\sum_{i,j} \ln(\alpha_{ij}) t_{ij} = \sum_{i,j} \ln(\alpha_{ij}) t'_{ij}$. The weighted sum of transfers is the same for all equilibria.

Taken together, Propositions 4 and 5 illustrate a principle of economy in transfers at work in altruistic networks. Even though each agent seeks to maximize his own utility function independently, the interplay of these decentralized giving decisions eliminates waste in transfers.

### C. Constant Absolute Risk Aversion

In this section, we obtain further results for CARA preferences. We show that the transfer game in this case possesses a best-response potential. This allows us to characterize the Nash equilibria as the solutions of a quadratic maximization problem. We then derive some implications of this reformulation.

Consider CARA preferences, $u_i(y) = -e^{-A_i y}/A_i$, $\forall y$ for some $A_i > 0$. Lemma 1 becomes: (1) $\forall i, j$: $A_i y_i \leq A_j y_j - \ln(\alpha_{ij})$ and (2) $\forall i, j$: $t_{ij} > 0$, $A_i y_i = A_j y_j - \ln(\alpha_{ij})$. Following Monderer & Shapley (1996) and Voorneveld (2000), a function $\varphi$ defined over transfer profiles is a best-response potential of the transfer game if $\arg\max_{\mathbf{T}_i} v_i(\mathbf{T}_i, \mathbf{T}_{-i}) = \arg\max_{\mathbf{T}_i} \varphi(\mathbf{T}_i, \mathbf{T}_{-i})$, $\forall i, \mathbf{T}_{-i}$.

**Proposition 6.** *Suppose that* $u_i(y) = -e^{-A_i y}/A_i$, $\forall i, \forall y$ *with* $A_i > 0$. *The function* $\varphi(\mathbf{T}) = \sum_{i,j} \ln(\alpha_{ij}) t_{ij} - \frac{1}{2} \sum_i A_i y_i^2$ *is a concave best-response potential of the transfer game.*

Proof: Since $\mathbf{y} \mapsto \sum_i A_i y_i^2$ is convex in $\mathbf{y}$ and $\mathbf{y}$ is linear in $\mathbf{T}$, $\varphi$ is concave in $\mathbf{T}$. First, compute $\partial\varphi/\partial t_{ij}$. Note that $\partial y_i/\partial t_{ij} = -1$, $\partial y_j/\partial t_{ij} = +1$ and $\partial y_k/\partial t_{ij} = 0$, for $k \neq i, j$. Therefore, $\partial\varphi/\partial t_{ij} = A_i y_i - A_j y_j + \ln(\alpha_{ij})$. Next, we have: $\partial v_i/\partial t_{ij} = -e^{-A_i y_i} + \alpha_{ij} e^{-A_j y_j} = e^{-A_i y_i}[e^{A_i y_i - A_j y_j + \ln(\alpha_{ij})} - 1]$. This shows that $\partial v_i/\partial t_{ij} > 0 \Leftrightarrow \partial\varphi/\partial t_{ij} > 0$ and $\partial v_i/\partial t_{ij} = 0 \Leftrightarrow \partial\varphi/\partial t_{ij} = 0$. The problems of maximizing $v_i$ and $\varphi$ over $\mathbf{T}_i$ have the same necessary and sufficient first-order conditions. □

**Corollary 2.** *Suppose that* $u_i(y) = -e^{-A_i y}/A_i$, $\forall i, \forall y$ *with* $A_i > 0$. *A matrix* $\mathbf{T}$ *is a Nash equilibrium of the transfer game if and only if it maximizes* $\varphi(\mathbf{T})$ *subject to:* $\forall i, j, t_{ij} \geq 0$ *and* $\forall i, j$: $\alpha_{ij} = 0$, $t_{ij} = 0$.

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Under CARA, agents act as if they are all trying to maximize the same objective function. Consider a binary network and common preferences: $\forall i, A_i = A$. Denote by $Var(\mathbf{x})$ the variance of profile $\mathbf{x}$ and observe that $\sum_i A_i y_i^2 = An[Var(\mathbf{y}) + (\bar{\mathbf{y}}^0)^2]$. Equilibrium behavior can be viewed as trading off a reduction in variance against a decrease in aggregate transfer. At one extreme, income variance is minimized by the equal income distribution: $y_i = \bar{y}^0$, $\forall i$. However, reaching this distribution on an arbitrary network typically requires a lot of indirect transfers. At the other extreme, aggregate transfer is minimized when there is no transfer; incomes are unchanged and the initial income distribution may have a high variance. Therefore, the equilibrium income distribution somehow lies inbetween the initial and the equal income distributions.

The existence of a best-response potential has nice implications. In particular, we can derive from Corollary 2 alternative proofs of Proposition 3 and Theorem 1. Convexity of the equilibrium set directly follows from the classical property that the set of maximizers of a concave function over a convex set is convex. Then, convexity of the equilibrium set can be shown to imply income uniqueness. In Section V.B below, we derive some further comparative statics implications

## V. Comparative statics

### A. Impact of initial incomes

In this section, we look at the effect of initial incomes. We show that equilibrium incomes are monotonically related to initial incomes. We illustrate how a shock on initial incomes gets transmitted throughout the network. We discuss the effect of redistributive policies and show that an ex-ante Pigou-Dalton redistribution can lead to an increase in ex-post inequality.

We first show that income after transfers varies monotonically with incomes before transfers.

**Proposition 7.** *For any utility functions and any altruistic network, $y_i$ is strictly increasing in $y_i^0$ and weakly increasing in $y_j^0$, $\forall j \neq i$.*

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Proof: Consider two initial income distributions $\mathbf{y^0}$ and $\mathbf{y^{0\prime}}$ such that $y_i^0 > y_i^{0\prime}$ and $y_j^0 = y_j^{0\prime}$ $\forall j \neq i$. Let $\mathbf{T}$ and $\mathbf{T}'$ be corresponding equilibria and $\mathbf{y}$ and $\mathbf{y}'$ the corresponding income distributions. Define $U = \{j : y'_j > y_j\}$ and suppose that $U \neq \emptyset$. We adapt arguments from the proof of Theorem 1. If $j \in U$ and $t_{jk} > 0$ or $t'_{kj} > 0$ then $k \in U$. This implies that $\sum_{j\in U} y_i = \sum_{j\in U} y_i^0 + t_{N-U,U}$ and $\sum_{j\in U} y'_i = \sum_{j\in U} y_i^{0\prime} - t'_{U,N-U}$. Therefore, $\sum_{j\in U} y_j \geq \sum_{j\in U} y_j^0 \geq \sum_{j\in U} y_j^{0\prime} \geq \sum_{j\in U} y'_j$, which establishes a contradiction. This implies that $U = \emptyset$, and hence $\forall j \in N$, $y_j \geq y'_j$. Next, define $V = \{j : y'_j \geq y_j\}$. Suppose that $i \in V$. By definition of $\mathbf{y^0}$ and $\mathbf{y^{0\prime}}$, $\sum_{j\in V} y_j^{0\prime} < \sum_{j\in V} y_j^0$. And through a similar reasoning, we can show that $\sum_{j\in V} y_j \geq \sum_{j\in V} y_j^0 \geq \sum_{j\in V} y_j^{0\prime} \geq \sum_{j\in V} y'_j$ which establishes a contradiction. Therefore, $i \notin V$ and $y'_i < y_i$. □

Suppose that an agent suffers an income shock. This shock is transmitted throughout the altruistic network and affects overall transfers and equilibrium incomes. Proposition 7 states that the agent necessarily bears a part of the shock and that the other agents cannot gain from it. Every other agent is either unaffected or affected negatively.

To gain some insight on how the shock is transmitted throughout the network, consider the following example. Agents have common preferences, the network is binary and all agents have initial income $y^0$ except for agent $i$ for whom $y_i^0 = y^0 - L$. Equilibrium incomes are characterized by the following properties. There are threshold levels $l_k$, $k = 1, 2, ...$ If $L < l_1$, there is no transfer and incomes are unchanged. If $l_1 < L < l_2$, $i$ is supported by his direct friends, who end up with income $y_j$ such that $u'(y_j) = \alpha u'(y_i)$, and agents at distance 2 or more do not give any money. If $l_2 < L < l_3$, $i$ is supported by his direct friends who are themselves supported by agents at distance 2 and agents at distance 3 or more do not give any money. If $j$ is a direct friend and $k$ is at distance 2, then $u'(y_k) = \alpha u'(y_j) = \alpha^2 u'(y_i)$. And so on. Equilibrium income is weakly decreasing in distance from $i$ and all agents at the same distance end up with the same income. Moreover, equilibrium incomes and threshold levels depend on the number of agents at finite distances from $i$, but do not depend on the link patterns over and above this number distribution. More generally, $\mathbf{y}$ is a continuous and piecewise differentiable function of $\mathbf{y}^0$ with a potentially large number of pieces. And an income shock affects socially closer agents first.

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The presence of an altruistic network affects the impact and design of redistributive policies. We illustrate the richness of the effects at work in what follows and leave a full-fledged study of this issue for future research. Consider the example depicted in Figure 2. Three agents are placed on a binary line and have CARA preferences, $u_i(y) = -e^{-Ay}/A$ with $\alpha$ and $A$ such that $-\ln(\alpha)/A = 2$. On the upper left panel, initial incomes are 4, 10 and 0. In equilibrium, on the lower left panel, the center gives 4 to the right and incomes are 4, 6 and 4 with $Var(\mathbf{y}) \approx 0.89$. Next, redistribute ex-ante 2 from the relatively rich center to the relatively poor left. Initial incomes, on the upper right panel, are now 6, 8 and 0. In the new equilibrium, in the lower right panel, the center gives 3 to the right and incomes are 6, 5 and 3 with $Var(\mathbf{y}) \approx 1.56$. The lowest income has decreased and variance has increased. The center here is the main source of support of the agent on the right. Any public policy that takes money away from the center without giving money to the agent on the right may lead to a deterioration of the situation of the poorest agent. Therefore, redistributing money from richer to poorer individuals can actually lead to an increase in inequality once altruistic transfers have been accounted for.

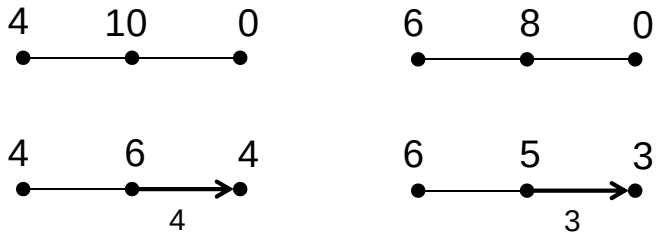


Figure 2: An ex-ante Pigou-Dalton redistribution can increase ex-post inequality

### B. Impact of the altruistic network

In this section, we study the impact of the altruistic network. We first show that any network reduces income inequality in the sense of second-order stochastic dominance. We then consider an increase in altruism. We find that an expansion of the altruistic network can increase income variance. Still, the maximal income spread tends to decrease and,

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under CARA, a linear combination of variance and transfers must decrease. Finally, we look at the relation between income homophily and income inequality.

Let us establish, first, that altruistic networks indeed reduce inequality.

**Proposition 8.** *On any altruistic network and for any utility functions, the distribution of equilibrium incomes second-order stochastically dominates the distribution of initial incomes.*

Proof: We show that the distribution of equilibrium incomes can be obtained from the initial distribution through a series of Pigou-Dalton transfers from richer to poorer agents. Consider an equilibrium $\mathbf{T}$. Order transfers $t_{ij}$ as follows. By Proposition 4, the transfer network is acyclic. Thus, there is an agent $i$ who does not receive. From the initial distribution, apply $i$'s transfers first, in any order. Then remove $i$ and repeat. Pick a second agent who does not receive from others in the remaining network. Apply this second agent's transfers, in any order. Repeat til no more agent is left. This procedure leads to an ordering of all pairwise transfers. Therefore, final incomes are indeed equal to equilibrium incomes. This ordering also guarantees that a transfer always takes place from a richer to a poorer agent. □

Proposition 8 shows that any network reduces inequality compared to the empty network. More generally, how does the shape of the altruistic network affects society's move towards equality?

We ask, first, whether Proposition 8 extends. Starting from an altruistic network, does an increase in altruism necessarily reduce inequality? The answer turns out to be negative. Consider the numerical example presented in Figure 3 with CRRA preferences, $\forall i, u_i(y) = \ln(y)$, and a binary network with $\alpha = 0.5$. The Nash conditions of Lemma 1 become: $\alpha_{ij} > 0 \Rightarrow y_i \leq 2y_j$ and $t_{ij} > 0 \Rightarrow y_i = 2y_j$. There are four agents with initial incomes 27, 6, 2 and 16. Number agents from left to right. In the initial network, on the left panels, 2 is connected with 1 and 3, and 4 is isolated. Equilibrium incomes, in the lower left panel, are 20, 10, 5 and 16 with $t_{12} = 7$ and $t_{23} = 3$ and $Var(\mathbf{y}) \approx 32.7$. Next, add a connection between 3 and 4 as depicted on the right panels. Equilibrium incomes on the lower right

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panel become 22, 11, 6 and 12 with $t_{12} = 5$, $t_{23} = 0$ and $t_{43} = 4$ and $Var(\mathbf{y}) \approx 33.7$. The new link connects the poorest agent, 3, with a relatively wealthy agent 4. Thanks to his new connection, 3 does not need support from 2 any more and this cuts down indirect gifts from 1. The largest income $y_1$ increases and overall variance increases. Thus, adding links to an altruistic network may lead to an increase in income variance.

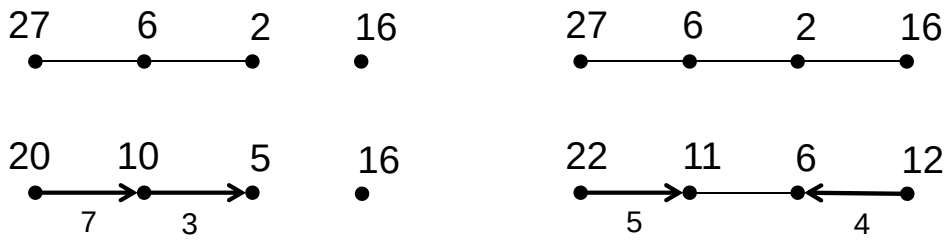


Figure 3: Adding an altruistic link can increase inequality

Next, assume common preferences $\forall i, u_i = u$ and focus on the relation between the lowest and highest incomes $y_{\min}$ and $y_{\max}$. Note that $u'(y_{\max})/u'(y_{\min})$ is lower than 1 and tends to be lower when the income spread $y_{\max} - y_{\min}$ is higher. For any pair $i, j$, define $\hat{\alpha}_{ij}$ as the highest altruistic strength among chains connecting $i$ and $j$. And let $\hat{\alpha}_{\min} = \min_{i,j} \hat{\alpha}_{ij}$ be the lowest of these pairwise coefficients. Thus, $\hat{\alpha}_{\min}$ captures the altruistic strength of the *weakest indirect link* in the network.

**Proposition 9.** *On any altruistic network and for any common utility function such that* $u'(\infty) < \hat{\alpha}_{\min} u'(0)$,

$$\min_{\mathbf{y}^0} \frac{u'(y_{\max})}{u'(y_{\min})} = \hat{\alpha}_{\min}$$

Proof: Consider an equilibrium and $i$ and $j$ such that $y_i = y_{\max}$ and $y_j = y_{\min}$. From the proof of Proposition 4, we know that $u'(y_{\max})/u'(y_{\min}) \geq \hat{\alpha}_{ij} \geq \hat{\alpha}_{\min}$. Next, let $i$ and $j$ be such that $\hat{\alpha}_{ij} = \hat{\alpha}_{\min}$. Consider the following distribution: $y_i^0 = Y$; $y_k^0 = 0$, $\forall k \neq i$. Note that $y_i = y_{\max}$ and hence $y_i \geq \bar{y}^0 = Y/n$. As $Y$ tends to $+\infty$, $y_i$ tends to $+\infty$. Since $u'(y_i) \geq \hat{\alpha}_{ij} u'(y_j)$, $u'(\infty) \geq \hat{\alpha}_{ij} u'(y_j)$. By assumption, $\hat{\alpha}_{ij} u'(0) > u'(\infty)$ and hence $y_j > 0$. Thus, if $Y$ is high enough, $j$ is a net receiver. Since $i$ is the only individual with positive initial income, money must flow somehow from $i$ to $j$ and hence $u'(y_i)/u'(y_j) = \hat{\alpha}_{ij}$. □

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This result says that the relation between the largest and the lowest equilibrium income is controlled by the strength of the weakest indirect link in the altruistic network. When all links have strength $\alpha$, $\hat{\alpha}_{\min} = \alpha^d$ where $d$ is the network's diameter, that is, the largest distance between any two agents. Under CARA preferences, $u(y) = -e^{-Ay}/A$, and Proposition 9 reduces to: $\max_{\mathbf{y}^0}(y_{\max} - y_{\min}) = d\ln(\alpha)/A$. The highest income difference sustainable in equilibrium is simply proportional to the network's diameter. Under CRRA preferences, $u(y) = y^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ and $u(y) = \ln(y)$ for $\gamma = 1$, and Proposition 9 becomes $\max_{\mathbf{y}^0} \ln(y_{\max}) - \ln(y_{\min}) = d\ln(\alpha)/\gamma$. The highest difference in log incomes is now proportional to the diameter.

An implication of Proposition 9 is that $\min_{\mathbf{y}^0} u'(y_{\max})/u'(y_{\min})$ is weakly increasing in $\boldsymbol{\alpha}$. When the altruistic network expands, the weakest link can only become stronger which can only reduce inequality as captured by $u'(y_{\max})/u'(y_{\min})$.

Next, we consider CARA preferences and build on the potential characterization to derive a comparative statics result.

**Proposition 10.** *Suppose that* $u_i(y) = -e^{-A_i y}/A_i$, $\forall i$, $\forall y$ *with* $A_i > 0$. *Consider an equilibrium* $\mathbf{T}$ *with income* $\mathbf{y}$ *for the network* $\boldsymbol{\alpha}$ *and an equilibrium* $\mathbf{T}'$ *with income* $\mathbf{y}'$ *for the network* $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$. *Then*

$$\sum_{i,j} \ln(\alpha_{ij}) t_{ij} - \frac{1}{2}\sum_i A_i y_i^2 \leq \sum_{i,j} \ln(\alpha'_{ij}) t'_{ij} - \frac{1}{2}\sum_i A_i y_i'^2$$

Proof: As $\boldsymbol{\alpha}$ increases, the objective function increases weakly and the constrained set expands. Therefore, the value of the objective function at the maximum increases weakly. □

For common preferences and binary networks, a linear combination of variance and aggregate transfer must decrease when altruism expands. This means, in particular, that increases in income variance are necessarily associated with relatively strong decreases in aggregate transfer.

To conclude this section, we provide a first look at the relation between homophily and inequality. We consider the following example. There are 20 agents divided in two groups:

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poor and rich. Ten agents are poor with $y_i^0 = 0$ and the other ten are rich with $y_i^0 = 10$. All links have strength $\alpha$ and agents have CARA preference with $-\ln(\alpha)/A = 2$. The network is built as follows. Start from the network where agents are fully connected within and there is no link between. Then, remove $l$ links at random within each income group and add $2l$ links at random between the poor and the rich. The overall number of links stays constant. As $l$ increases, the relative proportion of links between increases and hence homophily with respect to income decreases. We pick 1000 network realizations for each value of $l$ and for each network we compute equilibrium incomes.[9] We depict in Figure 4 the variance of equilibrium incomes as a function of $l$. The three curves depict the $5^{\text{th}}$ percentile, the median, and the $95^{\text{th}}$ percentile of the distribution of variances.

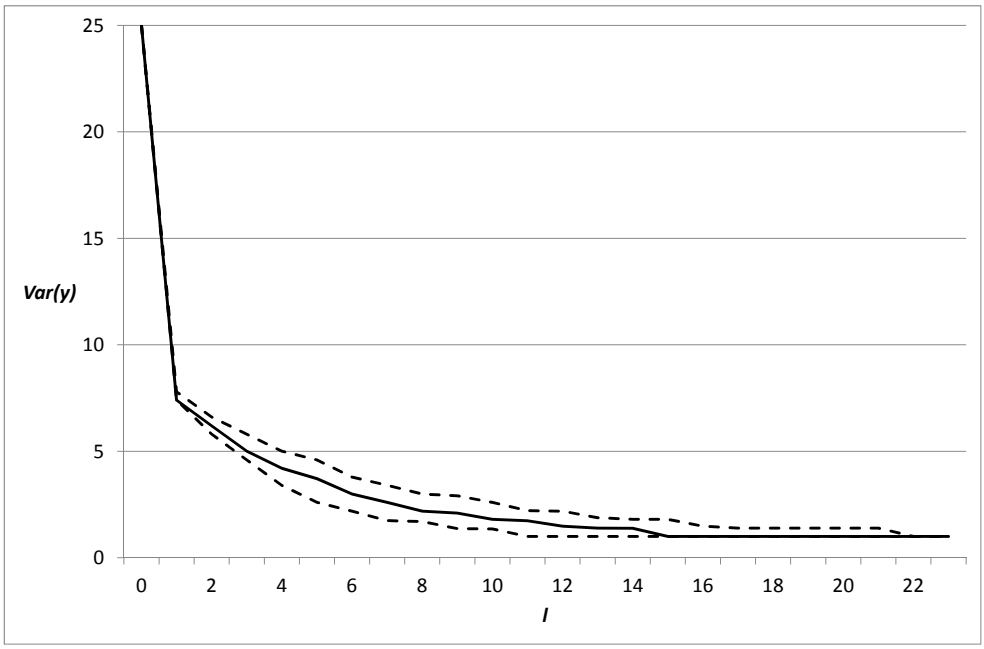


Figure 4: Homophily and Inequality

[9] We compute equilibrium incomes as follows: (0) Start from the profile of zero transfers. (1) Order agents in any way. (2) Have each agent in turn plays a best-reply. Repeat (1) and (2) til convergence, which is guaranteed by the existence of a concave best-response potential.

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In this example, altruistic networks with more income homophily clearly generate more inequality. More precisely, inequality reduction is lower on networks with higher homophily. In addition, we observe the emergence of small-world effects. The first altruistic links between the poor and the rich lead to a strong reduction in variance. Then, additional links have a much smaller and decreasing impact. This small-world effect is a consequence of indirect gifts. Even a single link between communities ends up affecting every agent, as poor agents receiving financial support from rich friends help out other poor agents lacking in such connections.

## VI. Conclusion

To conclude, we provide the first theoretical analysis of altruism in networks. Agents are connected through an arbitrary weighted network and care for their direct friends. We analyze the resulting transfer game. We show that equilibrium incomes are unique on any network and that equilibrium transfers are unique on trees. We show that the set of Nash equilibria is compact and convex. We uncover a principle of economy in transfers at work in altruistic networks. Equilibrium transfers flow through paths of highest altruistic strength and minimize a weighted sum of transfers needed to reach equilibrium incomes. A negative shock on one agent cannot lead to an increase in the income of another agent, and affects socially closer agents first. An equalizing redistribution of initial incomes can end up increasing inequality. Altruistic networks decrease inequality, but more altruistic or more homophilous networks can increase inequality.

A number of potentially important issues could be explored in future research.

First, *the risk sharing properties of altruistic networks.* Consider some joint distribution of stochastic incomes and suppose that transfers ex-post operate as described in our analysis. How would expected utilities depend on the network structure? Would friends' friends help by providing a source of support for direct friends, or would they reduce payoffs by acting as competitors for direct friends' gifts?

Second, *the effect of altruistic networks on incentives.* Under moral hazard, Alger & Weibull (2010) show that mutual altruism in pairs has two opposite effects: a negative,

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free-riding effect and a positive empathy effect.[10] The empathy effect may dominate when altruism is strong. How would their analysis extend to arbitrary social structures? What kind of networks would be better able to informally address moral hazard?

Third, *the design of public policies in altruistic societies.* How to optimize the targets of cash transfers programs in altruistic networks? How to avoid the kind of inequality-enhancing redistribution highlighted in Section V? How would crowding out of private gifts by public transfers depend on the network? How to introduce formal insurance in altruistic communities?

Fourth, *the empirical identification of real motives behind private transfers.* As mentioned in the Introduction, altruism generates specific predictions with respect to the size of shocks. In addition, many of our results have identifying power. With detailed data on transfers and incomes, they could potentially be tested or used to structurally estimate altruistic coefficients. This is true, in particular, of our characterization of the shape of transfer networks and of our comparative statics result with respect to initial incomes and to the network.

Fifth, *the interaction between altruism and other motives on networks.* In reality, altruism, exchange and social pressure likely all play a role in explaining private transfers. Few studies have explored the interaction between altruism and other motives. Foster & Rosenzweig (2001) looks at the impact of altruism on mutual insurance arrangements under limited commitment. Alger & Weibull (2008) study the combined effect of altruism and social pressure on incentives. These two studies consider individuals interacting in pairs. It would be interesting to look at the interaction between altruism, exchange and social pressure on social networks.

[10] An agent who knows that he will be helped may shirk and reduce hisloss-avoiding effort. This is the free-riding effect. An agent who knows that he may have to provide for those he care about may increase his effort to be in a better position to help. This is the empathy effect.

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# APPENDIX

**Proof of Proposition 1.**

Define $S_M = \{\mathbf{T} \in S : \sum_{i,j} t_{ij} \leq M\}$ for any $M > 0$. Then, $S_M$ is compact and convex. Agent $i$'s utility $v_i$ is concave in $\mathbf{T}_i$ and continuous in $\mathbf{T}$. By Theorem 1 in Rosen (1965), an equilibrium exists on $S_M$. In this equilibrium, the added constraint does not change the property that $t_{ij} > 0 \Rightarrow y_i > y_j$. This property implies that the transfer network is acyclic. Indeed, suppose that $t_{ii_2} > 0$, $t_{i_2i_3} > 0$,..., $t_{i_{l-1}j} > 0$. Then: $y_i > y_{i_2}$,...,$y_{i_{l-1}} > y_j$ and hence $y_i > y_j$ and there is no cycle.

Next, let us show by induction that aggregate transfers in acyclic networks are bounded from above. The induction hypothesis, $H_n$, is as follows: In an acyclic transfer network with $n$ agents, $\sum_{i,j} t_{ij} \leq (n-1)\sum_i y_i^0$. Suppose first that $n = 2$. Then $t_{12}t_{21} = 0$. If $t_{12} > 0$, then $t_{12} \leq y_1^0$. If $t_{21} > 0$, then $t_{21} \leq y_2^0$. In any case, $H_2$ holds. Suppose, next, that $H_{n-1}$ is true. Consider an acyclic transfer network with $n$ agents. Without loss of generality, suppose that agent 1 does not receive. This means that $\sum_j t_{1j} \leq y_1^0$. Remove 1 and apply $H_{n-1}$ to the resulting network: $\sum_{i,j=2}^n t_{ij} \leq (n-2)\sum_{i=2}^n (y_i^0 + t_{1i})$. Therefore, $\sum_{i,j} t_{ij} = \sum_j t_{1j} + \sum_{i,j=2}^n t_{ij} \leq (n-1)y_1^0 + (n-2)\sum_{i=2}^n y_i^0 \leq (n-1)\sum_i y_i^0$ and $H_n$ holds.

Finally, choose $M > (n-1)\sum_i y_i^0$. An equilibrium on $S_M$ is an equilibrium on $S$. □

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# REFERENCES

Agarwal, Reena and Andrew W. Horowitz. 2002. "Are International Remittances Altruism or Insurance? Evidence from Guyana Using Multiple-Migrant Households." *World Development* 30(11): 2033-2044.

Alger, Ingela and Jörgen W. Weibull. 2008. "Family Ties, Incentives and Development. A Model of Coerced Altruism." in *Arguments for a Better World: Essays in Honor of Amartya Sen*, Kaushik Basu and Ravi Kanbur eds., Oxford University Press.

Alger, Ingela and Jörgen W. Weibull. 2010. "Kinship, Incentives, and Evolution." *American Economic Review* 100(4): 1727-1760.

Ambrus, Attila, Mobius, Markus and Adam Szeidl. 2013. "Consumption Risk-Sharing in Social Networks." *American Economic Review* forthcoming.

Angelucci, Manuela and Giacomo De Giorgi. 2009. "Indirect Effects of an Aid Program: How Do Cash Injections Affect Non-Eligibles' Consumption." *American Economic Review* 99(1): 486-508.

Angelucci, Manuela, De Giorgi, Giacomo and Imran Rasul. 2012. "Resource Pooling within Family Networks: Insurance and Investment." *working paper* Stanford University.

Arrondel, Luc and André Masson. 2006. "Altruism, Exchange, or Indirect Reciprocity: What do the Data on Family Transfers Show?" chapter 14 of the *Handbook of the Economics of Giving, Altruism and Reciprocity*, Serge-Christophe Kolm and Jean Mercier Ythier eds., North Holland.

Arrow, Kenneth. 1981. "Optimal and Voluntary Income Distribution." in Steven Rosefielde ed., *Economic Welfare and the Economics of Soviet Socialism: Essays in Honor of Abram Bergson*, Cambridge University Press.

Baland, Jean-Marie, Guirkinger, Catherine and Charlotte Mali. 2011. "Pretending to Be Poor: Borrowing to Escape Forced Solidarity in Cameroon." *Economic Development and Cultural Change* 60(1): 1-16.

Ballester, Coralio, Calvó-Armengol, Antoni and Yves Zenou. 2006. "Who's Who in Networks. Wanted: the Key Player." *Econometrica* 74(5): 1403-1417.

Becker, Gary S. 1974. "A Theory of Social Interactions." *Journal of Political Economy* 82(6): 1063-1093.

Bernheim, B. Douglas and Oded Stark. 1988. "Altruism within the Family Reconsidered: Do Nice Guys Finish Last?" *American Economic Review* 78(5): 1034-1045.

Bloch, Francis, Genicot, Garance and Debraj Ray. 2008. "Informal Insurance in Social

halshs-00881451, version 1 - 8 Nov 2013

Networks." *Journal of Economic Theory* 143: 36-58

Bramoullé, Yann and Rachel Kranton. 2007a. "Risk Sharing Networks." *Journal of Economic Behavior and Organization* 64: 275-294.

Bramoullé, Yann and Rachel Kranton. 2007b. "Risk Sharing Across Communities." *American Economic Review, Papers and Proceedings* 97(2): 70-74.

Bramoullé, Yann, Rachel Kranton and Martin D'Amours. 2013. "Strategic Interaction and Networks." *American Economic Review* forthcoming.

Cox, Donald and Marcel Fafchamps. 2008. "Extended Family and Kinship Networks: Economic Insights and Evolutionary Directions." chapter 58 of the *Handbook of Development Economics,* T. Paul Schultz and John Strauss eds., North Holland.

De Weerdt, Joachim and Stephan Dercon. 2006. "Risk Sharing Networks and Insurance against Illness." *Journal of Development Economics* 81: 337-356.

De Weerdt, Joachim and Marcel Fafchamps. 2011. "Social Identity and the Formation of Health Insurance Networks." *Journal of Development Studies* 47(8): 1152-1177.

Dubois, Pierre, Jullien, Bruno and Thierry Magnac. 2008. "Formal and Informal Risk Sharing in LDC's: Theory and Empirical Evidence." *Econometrica* 76(4): 679-725.

Fafchamps, Marcel and Flore Gubert. 2007. "The Formation of Risk Sharing Networks." *Journal of Development Economics* 83(2): 326-350.

Fafchamps, Marcel and Susan Lund. 2003. "Risk Sharing Networks in Rural Philippines." *Journal of Development Economics* 71: 261-287.

Foster, Andrew D. and Mark R. Rosenzweig. 2001. "Imperfect Commitment, Altruism, and the Family: Evidence from Transfer Behavior in Low-Income Rural Areas." *Review of Economics and Statistics* 83(3): 380–407.

Galeotti, Andrea, Goyal, Sanjeev, Jackson, Matthew O., Vega-Redondo, Fernando and Leeat Yariv. 2010. "Network Games." *Review of Economic Studies* 77(1): 218-244.

Jackson, Matthew O. 2008. *Social and Economic Networks.* Princeton University Press.

Kazianga, Harounan. 2006. "Motives for Household Private Transfers in Burkina Faso." *Journal of Development Economics* 79: 73-117.

Kolm, Serge-Christophe and Jean Mercier Ythier, eds. 2006. *Handbook of the Economics of Giving, Altruism and Reciprocity.* North Holland.

Lee, Ronald and Gretchen Donehove. 2011. "Private Transfers in Comparative Perspective", chapter 8 in *Population Aging and the Generational Economy*, eds. Ronald Lee and Andrew Mason, Edward Elgar.

halshs-00881451, version 1 - 8 Nov 2013

Leider, Stephen, Möbius, Markus M., Rosenblat, Tanya and Quoc-Anh Do. 2009. "Directed Altruism and Enforced Reciprocity in Social Networks." *Quarterly Journal of Economics* 124(4): 1815-1851.

Ligon, Ethan and Laura Schechter. 2012. "Motives for Sharing in Social Networks." *Journal of Development Economics* 99: 13-26.

Ligon, Ethan, Thomas, Jonathan P. and Tim Worrall. 2002. "Mutual Insurance and Limited Committment: Theory and Evidence in Village Economies." *Review of Economic Studies* 69: 115-139.

Mazzocco, Maurizio and Shiv Saini. 2012. "Testing Efficient Risk Sharing with Heterogeneous Risk Preferences." *American Economic Review* 102(1): 428-468.

McPherson, Miller, Smith-Lovin, Lynn and James M. Cook. 2001. "Birds of a Feather: Homophily in Social Networks." *Annual Review of Sociology*, 27: 415-444.

Mercier Ythier, Jean. 1993. "Équilibre Général de Dons Individuels." *Revue Économique* 44(5): 925-950.

Mercier Ythier, Jean. 2006. "The Economic Theory of Gift-Giving: Perfect Substituability of Transfers and Redistribution of Wealth." chapter 5 in *Handbook of the Economics of Giving, Altruism and Reciprocity*, Serge-Christophe Kolm and Jean Mercier Ythier eds., North Holland

Mitrut, Andreea & Katarina Nordblom. 2010. "Social Norms and Gift Behavior: Theory and Evidence from Romania." *European Economic Review* 54: 998-1015.

Monderer, Dov and Lloyd Shapley. 1996. "Potential Games." *Games and Economic Behavior* 14: 124-143.

Rosen, J B. 1965. "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games." *Econometrica* 33(3): 520-534.

Stark, Oded. 1995. *Altruism and Beyond: an Economic Analysis of Transfers and Exchanges within Families and Groups.* Cambridge University Press.

Townsend, Robert. 1994. "Risk and Insurance in Village India." *Econometrica* 62(3): 539-591.

Viessman Jr., Warren and Gary L. Lewis. 2002. *Introduction to Hydrology.* Prentice Hall.

Voorneveld, Mark. 2000. "Best-Response Potential Games." *Economics Letters* 66: 289-295.

Worldbank. 2011. *Migration and Remittances Factbook 2011, 2$^{\mathrm{d}}$ edition.* Washington DC.