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THE PERILS OF NOMINAL TARGETS

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Abstract

A monetary authority can be committed to pursuing an inflation, price-level, or nominal output target yet systematically fail to achieve the specified goals. Constrained by the zero lower bound on the policy rate, the monetary authority is unable to implement its objectives when private-sector expectations stray away from the target in the first place. Low-inflation expectations become self-fulfilling, leading to multiple Markov equilibria. Private-sector expectations are anchored on a unique Markov equilibrium if the monetary authority is given a strong stabilization goal for the policy rate. However, policy-rate stabilization may not improve welfare as the resulting policy is severely distorted.

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1 Introduction

What is the right framework for the conduct of monetary policy? Academic and policy debates on this decades-old question have increasingly been dominated by nominal targets in one form or another. The prime example is inflation targeting, which enjoys the rare distinction of having a proven record in both theory and practice. The darling of theory is actually price-level targeting since it introduces the right amount of history dependence and improves stabilization policy, especially when the policy rate is close to its zero lower bound. Nominal GDP targeting is also a safe bet to surface whenever the design of the monetary policy is called into question. All of these monetary frameworks have been praised, to varying degrees, for their ability to enhance the central bank's credibility, transparency, and overall performance.

It is widely recognized that nominal targets require a firm commitment by the monetary authority to pursue the specified target. A starting assumption in Svensson (1997) is that society "can commit to targets for the central bank."[1] Eggertsson and Woodford (2003) write that it is necessary "for the central bank to be committed (and be understood to be committed)."[2] In practice, adopters of inflation targeting have written legislation detailing the monetary authority's objectives and putting in place procedures to ensure compliance.

In this paper I argue that, unfortunately, a monetary authority cannot commit to achieve an inflation, price-level, or nominal GDP target any more than the Philadelphia Eagles can commit to winning the Super Bowl; there remains a question of ability. Attaining the target may simply be *unfeasible* when private-sector expectations depart from the target in the first place: The monetary authority, constrained by the zero lower bound on the policy rate, cannot disprove the expectations and steer the inflation or price-level back to target—no matter how *willing* the central bank is to do so. The main results in this paper show that multiple equilibria are pervasive, typically including a low-inflation equilibrium where the policy rate spends long spells at the zero lower bound and yet the monetary authority systematically fails to achieve its targets. Thus I conclude that an irrevocable commitment by the monetary authority is a necessary, but not sufficient, condition to achieve the desired target.

The setting for my analysis is a simple New Keynesian model that observes the zero lower bound (ZLB) for the policy rate. In the spirit of Rogoff (1985) and Walsh (1995), society designs the monetary authority's loss function to include the nominal target of choice, typically combined with a weight on output stabilization, but the central bank retains discretion in setting the policy rate.[3] There is no chance for society to revisit the central bank's goals: Thus the monetary authority's willingness to pursue the specified

[1] Page 99.

[2] Page 181.

[3] Svensson (1997) puts it succinctly: "The government delegates monetary policy to an instrument independent central bank that is assigned a particular loss function."

goals is beyond any doubt. My analysis considers only Markov equilibria, abstracting from the possibility of history-dependent or sunspot equilibria.

The key to the result is how private-sector expectations constrain the feasibility set of the monetary authority. Because of the ZLB, inflation expectations limit the range of real interest rates, which in turn cap the inflation and output levels the monetary authority can engineer. If inflation expectations are low enough to start with, the monetary authority finds that it can only disprove the inflation and output expectations on the downside—and thus driving them both further away from their respective targets—by raising the policy rate above the ZLB. The least damaging option is to validate the private-sector expectations, and thus multiple equilibria arise.

The most complete set of results belong to inflation targeting. I show that an equilibrium is, generically, not unique for any stochastic process for cost-push and real-rate shocks. I complement the analytic results with a numerical exploration of the simple model presented here. A typical case has two Markov equilibria: 1) a "stabilization" equilibrium, where inflation and output fluctuate around their targets and the ZLB binds only occasionally; and 2) a "low inflation" equilibrium, where inflation is consistently far below the target and the policy rate is often at the ZLB—though not necessarily all the time.[4] It is, though, possible to have no or more than two equilibria as well.[5]

Price-level targeting does not resolve the equilibrium multiplicity despite introducing an endogenous state variable, namely, the price level (or its deviation from a predetermined path). There are always two equilibria in a non-stochastic economy. In one equilibrium the monetary authority perfectly stabilizes the price level at the desired target (or targeted path). In the other equilibrium, the price level falls by an ever-increasing distance from the target. I also show that there are convergent dynamics to the falling price-level equilibrium. The same set of results applies to nominal output targeting.

Not all hope is lost for nominal targets, as a relatively minor modification to the central bank's objectives can effectively anchor expectations on a single Markov equilibrium. I show that if the monetary authority is given a strong goal for policy-rate stabilization then there is a unique Markov equilibrium. This modification naturally induces the monetary authority to stay away from the ZLB unless there are large shocks. If the penalty term is large enough, the monetary authority will *increase* the policy rate when faced with deflationary expectations—even though doing so will send both output and inflation further below target. Policy-rate stabilization, though, does not ensure that the monetary authority will achieve its output and inflation targets—actually, the resulting unique Markov equilibrium may be welfare dominated by every equilibrium under inflation targeting. There are several downsides to policy-rate stabilization. Numerical results confirm these fears, as the equilibrium under policy-rate stabilization does not inherit any of the

[4]Indeed, the departures from the ZLB are long-lived enough that I hesitate to label the equilibrium a "liquidity trap."

[5]An additional contribution is to provide an algorithm ensuring that no Markov equilibrium is missed, thus improving on alternative approaches like adaptive expectations.

desirable properties of inflation targeting.

I should also note that committing to keep the policy rate low for a long period does neither lift the economy from the ZLB nor sets the inflation rate (or the price level) on a path to the target. The reason is that a persistently low policy rate is already the expected path in the low-inflation equilibrium, so this form of forward guidance does little but validate the private-sector expectations. This is in sharp contrast with the result in Eggertsson and Woodford (2003), where forward guidance is effective: The difference is that in Eggertsson and Woodford (2003) the economy finds itself against the ZLB due to a real interest shock rather than private-sector expectations.

If we are free to design the monetary authority's objectives as we wish, why not take it one step further and specify a loss function in terms of a state-contingent, or even history-contingent, plan for the policy rate? Nothing prevents us from doing so in the context of the model analyzed here—yet it is hard to see this possibility as a practical option.[6] First, monetary economists widely agree that simple targets or rules are preferable: they are easier to communicate to the public and possibly more robust given the uncertainty regarding the actual economy. Second, a complex target criterion may strain the legal and political mechanisms available to ensure a credible commitment by the monetary authority. And third, we have yet to see a central bank let its policy instrument be completely set by a rule, simple or otherwise.

There is no choice but to start the literature review with the paper whose title this paper pays homage to. Benhabib et al. (2000) shows how expectations are not anchored by a standard Taylor rule when the ZLB is taken into account. In short, this paper shows that the inflation and price-level targeting share the same perils as Taylor rules. The conclusions are similar, though, despite some very important conceptual differences on how policy is specified. While in Benhabib et al. (2000) the policy rate is simply given by an exogenous rule, here the monetary framework determines the central bank's objectives and the policy rate is determined by the monetary authority's best response function. In this setting the key distinction between a commitment to pursue a goal and the ability to do so arises: The equilibrium multiplicity is triggered because the monetary authority's goals become unfeasible. I should also note that it is not possible to summarize policy with a Taylor rule: Even a linear approximation typically sees the coefficients on output and inflation change across equilibria.[7] My analysis also shows that introducing history dependence through a price-level target fails to anchor expectations. It is instead a stabilization goal for the policy rate that does it. As far as I know, neither result has been documented in the context of Taylor rules.

There is abundant work on New Keynesian models regarding policy when the nominal

[6] Svensson (1997), again succinctly, simply states that "it is taken for granted that commitment to a complicated state-contingent rule for the central bank's instrument is infeasible."

[7] It is also possible, if unfortunate, that no Markov equilibria exist under some parameters, while Taylor rules do not fail to produce an equilibrium.

interest rate is at or close to the ZLB. However, most of the literature has ignored the possibility that expectations are not anchored. An early exception is Eggertsson (2006) who also emphasize the role that policy constraints play in the face of a large demand shock. More recently, Aruoba and Schorfheide (2013) estimate a rich New Keynesian model to see whether the data favors expectations-driven dynamics to explain U.S. data (it does). See also Mertens and Ravn (2013). These papers assume a Taylor rule, subject to the perils highlighted in Benhabib et al. (2000), and share a focus on the implications for fiscal policy.

This paper is also closely related to the work on "expectations traps," as branded by Chari et al. (1998). The standard setup is a game between the private sector and the monetary authority, which is assumed to maximize social welfare. Most of the literature has typically ignored the ZLB, yet multiple Markov equilibria arise in several settings as shown in Albanesi et al. (2003), Armenter and Bodenstein (2008), and Siu (2008), among others.[8] Armenter (2008) finds that very weak conditions are sufficient for multiple Markov equilibria to arise. The two key points of departure here from the literature are the focus on the design of the monetary authority's objectives as well as the role of the ZLB—if the latter were to be dropped, the model presented here will be linear with an unique equilibrium.

The paper is organized as follows. The next section details the model and equilibrium. Section 3 provides some insight on the key results by deriving the correspondence between inflation expectations and the monetary authority's feasibility set under some simplifying assumptions. The subsequent three sections tackle inflation targeting, policy-rate stabilization, and price-level targeting, respectively. The conclusions include a brief discussion of possible arguments to dismiss the perils. All proofs as well as some additional definitions are included in the appendix.

2 Model

2.1 The economy

The economy is given by a simple New Keynesian model that observes the ZLB for the policy rate. The log-linearized equations are taken to be the primitives of the model, sidestepping the possibility that additional equilibria arise in the nonlinear specification.[9]

[8] These papers posit the nominal interest rate as the sole monetary policy instrument, abstracting from money growth. This is not without loss of generality, as the set of equilibria may change depending on the instrument choice of the monetary authority as the results in King and Wolman (2004) and Dotsey and Hornstein (2011) exemplify.

[9] See Braun et al. (2012). Variables are expressed as log-deviations from their efficient levels or, for nominal variables, from their steady-state values. See Woodford (2003) for the micro-foundations of the model.

Time is discrete and infinite $t = 0, 1, \ldots$. Let π_t and y_t denote the log-deviations in the inflation rate and output (or, simply, output gap). The first equation is the standard New Keynesian Phillips curve (NKPC)

$$\pi_t = \kappa y_t + \beta \pi^e_{t+1} + u_t, \tag{1}$$

where π^e_{t+1} denotes private-sector expectations for inflation at period $t+1$, formed at date t, and u_t is a cost-push shock. The slope of the NKPC and the intertemporal discount rate satisfy $\kappa > 0$ and $\beta \in (0, 1)$, respectively.[10]

The second equation in the model is the intertemporal first-order condition, the Euler equation,

$$R_t = \sigma \left(y^e_{t+1} - y_t \right) + \pi^e_{t+1} + v_t, \tag{2}$$

where R_t is the one-period nominal rate, or policy rate, y^e_{t+1} denotes private-sector expectations for the output gap at period $t + 1$, formed at date t, and v_t is a real-rate shock. The intertemporal elasticity of substitution satisfies $\sigma > 0$.

The zero lower bound for the nominal interest rate is given by

$$R_t \geq -Z, \tag{3}$$

where $Z > 0$ is the distance to the zero rate in logs from whatever long-run level for the nominal interest rate is assumed to arise under the efficient allocation.

Finally, society evaluates inflation and output gap deviations according to a quadratic loss function,

$$l_t = \pi_t^2 + \lambda y_t^2, \tag{4}$$

where the weight on output gap satisfies $\lambda > 0$. It is possible to relate λ to the other parameters of the model using its micro-foundations.[11] Total welfare is given by the loss function $L_t = E_t \sum_{j=t}^{\infty} \beta^{j-t} l_j$.

The description of the economy is closed by the shock specification. Let $s_t = \{u_t, v_t\}$. The exogenous state is governed by a first-order Markov process $F(s'|s)$ over finite support S, with cardinality n. I assume that $F(s'|s) > 0$ for all pairs $\{s, s'\} \in S^2$ and that there is a unique ergodic distribution such that the unconditional mean of both shocks is zero. Let $F^j(s'|s)$ be the conditional distribution j steps ahead. The realization of s_t is observed by all agents—including the monetary authority—at the beginning of period t.

[10] The specification of the NKPC assumes prices are indexed to a steady-state level of inflation that would prevail in the efficient allocation. This ensures that the efficient allocation is always an equilibrium in a non-stochastic economy for all the monetary frameworks considered here.

[11] Note that social welfare loss is minimized at $\pi = 0, y = 0$, the efficient allocation. This assumes there are no permanent differences between the efficient and natural levels of output, which in turn means there will be no inflationary bias when we consider policy discretion.

2.2 The monetary authority

I model the monetary framework as one of delegation. Society determines the central bank's objectives by designing its loss function, $l^p(\pi, y)$, which needs not coincide with society's welfare function, l_t. The monetary authority is instrument independent, that is, it retains discretion to set the nominal interest rate, R_t, or policy rate. This approach traces back to Rogoff (1985) and has been prevalent in the literature on nominal targets.[12] I assume the monetary authority has the same discounting as society, and thus total welfare loss is simply given by $L_t^p = \sum_{j=t}^{\infty} \beta^{j-t} l_j^p$.

A key assumption is that society is perfectly committed to the central bank's objectives, that is, the loss function $l^p(\pi, y)$ is not revised at any date. In this sense, the monetary authority's pursuit of the objectives prescribed by society is beyond any doubt, that is, there is no question of the central bank's *willingness* at all dates and states of the world to set the policy rate in the manner most congruent with the prescribed targets.

2.3 Markov equilibrium

All is set for the equilibrium definition. Formally, the model can be viewed as a game between a strategic player, the monetary authority, and a nonstrategic player, the private sector. Since the economy is infinitely lived, it is well known that history-dependent equilibria arise, based on trigger strategies. I choose to restrict the set of equilibria under analysis to Markov equilibria, that is, allocations and policy are a function of the state of the economy only.[13]

Definition 1. *A Markov equilibrium is a set of vectors in* $\Re^n$,

$$\{R, \pi, y, \pi^e, y^e\},$$

such that for all $s \in S$*:*

1. *Equilibrium equations (1)-(3) are satisfied,*
2. *Rational expectations hold,* $\pi(s) = \pi^e(s), y(s) = y^e(s)$,
3. *The nominal interest rate solves*

$$\min_R L_t^p$$

subject to (1)-(3) and taken as given private-sector expectations, π^e, y^e.

Markov equilibria are usually the equilibrium definition of choice in the literature on expectation traps; see Albanesi et al. (2003) and Armenter (2008), for example. An exception is Chari et al. (1998).

[12] See Walsh (1995) for a careful formalization of the delegation problem. It is also possible to state instead a policy rule designed to implement the desired target: see Svensson (2002) for a discussion.

[13] When studying price-level targeting it will be necessary to restate the equilibrium definition, as the state of the economy will then include the price level.

2.4 Generic economies

Let ξ be a vector collecting all the structural parameters in the model

$$\xi = \{\kappa, \beta, \sigma, Z, F, S\},$$

and Ξ the corresponding set of admissible parameter vectors constructed from the parameter restrictions given in the description of the model.[14] A property of an economy $\xi \in \Xi$ is "generic" if it holds in a neighborhood of ξ, that is, there exists $\delta > 0$ such that for any $\xi' \in \Xi$ with $||\xi - \xi'|| < \delta$ the property is also satisfied. Conversely, a property of economy ξ is not generic if there is an arbitrarily close economy $\xi' \in \Xi$ that does not satisfy it.

For all purposes here, whether a property is generic or not is settled on the basis of the number of solutions a linear equation system has. A system of n distinct linear equations will generically have exactly one solution. If n equations are combined with an additional condition, then generically the system will have no solution. Intuitively, a system of $n+1$ equations with n unknowns that has a solution implies that the system has rank n and thus implies an exact relationship between coefficients—that is, parameters in the model. It is thus a measure-zero case.

3 Willing but unable

The key to this paper's results is how private-sector expectations, together with the ZLB, constrain the range of inflation and output the monetary authority can achieve. Under some simplifying assumptions, the correspondence between the private-sector expectations and the monetary authority's feasibility set is easily derived. I show how low-inflation expectations render the inflation or price-level target unfeasible and the central bank, despite being committed to pursuing its targets, systematically fails to achieve them. I also include a brief discussion on how policy-rate stabilization may anchor expectations.

Abstract from shocks and assume the private sector believes inflation to be constant at π^e from period $t+1$ onward. The private-sector expectation for the output gap must be consistent with the inflation expectations, so it satisfies (1):

$$x^e = \frac{1-\beta}{\kappa}\pi^e$$

and is also constant over time from period $t+1$ onward. Let us now characterize the set of present-period inflation rates and output levels π_t, x_t that the monetary authority can engineer given private-sector expectations. The ZLB (3) constrains the policy rate. Substituting the Euler equation (2), we obtain

$$R_t = \sigma\left(x^e - x_t\right) + \pi^e \geq -Z.$$

[14] From these restrictions it is also immediate that Ξ is a convex set.

Inflation expectations are given, so the real rate will move with the policy rate one-to-one, which in turn will induce the output gap to adjust in the present period. However, the real rate inherits the ZLB, with its exact lower bound pinned down by inflation expectations. If the latter are high, the real rate can go very low. But if inflation expectations are low, then the real rate may not decrease much:

$$\sigma\left(x^e - x_t\right) \geq -Z - \pi^e.$$

Since the expectation for the next-period output gap is also given, we also have an upper bound on the output gap that can be achieved:

$$\begin{aligned} x_t &\leq \frac{1}{\sigma}\left(Z + \pi^e\right) + x^e \\ &\leq \frac{1}{\sigma}Z + \left(\frac{1}{\sigma} + \frac{1-\beta}{\kappa}\right)\pi^e, \end{aligned}$$

where in the second line I substituted for the output expectation solved before. The upper bound in the output gap at period t, in turn, implies an upper bound on inflation at date t, as given by (1):

$$\pi_t \leq \frac{\kappa}{\sigma}Z + \left(\frac{\kappa}{\sigma} + 1\right)\pi^e.$$

Thus the ZLB imposes an upper bound on the inflation and output the monetary authority can engineer. The bound is tighter the lower inflation expectations are—as the real interest rate cannot drop below the ZLB minus inflation expectations.

Consider two hand-picked levels of inflation expectations, $\pi^e = 0$ and $\pi^e = -Z$. In the former case, the private sector expects inflation and output to be fully stabilized from the next period onward. It is straightforward to check that the monetary authority could stabilize inflation and output in the present period as well, $\pi_t = x_t = 0$, simply by setting $R_t = 0$. As long as the monetary authority's objectives value full stabilization—and it would be extravagant not to—this will be the basis for the "good" Markov equilibrium.

Now consider the possibility that inflation expectations are quite low, $\pi^e = -Z$. The monetary authority finds it cannot implement a higher inflation rate than the private sector expected, $\pi_t \leq \pi^e$. Similarly, its options regarding the output gap are limited, $x_t \leq x^e$. Regardless of the monetary authority's policy rate decision, inflation and output will *both* be below their efficient levels. If it validates the private-sector expectations, the nominal interest rate will be right at the ZLB. In other words, the central bank's only chance to disavow expectations is on the downside, that is, further away from full stabilization, by setting a higher policy rate than expected.

It is finally time to introduce the monetary authority's objectives. Consider first inflation targeting, loosely defined here as a joint goal of output and inflation stabilization, with an arbitrary weight on one over the other goal. If the private sector expects full

stabilization, $\pi^e = 0$, the central bank has no trouble achieving its objectives.[15] However, if inflation expectations are sufficiently low, the monetary authority will simply be unable to achieve its goals—no matter how committed it is to pursuing the specified targets. It is also irrelevant how output and inflation deviations are weighted, since both will fall short of their targets. The monetary authority will then do its best to minimize the deviations, which is to set the nominal rate at zero and validate the private-sector expectations.

In a stochastic economy the feasibility set of the monetary authority is not so tightly set, as shocks can give the central bank some space to maneuver the policy rate. In Section 4, I show that equilibrium multiplicity arises under any shock process.[16] The dynamics of the policy rate are also somewhat more complex. The policy rate may be at the ZLB in the stabilization equilibrium, though typically only briefly. In the low-inflation equilibrium the economy spends long spells at the liquidity trap, though the policy rate may rise also above the ZLB for substantial periods.

What does it take to rule out the low-inflation equilibrium? To disprove low-inflation expectations, the monetary authority should be willing to decrease inflation and output further. Such a response can be induced if we include a stabilization goal for the policy rate—i.e., a penalty term for policy rate deviations. If the weight on the latter is large enough in policy loss function, the monetary authority will increase the policy rate despite depressing output and inflation further in doing so. In Section 5, I provide a threshold condition on the weight on policy-rate stabilization that it is sufficient to anchor expectations, that is, the Markov equilibrium is unique. Unfortunately, this is not without costs, as it distorts stabilization policy overall.[17]

Turning to price-level targeting, it is clear that if the price level is at or below target, the situation is essentially identical to inflation targeting: Faced with low-inflation expectations, the monetary authority will fall short of both the output and price-level targets. However, if the price level is *above* target, the monetary authority may actually want to lower inflation below the private-sector expectations. Does this rule out the low-inflation equilibrium? Unfortunately, it does not. I show in Section 6 that there is an equilibrium such that, for any price level, the economy eventually converges to the ZLB. Along the path, both inflation and output stay below target.

4 Inflation targeting

Inflation targeting was introduced in New Zealand in 1990. Close to 25 years later, it is still the monetary framework of choice in many developed economies. Researchers have found strong theoretical foundations to back up inflation targeting's success. Bernanke

[15] If there are cost-push shocks, the monetary authority will need to trade off output and inflation deviations, but both will fluctuate around their efficient levels.

[16] Unfortunately, it is also possible to have no equilibria or to have more than two.

[17] Moreover, sunspot equilibria could arise as the Taylor principle will not be satisfied.

and Mishkin (1997) described inflation targeting as "constrained discretion," where the monetary authority operates with well-defined goals under a high degree of transparency and accountability. In this context, inflation targeting has been repeatedly shown to be preferable to "full" discretion.[18]

The literature is unanimous that an institutional commitment is a necessary condition for inflation targeting. Svensson (2002) goes as far as to consider commitment as "essential for inflation targeting to have much meaning."[19] The results below show that, unfortunately, such a commitment is not sufficient to ensure that an inflation-targeting central bank achieves its goals.

4.1 The monetary authority's objectives

I adopt the standard specification for a flexible inflation-targeting central bank,

$$l_t^p = \pi_t^2 + \psi y_t^2, \tag{5}$$

where $\psi \geq 0$. The case $\psi = \lambda$ corresponds to a central bank pursuing exactly society's welfare, while the case $\psi = 0$ is what has been termed an "inflation nutter." Because of the underlying assumptions in the economy, there is no "inflationary bias" problem as in Barro and Gordon (1983). There is, however, a "stabilization bias," as defined in Svensson (1997) and Clarida et al. (1999), which provides a rationale for why society would like to set $\psi < \lambda$.

The necessary and sufficient first-order condition associated with the monetary authority's problem is simply

$$\kappa\pi(s) + \psi y(s) \leq 0 \tag{6}$$

with strict equality if $R(s) > -Z$. Equation (6) summarizes what the monetary authority intends to achieve and describes what it will actually achieve. Ignore first the possibility that the ZLB is binding and thus the first-order condition will hold with strict equality. Real-interest rate shocks pose no obstacle to fully stabilizing output and inflation. Cost-push shocks, though, will force the monetary authority to trade off inflation deviations for output deviations: It will do so at rate ψ/κ, being κ the marginal rate of "transformation" of inflation to output deviations, and ψ the corresponding marginal rate of substitution as dictated by the policy loss function. The monetary authority will end up missing both targets, but never on the same side. That is, whenever inflation is above target, output will be below, and vice versa.

The monetary authority may also miss its targets because of the ZLB. Now both cost-push and real-interest rate shocks can create inflation and output deviations. Moreover, it is possible that both inflation and output are simultaneously below target, that is, (6)

[18] Svensson (1997) is credited as a keystone in the literature, yet research on inflation targeting remains active on several fronts: see Bernanke and Woodford, eds (2005) for an overview.

[19] Page 772.

is slacked. These possibilities and their resulting dynamics have been extensively studied elsewhere, most recently in medium- to large-scale macroeconomic models.[20] The ZLB, though, also opens the possibility that the private-sector expectations, by themselves, render the monetary authority unable to achieve its prescribed goals.

4.2 Equilibrium characterization

It is possible to solve the model exclusively in terms of actual and expected inflation dynamics. I proceed to do so. From (1), the output gap can be expressed as

$$\kappa y(s) = \pi(s) - \beta E_1(\pi|s) - u,$$

where

$$E_1(\pi|s) = \sum_S \pi(s') F(s'|s)$$

is the one-step-ahead inflation expectation. Then I express the ZLB condition (3) using the Euler equation (2) as

$$\sigma(E_1(y|s) - y(s)) + E_1(\pi|s) + v \geq -Z,$$

where $E_1(y|s)$ follows the same shorthand notation as $E_1(\pi|s)$. Let $\nu = \kappa\sigma^{-1}$. Substituting for actual and expected output, we arrive at

$$\pi(s) \leq \pi^b(s) \equiv (1+\nu+\beta) E_1(\pi|s) - \beta E_2(\pi|s) + A(s), \tag{7}$$

where $A(s) = u - E_1(u|s) + \nu v + \nu Z$ and

$$E_2(\pi|s) = \sum_S E_1(\pi|s') F(s'|s)$$

is the two-steps-ahead inflation expectation.

Returning to (6), I obtain

$$\pi(s) \leq \pi^u(s) \equiv \frac{\psi}{\kappa^2+\psi}(\beta E_1(\pi|s) + u). \tag{8}$$

At every state, either (7) or (8) must hold with strict equality. Thus whether the ZLB is binding or not is resolved by

$$\pi(s) = \min\left\{\pi^u(s), \pi^b(s)\right\}. \tag{9}$$

The equilibrium inflation vector π is thus characterized by a system of n piece-wise linear equations from a total of $2n$ linear functions. There is no difficulty in retrieving the equilibrium values for the interest rate and the output gap.

[20] See Fernandez-Villaverde et al. (2012) and Lopez-Salido et al. (2010) among many others.

4.3 Analytic results

For the centerpiece result of this paper, I show that there is, generically, no unique equilibrium under inflation targeting. The result is surprisingly broad, as it does not require any further restriction on parameters and encompasses any shock process we wish to impose. It also applies to all inflation-targeting regimes, from a central bank with very high output weight to an inflation nutter, that is, any policy loss function $\psi \geq 0$.

An analytic result is particularly useful because the system of nonlinear equations described by (9) poses some challenges. Equilibria can be characterized as fixed points of a vector-valued function. However, the resulting mapping is not well behaved due to the second-order difference equation, (7), governing inflation dynamics whenever the ZLB is binding. It is indeed possible that there may be no Markov equilibria at all for some economies. Numerical methods inherit these difficulties: Iteration by adaptive expectations, for example, may fail to converge even if an equilibrium does exist.

Proposition 4.1. *Let $\pi^* \in \Re^n$ be a Markov equilibrium. Then, generically, there exists at least an additional distinct Markov equilibrium, $\tilde{\pi} \neq \pi^*$.*

Proof. In the appendix □

The proof takes a novel approach that exploits the piece-wise linear structure of the equilibrium equations. For each state, the two equilibrium conditions (7) and (8) can be viewed as describing two half-spaces in $\Re^n$, namely,

$$H^u(s) = \{x \in \Re^n : x \geq (1 + \nu + \beta) E_1(x|s) - \beta E_2(x|s) + A(s)\}$$

and

$$H^b(s) = \left\{x \in \Re^n : x \geq \frac{\psi}{\kappa^2 + \psi}(\beta E_1(x|s) + u)\right\}.$$

Note their intersection naturally satisfies (9). I show that the $2n$ half-spaces describe a solid polytope in $\Re^n$: A Markov equilibrium must be a vertex, as it is a point where the boundaries of n of those half-spaces intersect.[21] Not all vertexes are equilibria, though. There are "kinks" defined by the intersection of the boundary of two half-spaces corresponding to the same equilibrium equation. The final step of the proof shows that it is not possible that all vertexes but one are "kinks," and thus there are at least two Markov equilibria.

One shortcoming of Proposition 4.1 is that while it rules out a unique equilibrium, it does not establish the actual number of equilibria—no equilibria being an open possibility. Fortunately, the proof of Proposition 4.1 shows that characterizing the full set of equilibria is a vertex enumeration problem, for which readily available algorithms already exist. While computationally expensive, these algorithms guarantee that no Markov equilibria will be missed and thus that can be used to establish equilibrium nonexistence as well.

[21] Incidentally, this proves that each Markov equilibrium is locally unique.

While it may not be immediately apparent, Proposition 4.1 does not apply to economies with i.i.d. shocks, as such restriction is "non-generic" from the point of view of the general set of possible shock processes. This is no major problem as an equivalent result exists for i.i.d. economies, with a much simpler proof.

Proposition 4.2. *Let $F(s'|s) = F(s')$ for all $s, s' \in S$. Then if there exists a Markov equilibrium, there is, generically, another distinct equilibrium.*

Proof. In the appendix □

4.4 Numerical example

The analytic results are admittedly limited as they are silent regarding each of the equilibrium properties. I explore here numerically an economy based on standard parameter values in the literature—the model, however, is too simple for this exercise to be a proper calibration.

The model is evaluated at the quarterly frequency. I assume log-preferences so $\sigma = 1$ and set the intertemporal discount rate β at .994 such that the annual real interest rate is 2.5 percent. The slope of the NKPC, κ, is .024. This is the value used in Rotemberg and Woodford (1997) and followed countless times for small-scale New Keynesian models. I assume a target inflation of 2 percent annual and set the zero lower bound accordingly. The weight on the output gap in the social welfare function, λ, is given by the structural parameters assuming an elasticity of substitution across goods of 5.[22] Regarding the shock processes, I approximate an independent Gaussian auto-regressive process for each shock, with standard deviations σ_u and σ_v set at .07 and .6, respectively, and auto-correlation coefficients at $\rho_u = .8$ and $\rho_v = .9$.[23] These values roughly approximate the volatility and auto-correlation of the output gap and inflation in the U.S., as given by the CBO output gap and core PCE, when the model is evaluated under discretion, i.e., $\psi = \lambda$, and ignoring the ZLB.

The first two columns in Table 1 correspond to an economy without the ZLB and thus a unique equilibrium. In the first column I compute the optimal policy, from a timeless perspective, as given by the equation $\pi_t = \lambda/\kappa\,(y_t - y_{t-1})$. The first column reports the welfare loss, according to society. All inflation moments are given in annualized percentages—output moments, in percentage deviations from the efficient level (output gap). There is not much to report on the optimal policy except in contrast to inflation targeting. The second column reports the result of inflation targeting, setting the output weight arbitrarily at just half the weight in the social welfare loss function, $\psi = .5\lambda$—still ignoring the ZLB. Inflation targeting approximates quite well the optimal policy outcome,

[22] That is, $\lambda = \kappa/5 = .005$. See Woodford (2003) for details on the derivation.

[23] I use the Rouwenhorst method 16 grid points for the state space. See Kopecky and Suen (2010) for a description and discussion of the Rouwenhorst method.

	Optimal Policy	**Inflation Targeting**		
	No ZLB	**No ZLB**	**ZLB**	
L_t	0.023	0.045	0.047	0.82
$E\pi_t$	2.00	2.00	1.96	0.38
Ey_t	0.00	0.00	-0.01	-0.10
σ_y	1.34	1.35	1.32	3.07
σ_π	0.287	0.542	0.569	2.13
$\rho(\pi_t, y_t)$	-0.134	-1	-0.954	0.746
$\rho(y_t, y_{t-1})$	0.964	0.804	0.796	0.822
$\rho(\pi_t, \pi_{t-1})$	0.541	0.804	0.799	0.857
$\Pr(R_t = -Z)$	-	-	0.0625	0.453
$E\tau_Z$	-	-	3.95	11.6

Table 1: Welfare loss and moments across policies and economies

with only slightly more output volatility and a more sizable, yet still modest, increase in inflation volatility. Since there is no ZLB, real-rate shocks are perfectly stabilized and the only variation comes from cost-push shocks, which send inflation and output in opposite directions.

Upon reintroducing the ZLB, I find there are two Markov equilibria, reported in the last two columns of Table 1.[24] There is no mistaking which equilibrium is the stabilization or "good" equilibrium. There is indeed little difference between the second and third column, as the ZLB binds only rarely and for short spells—about 6 percent of the time and for an average of four quarters, respectively. In contrast, the last column features quite large volatility of both output and inflation, as well as a low inflation rate average—though not quite outright deflation. The economy spends close to half of the time at the ZLB, with spells at the ZLB averaging close to three years.

Once the ZLB is observed, the economy is not linear and thus the first and second moments reported in Table 1 do not tell the whole story. I simulate the economy for 100 periods and plot the results in Figure 1 for both Markov equilibria: the stabilization equilibrium in blue and the low-inflation equilibrium in green. Clockwise, from the top left, Figure 1 displays the output gap, the inflation rate, the nominal rate, and the real rate. All variables but the output gap are in annualized percentage rates; the output gap is in percentage deviations from its efficient level.

It is immediately apparent from Figure 1 that what sets both equilibria apart are infrequent yet steep deflation episodes in the low-inflation equilibrium. Output and inflation collapse as the real rate shoots up since the policy rate is already at the ZLB. In the stabilization equilibrium, these episodes are much milder as the monetary authority has

[24] I explored a range of parameters and pervasively found two Markov equilibria. For very persistent shock processes there were occasionally no equilibrium.

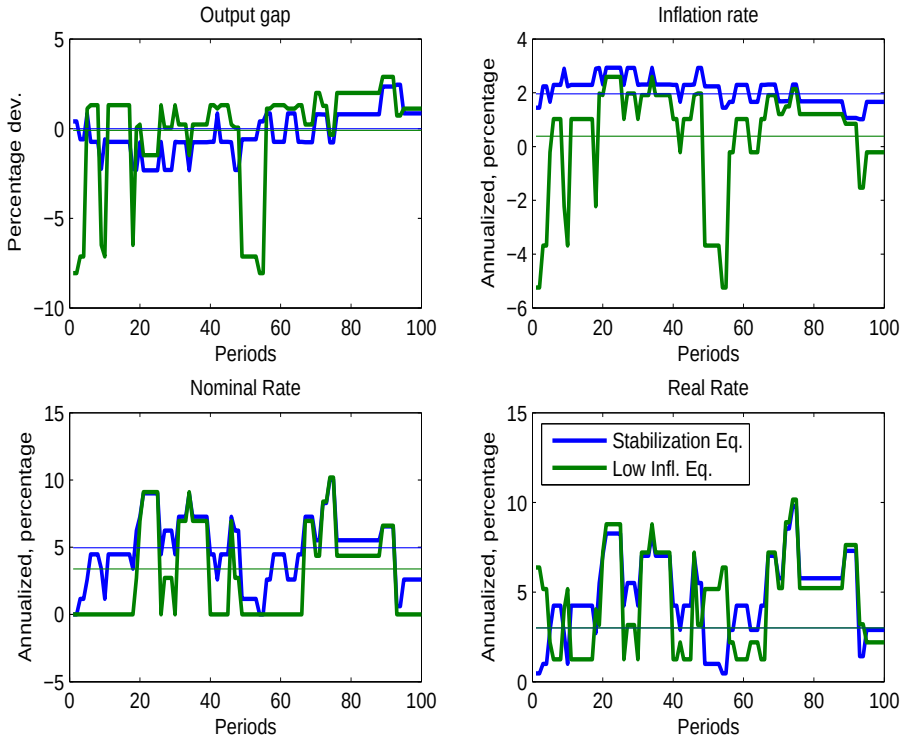

Figure 1: Simulations for inflation targeting Markov equilibria

room to maneuver and can decrease the policy rate aggressively, bringing the real rate down.

With the exception of these deflation episodes, both equilibria look strikingly similar. The difference in average inflation is noticeable, but there is substantial overlap across equilibria. Output gap, similarly, fluctuates on a narrow range around its efficient level most of the time in both equilibria. Also note how the nominal rate in the low-inflation equilibrium has long spells far away from the ZLB, occasionally spiking at pretty high rates. Conversely, the policy rate does touch on the ZLB in the stabilization equilibrium, though only rarely.

5 Policy-rate stabilization

In this section I show that private-sector expectations can be anchored into a unique Markov equilibrium by introducing a stabilization goal for the policy rate, that is, a penalty term for policy rate deviations in the monetary authority's objective function. However, there is no guarantee that policy-rate stabilization allows the monetary authority to achieve its inflation and output targets or improves welfare upon any of the Markov

equilibria under inflation targeting.

5.1 The monetary authority's objectives

The monetary authority's inflation target is combined with a penalty term for deviations in the policy rate from its steady-state level—in other words, the stabilization of the nominal interest rate is made an explicit goal of the central bank. The policy loss function is specified as

$$l_t^p = \pi_t^2 + \psi y_t^2 + \rho R_t^2, \tag{10}$$

where $\psi, \rho \geq 0$. I should emphasize that the term ρR_t^2 is taken to be a deviation from the social welfare, which remains given by (4). However, there are foundations for such a term to be included in a micro-founded social welfare loss function if, for example, there are transaction frictions.[25] It should also be noted that the stabilization goal is in terms of the level of the policy rate, not its changes from period to period.

The necessary and sufficient first-order condition associated with the monetary authority's problem is

$$\kappa\pi\left(s\right) + \psi y\left(s\right) \leq \rho\sigma R\left(s\right) \tag{11}$$

with strict equality if $R\left(s\right) > -Z$. The comparison with the first-order condition in the case of inflation targeting (6) is straightforward: The monetary authority is balancing the stabilization of output and inflation—the terms in the left-hand side of equation (11)—with the stabilization of the policy rate. Note that the resulting expression is reminiscent of a simple Taylor rule without persistence.

5.2 Equilibrium characterization

The steps to characterizing an equilibrium are very similar to those in Section 4. Whenever the ZLB is binding, there is no policy decision, and thus the equilibrium condition is given by the Euler equation, or equation (7) once the output gap has been substituted for. When the ZLB is not binding, the first-order condition (11) dictates policy. Substituting the nominal interest rate using (2), I obtain

$$\kappa\pi\left(s\right) + \psi y\left(s\right) \leq \rho\sigma\left(\sigma\left(E_1(y|s) - y\left(s\right)\right) + E_1\left(\pi|s\right) + v\right).$$

The output gap can be solved for in terms of actual and expected inflation using the NKPC (1). After some algebra, the equilibrium condition is

$$\pi\left(s\right) \leq \pi^u\left(s\right) \equiv b_1 E_2(\pi|s) + b_2 E_1(\pi|s) + B\left(s\right), \tag{12}$$

[25] See Woodford (2003) and Rudebusch (2006) for a discussion.

where

$$\begin{aligned} b_1 &= -\frac{\sigma^2\rho\beta}{\kappa^2+\psi+\sigma^2\rho}, \\ b_2 &= \frac{\beta\psi+\sigma^2\rho(1+\beta)+\rho\sigma\kappa}{\kappa^2+\psi+\sigma^2\rho}, \\ B(s) &= \frac{\rho\sigma\kappa v+(\psi+\sigma^2\rho)u-\sigma^2\rho E_1(u|s)}{\kappa^2+\psi+\sigma^2\rho}. \end{aligned}$$

The two equations are combined in (9), automatically determining whether the ZLB is binding or not.

5.3 Anchoring expectations on a single Markov equilibrium

The main result in this section is a sufficient condition to anchor private-sector expectations: Proposition 5.1 states that if the weight on interest-rate stabilization is larger than a threshold, then there is a unique Markov equilibrium. Unfortunately, an analytic result is available only for i.i.d. shocks and I have to resort to numerical methods to explore the case of persistent shocks.

Proposition 5.1. *Let $F(s'|s) = F(s')$ for all $s, s' \in S$. If $\rho > \frac{\kappa^2+\psi(1-\beta)}{\sigma\kappa}$ then there is, generically, a unique Markov equilibrium.*

Proof. In the appendix □

The threshold depends on the parameters governing the monetary policy transmission, κ, σ, β, as well as the relative weight given to output in the policy loss function. Conspicuously absent are any terms regarding the volatility of the shock process or, more surprisingly, the distance to the ZLB, Z. The reason is that the threshold is not designed to avoid the ZLB at all costs. As the numerical example below shows, the ZLB can be binding in response to shocks. Emphasizing interest-rate stability, though, ensures that the monetary authority does not accommodate any permanent shift in inflation expectations, choosing to keep the nominal interest rate close to the steady state.

Unfortunately, there is no guarantee that policy-rate stabilization allows the monetary authority to achieve its output and inflation targets and, more worryingly, that it actually improves welfare upon the low-inflation equilibrium that arises from inflation targeting. By construction, the monetary authority's response to shocks will be muted—including to real-interest rate shocks, which can typically be stabilized completely if the ZLB is not binding. Moreover, Proposition 5.1 proves uniqueness of Markov equilibria, but it is likely that other rational-expectations equilibria exist. In particular, there may exist equilibria that feature history dependence or sunspots. The latter is a likely possibility since the interest-stabilization goal leads the nominal interest rate to react less than one-to-one with inflation, thus violating the Taylor principle. The threshold in Proposition

5.1 implies that the inflation coefficient in the first-order condition, (11), is less than one; however, since the economy is not linear, it is not immediate that sunspots arise. In any case, a characterization of all rational-expectations equilibria is beyond the scope of this paper.

5.4 Numerical example

I return to the previous numerical example to explore the implications of a policy-rate stabilization goal. The key parameter, ρ, is set to the minimum value to guarantee equilibrium uniqueness: Its value is virtually identical to the threshold presented in Proposition 5.1. All other parameters remain as documented earlier.

Unfortunately, the remedy turns out to be worse than the disease. Table 2 reports the results as well as the policy parameters used in the computation in the first column, labeled "Framework 1." The welfare loss is higher than in the low-inflation equilibrium (reported in Table 1). Inflation is artificially high, so the monetary authority misses its inflation target systematically. The standard deviations for output and inflation are way above their values under inflation targeting. There are two reasons for the poor stabilization policy. First, the monetary authority seeks to undo real-interest rate shocks with output and inflation: Under inflation targeting, real-interest rate shocks destabilize only if the ZLB is binding. Second, inflation and output co-move strongly, which only exacerbates the impact of cost-push shocks. Note that, despite the policy-rate stabilization goal, the economy does visit the ZLB with positive, if small, probability.

Now there is no good reason to introduce a policy-rate stabilization goal in isolation. In the second column in Table 2 the weight on the output gap, ψ, has been doubled. This monetary framework dominates the low-inflation equilibrium under inflation targeting, though it is nowhere close to replicating the results of the stabilization equilibrium.[26] Inflation remains well above target and, as matter of fact, the monetary authority is not even able to stabilize the policy rate properly.

Overall, the numerical results are quite discouraging: While policy-rate stabilization effectively anchors expectations on a unique Markov equilibrium, the latter has none of the desirable properties of the stabilization equilibrium under inflation targeting. Further research may show how to implement policy-rate stabilization without conflicting with output and inflation stabilization: Perhaps it should be designed as an "escape" clause from the ZLB but not be active away from the ZLB.

[26] I explored a wide range of output weights, adjusting the weight on policy-rate stabilization accordingly: I could find only very modest improvements upon the value reported, $\psi = \lambda$.

	Framework 1	**Framework 2**
Policy parameters		
ψ	$.5\lambda$	λ
ρ	.026	.026
Results		
L_t	1.41	0.79
$E\pi_t$	3.84	3.33
Ey_t	0.11	0.08
σ_y	3.12	2.17
σ_π	3.06	2.28
$\rho(\pi_t, y_t)$	0.92	0.86
$\rho(y_t, y_{t-1})$	0.90	0.90
$\rho(\pi_t, \pi_{t-1})$	0.89	0.87
$\Pr(R_t = -Z)$	.0625	.0625
$E\tau_Z$	3.94	3.94

Table 2: Welfare loss and moments under policy-rate stabilization

6 Price-level targeting

Inflation targeting does not anchor private-sector expectations. Introducing a stabilization goal for the nominal rate does so, but it is unsatisfactory on several counts. Perhaps price-level targeting fares better? Unfortunately, it does not.

Svensson (1999) was among the first to analyze price-level targeting at par with inflation targeting. In a key contribution, Vestin (2006) shows that a monetary authority committed to pursuing a price-level target would actually mimic the optimal policy. His analysis ignored the ZLB on the policy rate: It turns out that what really sets apart price-level targeting is its ability to approximate the optimal policy at the ZLB, as first pointed out by Eggertsson and Woodford (2003).[27]

6.1 The monetary authority

The price level, in terms of deviations from some steady-state underlying trend, is given by

$$p_t = \pi_t + p_{t-1}. \tag{13}$$

[27]See also Wolman (2005), Jung et al. (2005), Adam and Billi (2006) and Nakov (2008). There is very limited empirical evidence on price-level targeting because it has rarely been implemented. An exception is Berg and Jonung (1999), who provide an account of Sweden's experience with price-level targeting in the decade of the 1930s.

Note the timing regarding the definition of the price level: p_t is the end-of-period t price level. The price-level target is implemented by endowing the monetary authority with an objective function

$$l_t^p = p_t^2 + \psi y_t^2, \tag{14}$$

where $\psi \geq 0$ allows for flexible price-level targeting, i.e., the monetary authority also has a stabilization goal for output. The initial value for the price level, p_0, is taken as given. The specification (14) for the price-level targeting nests nominal GDP targeting as a special case with $\psi = 1$.

The price level is now an endogenous state variable in this economy.[28] As a result, the monetary authority's decision has an explicit intertemporal dimension: When setting the nominal interest rate in the present period, the monetary authority realizes that it will impact the price level, which will, in turn, shape future rate decisions. The analysis is further complicated because one should not expect the continuation value function to be differentiable with respect to the price level. In order to be able to characterize the law of motion for the price level, I have no choice but to restrict the analysis to perfect-foresight economies.

6.2 Results

Here I show that there is a Markov equilibrium such that, for any initial price level, p_0, the nominal interest rate converges to the ZLB in finite time, with output and the price level falling permanently below target after that. There is, of course, a more benign Markov equilibrium where inflation, output, and the price level converge to full stabilization—see Vestin (2006) and Eggertsson and Woodford (2003) for details.

Proposition 6.1. *Consider a perfect-foresight economy, $F(\{0,0\}) = 1$. There exists a Markov equilibrium and a finite time $t^*(p_0)$ such that, for all $t \geq t^*(p_0)$, the nominal interest rate is at the ZLB, $R_t = -Z$, and both inflation and output are below target, $\pi_t, y_t < 0$.*

Proof. In the appendix □

As discussed in Section 3, the steady-state inflation and output levels are the same as in the low-inflation equilibrium under inflation targeting. The key contribution of Proposition 6.1 is to show that there is global convergence to the low-inflation steady state. The proof proceeds by construction, conjecturing an upper bound on the law of motion for the price level as well as the aforementioned steady-state level for inflation and output. Note that since inflation is systematically below target, the price level is drifting further and further away from its target. Since the policy loss function is unbounded, the proof must also show that the monetary authority's value function remains well defined.

[28] The appendix contains an updated definition of a Markov equilibrium.

The result in Proposition 6.1 stands in contrast to some claims in the literature. Eggertsson and Woodford (2003) write that "eventually it [the central bank] will be able to hit the target" when discussing price-level targeting.[29] What is behind such contradicting conclusions? In Eggertsson and Woodford (2003) the economy finds itself against the ZLB due to a real-interest rate shock. Once the shock unwinds, the economy lifts from the ZLB and the monetary authority has no trouble returning the price level to the desired target. In my analysis, though, the culprit of the economy's dire situation is private-sector expectations that do not unwind, and thus the monetary authority never has a chance to return the price level (or inflation) to target.[30]

7 Conclusions

In this paper I have argued that nominal targets present some perils along with their many virtues—the latter being numerous enough to wish the former could be safely dismissed. After all, the experience with inflation targeting has been very successful. Several inflation-targeting central banks have kept their policy rates virtually at zero for extended periods, yet inflation expectations have remained anchored. The one country that appears perennially marooned on a liquidity trap, Japan, has no nominal target. I briefly ask here whether there is a valid basis to conclude that expectations are anchored after all.[31]

One possibility is that the ZLB is not such a constraint on policy as I made it out to be. Several central banks embarked on asset purchase programs as an additional policy tool when the nominal rate hit the ZLB. Were we to find these policies as effective as the policy rate, the central bank's hands would not be tied in the face of low-inflation expectations. However, the logic of the rational expectations is quite demanding: The effect of the asset purchases must be unbounded to ensure a unique equilibrium. Note that my results did not hinge on the particular value of the lower bound on the policy rate, Z. We could conceivably set Z far below zero, capturing a policy-rate equivalent impact of asset purchases, and we would yet face the same perils.

More broadly, we can also question the model used here. This poses a conundrum, since it is the very same framework universally used to argue for the merits of inflation targeting. A criticism of the equilibrium concept may fare better. While Markov equilibria are a narrow subset of all rational expectations equilibria, there is a question of whether all the equilibria are stable under some form of learning by the private sector. If the literature on Benhabib et al. (2000) is any indication, all steady states may have

[29] Page 184.

[30] While already noted, it is perhaps worth repeating that a commitment to keep policy rates low for an extended period does not return the economy to the target: Policy rates are expected to remain extremely low anyway in the low-inflation equilibrium.

[31] See Bullard (2010) for a similar discussion in the context of U.S. monetary policy and the results in Benhabib et al. (2000).

some converging learning dynamics associated.[32] Bullard (2010) also points out that the Japanese experience suggests a liquidity trap is a real possibility.

In my opinion, the most appealing argument is that a nominal target may bring coordination beyond the underlying commitment—which, as I have shown, is not sufficient. Private-sector agents may simply pencil in the central bank's targets and forgo developing their own forecast and, as the central bank is then enabled to achieve its targets, and these same agents find no reason to revise their trust. This, though, brings more questions than answers. How is such a concept of credibility acquired? How may it be lost? I must then close my paper with the ever-exasperating conclusion that further research on the subject is needed.

[32]See Eusepi (2007) and Evans et al. (2008).

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A Appendix

A.1 Section 4

Proof of Proposition 4.1. Assume economy ξ has one equilibrium. I will show that, generically, there is at least another equilibrium.

Consider the vector-valued function $\Gamma : \Re^n \to \Re^n$ given by

$$\Gamma_i(g;\xi) = \min\left\{\pi_i^u(g;\xi), \pi_i^b(g;\xi)\right\} \tag{15}$$

for each $i = 1, 2, \ldots, n$, where, in a slight abuse of notation,

$$\begin{aligned} \pi_i^b(g;\xi) &= (1+\nu+\beta)E_1(g|s_i) - \beta E_2(g|s_i) + A(s_i), \\ \pi_i^u(g;\xi) &= \frac{\psi}{\kappa^2+\psi}\left(\beta E_1(g|s_i) + u(s_i)\right). \end{aligned}$$

Each vector-valued function can be written concisely in matrix form as

$$\begin{aligned} \pi^b(g;\xi) &= C^b g + A, \\ \pi^u(g;\xi) &= C^u g + u. \end{aligned}$$

Clearly g is an equilibrium if and only if $\Gamma(g;\xi) = g$. Let g^* be an equilibrium whose existence is the premise of the proof. Let $h(g;\xi) = \Gamma(g;\xi) - g$ and

$$H = \{g \in \Re^n : h(g;\xi) \geq 0\}.$$

The set H is non-empty, since it includes g^*. It is also closed since h is continuous. It is also convex since Γ is the minimum of an ensemble of linear functions and thus weakly concave. In the next two lemmas I prove that H has a non-empty interior in a generic economy and is bounded.

Lemma A.1. *The set H has, generically, a non-empty interior.*

Proof. Let $e \in \{0,1\}^n$ with $e_i = 1$ if $\pi_i^b(g^*;\xi) \leq \pi_i^u(g^*;\xi)$, $e_i = 0$ otherwise, for all $i = 1, 2, \ldots, n$. Since $\Gamma(g^*;\xi) = g^*$, the equilibrium satisfies

$$e \circ \left(C^b g^* + A\right) + (1-e) \circ (C^u g^* + u) = g^*. \tag{16}$$

Generically, $\pi_i^b(g^*;\xi) < \pi_i^u(g^*;\xi)$ for all i such that $e_i = 1$: otherwise, adding $\pi_i^b(g^*;\xi) = g_i$ to (16) delivers a system of $n+1$ linear equations and n unknowns, which has generically no solution. Thus there exists $\delta > 0$ such that for all $g \in B_\delta(g^*) = \{g \in \Re^n : ||g - g^|| < \delta$, $\Gamma(g;\xi) = e \circ \left(C^b g^* + A\right) + (1-e) \circ (C^u g^* + u)$. Thus h is linear in $B_\delta(g^*)$ with $D\Gamma(g;\xi) = e \circ C^b + (1-e) \circ C^u$. Generically $D\Gamma(g^*;\xi)$ has an inverse and thus for any $\delta > 0$ there exists $x \in \Re^n$, $||x|| < \delta$ such that $D\Gamma(g^*;\xi)x > \mathbf{1}$, and hence $h(g+x;\xi) > 0$. Hence the interior of H is non-empty □

Lemma A.2. *The set H is, generically, bounded.*

Proof. By Lemma A.1 there exists, generically, a point g' such that $h(g';\xi) > 0$. Consider the subset $X \in \Re^n$ of normalized vectors $x \in \Re^n$ of length 1 such that $\max\{x\} \geq 0$. For any $x \in X$, pick $j \in \arg\max_i \left\{ x_i - \frac{\beta\psi}{\kappa^2+\psi} E_1(x|s_i) \right\}$ and let $\delta : X \to \Re$ be

$$\delta(x) = \frac{\pi^u_j(g';\xi)}{x_j - \frac{\beta\psi}{\kappa^2+\psi} E_1(x|s_j)}.$$

To show that $\delta(x)$ is finite and strictly positive for all $x \in X$, note first that if $x_i = \alpha$ for all i then $\alpha > 0$ by $x \in X$ and

$$x_j - \frac{\beta\psi}{\kappa^2+\psi} E_1(x|s_j) = \left(1 - \frac{\beta\psi}{\kappa^2+\psi}\right)\alpha > 0.$$

If $x_i \neq x_k$ for some pair in x, then $\max\{x\} > E_1(x|s_j)$ since $F(s_i|s_j) > 0$ for all pairs. Thus

$$x_j - \frac{\beta\psi}{\kappa^2+\psi} E_1(x|s_j) \quad > \max\{x\} - \tfrac{\beta\psi}{\kappa^2+\psi}\max\{x\} \geq 0.$$

Thus $x_j - \frac{\beta\psi}{\kappa^2+\psi} E_1(x|s_j) > 0$, which implies both that $\delta(x) > 0$ since $\pi^u(g';\xi) \geq h(g';\xi) > 0$ and $\delta(x)$ is bounded above. Since X is a compact space, $\delta(x)$ achieves its maximum in X, $\bar{\delta} = \max_{x\in X} \delta(x)$. By construction, for any $\varepsilon > 0$ and any $x \in X$,

$$\pi^u_j\left(g' + x\left(\bar{\delta}+\varepsilon\right);\xi\right) - \left(g' + x\left(\bar{\delta}+\varepsilon\right)\right) < 0$$

and thus $g' + x\left(\bar{\delta}+\varepsilon\right) \notin H$ since $h(g;\xi) \leq \pi^u(g;\xi) - g$. This proves the set H is bounded in all quadrants of R^n except the strictly negative, i.e., $x < 0$. For this last step, note that for any g and scalar τ,

$$h(g-\tau;\xi) \leq \pi^b(g-\tau;\xi) - (g-\tau) = \pi^b(g;\xi) - g - \nu\tau.$$

Consider the hyperplane $Z = \{g \in \Re^n : \sum g_i = \sum g'_i\}$. By construction, $g' \in Z$ so $Z \cap H$ is non-empty. In addition, $Z \cap H$ is also compact since any $g \in Z \cap H$ can be expressed as $g = g' + x||g - g'||$ for some $x \in X$. Thus each component of π^b achieves a maximum in $Z \cap H$ and

$$\bar{\tau} = \frac{1}{\nu} \min_i \max_{g \in Z\cap H} \pi^b_i(g;\xi)$$

is well defined. Then for any $\varepsilon > 0$, $g \in \{Z - (\bar{\tau}+\varepsilon)\}$ implies $g \notin H$ as $\pi^b_j(g;\xi) - g_j < 0$ for some j. Thus for any $g \in H$ with $g < g'$, $||g - g'|| \leq \bar{\tau}$. Thus H is contained in a ball of radius $\max\{\bar{\delta}, \bar{\tau}\}$ centered at g' □

The set H is the intersection of the collection of the $2n$ half-spaces given by the n equilibrium equations, each evaluated with the ZLB or not. Let K be an index set over $\{1,2,\ldots,n\}\times\{u,b\}$ with functions $i:K\to\{1,2,\ldots,n\}$ denoting the state and $j:K\to\{u,b\}$ whether the ZLB binds or not. Define the half-space H_k,

$$H_k=\left\{g\in\Re^n:\pi_{i(k)}^{j(k)}\left(g\right)-g\geq 0\right\}.$$

Formally, $H=\bigcap_K H_k$ and thus H is a polytope. By the previous two Lemmas, H is also non-empty and compact and therefore H is a solid (or proper) polytope in $\Re^n$. Every equilibrium is a vertex, yet not every vertex is an equilibrium: There are "kinks" defined by the intersection of the boundary of the two half-spaces of the same equilibrium equation. The proof proceeds by induction, eventually ruling out the possibility that all vertexes but one are kinks.

For any $X\subset\Re^n$, let $\Upsilon\left(X\right)=\{k\in K:\partial H_k\cap X\neq\emptyset\}$. A point $g\in\Re^n$ is an equilibrium if and only if $\Upsilon(g)$ has n elements and for any two distinct elements $k,k'\in\Upsilon(g)$, $i(k)\neq i(k')$.

Consider first the case that $\Upsilon(H)$ has $2n$ elements. Generically, this implies H has $2n$ $(n-1)$-facets. Let A be the convex hull from $\{H_i\cap\partial H:i\in\Upsilon(g^*)\}$. There exists a point $g\in\mathtt{int}(H),g\notin A$—otherwise, $H=A$ and $\Upsilon\left(H\right)$ would have only $n+1$ elements. By the separating hyperplane theorem, there exists a half-space C such that $g\in C,C\cap A=\emptyset$. The intersection $C\cap H$ is a polytope and thus contains at least one vertex $\tilde{g}\notin\partial C$. By construction, $\Upsilon\left(C\cap H\right)\cap\Upsilon\left(g^*\right)=\emptyset$, that is, none of the $(n-1)$-facets of $C\cap H$ belong to the supporting hyperplane of the $(n-1)$-facets of A. Thus $\Upsilon\left(\tilde{g}\right)$ is equal to $K/\Upsilon\left(g^*\right)$ and is an equilibrium. Hence, g^* is not an unique equilibrium.

Consider the case that $\Upsilon(H)$ has strictly less than $2n$ elements. Then there exists $k\in\Upsilon(H)$ such that for any $k'\neq k,k'\in\Upsilon(H)$, $i(k)\neq i(k')$. Since H is a proper polytope in $\Re^n$, then $H_k\cap\partial H$ is a proper polytope in $\Re^{n-1}$ and obviously $g^*\in H_k$. By considering the subspace containing $H_k\cap\partial H$ the previous steps can be repeated, substituting n by $m=n-1$ and considering only the remaining half-spaces with $i\neq i(k)$. Then either there exists another equilibrium or proceed with $m=n-2$ and so on. If $m=1$, then there are two vertexes that are both an equilibrium

□

Proof of Proposition 4.2. Clearly $E_1\left(\pi|s\right)=E_2\left(\pi|s\right)=E\pi$. Equation (7) can then be simplified to

$$\pi^b\left(s\right)=\left(1+\nu\right)E\pi+A\left(s\right).$$

Define the function $\Gamma(x)$ as

$$\Gamma(x)=\sum_S F(s)\min\left\{\frac{\psi}{\kappa^2+\psi}\left(\beta x+u\right),\left(1+\nu\right)x+A\left(s\right)\right\}.$$

$E\pi = x^*$ is an equilibrium, with

$$\pi(s) = \min\left\{\frac{\psi}{\kappa^2+\psi}\left(\beta x^* + u\right), (1+\nu)\, x^* + A\left(s\right)\right\}$$

for all $s \in S$, if and only if it is a fixed point of $\Gamma\left(x^*\right) = x^*$.

Lemma A.3. *If $\Gamma(x) \leq x$ for all x, then, generically, no equilibrium exists.*

Proof. The function Γ is differentiable almost everywhere. More precisely, it is only not differentiable at the points such that, for some state, both (7) and (8) hold with strict equality. Let x_i, $i = 1, \ldots, n$ solve

$$\frac{\psi}{\kappa^2+\psi}\left(\beta x_i + u_i\right) = (1+\nu)\, x_i + A(s_i).$$

Assume there exists a fixed point $\Gamma\left(x^*\right) = x^*$. If $x^* = x_i$ for any $i = 1, \ldots, n$ then the single unknown x^* satisfies two distinct linear equations. Thus, generically, Γ is differentiable at x^*. If $\Gamma(x) \leq x$ for all x, the derivative at x^* must be exactly zero, which is an additional equation and thus can be ruled out as a non-generic case □

Let $\alpha_0 = \nu^{-1} \sum_S F\left(s\right) A\left(s\right)$. For $x < \alpha_0$, $\Gamma(x) < x$ since $\Gamma(x) \leq \sum_S F\left(s\right)\left((1+\nu)\, x + A\left(s\right)\right)$. Let $\alpha_1 = \left(1 - \frac{\psi\beta}{\kappa^2+\psi}\right)^{-1} \sum_S F\left(s\right) u$. For $x > \alpha_1$, $\Gamma(x) < x$ since $\Gamma(x) \leq \sum_s F\left(s\right) \frac{\psi}{\kappa^2+\psi}\left(\beta x + u\right)$.

By the previous lemma, the only case left to be characterized is that there exists $\tilde{x}$ such that $\Gamma(\tilde{x}) > \tilde{x}$. It must be that $\tilde{x} \in [\alpha_0, \alpha_1]$ and, by continuity of Γ, $\tilde{x} \in (\alpha_0, \alpha_1)$ without loss of generality. Then, by the intermediate value theorem, there must be at least one fixed point between α_0 and $\tilde{x}$ and at least one fixed point between $\tilde{x}$ and α_1. Therefore there are generically at least two distinct equilibria □

A.2 Section 5

Proof of Proposition 5.1. Clearly $E_1\left(\pi|s\right) = E_2\left(\pi|s\right) = E\pi$. Equation (7) can then be simplified to

$$\pi^b\left(s\right) = (1+\nu)\, E\pi + A\left(s\right).$$

Equation (12) becomes

$$\pi^u\left(s\right) = b_3 E\pi + B\left(s\right),$$

where

$$b_3 = b_1 + b_2 = \frac{\beta\psi + \rho(\sigma^2 + \sigma\kappa)}{\kappa^2 + \psi + \sigma^2\rho}.$$

Define $\Gamma(x)$ as

$$\Gamma(x) = \sum_S F(s) \min\left\{b_3 x + B\left(s\right), (1+\nu)\, x + A\left(s\right)\right\}.$$

$E\pi = x^*$ is an equilibrium, with

$$\pi(s) = \min\left\{b_3 x^* + B\left(s\right), \left(1+\nu\right)x^* + A\left(s\right)\right\}$$

for all $s \in S$, if and only if it is a fixed point of $\Gamma\left(x^*\right) = x^*$.

If $\rho > \frac{\kappa^2+\psi(1-\beta)}{\sigma\kappa}$ then $b_3 > 1$. Since $\nu > 0$, the function $h(x) = \Gamma(x) - x$ is strictly increasing. Thus it has a single zero and there is a unique fixed point $\Gamma\left(x^*\right) = x^*$. Thus there is a unique equilibrium □

A.3 Section 6

Definition 2. *A **Markov equilibrium** given ψ consists of:*

- *A policy function $R(p) : \Re \to \Re$,*
- *A value function $V(p) : \Re \to \Re$,*
- *Allocation functions $f_\pi(p), f_y(p), f_p(p) : \Re \to \Re$, and*
- *Private-sector expectations functions $g_\pi(p), g_y(p) : \Re \to \Re$,*

such that for all $p_{-1} \in \Re$,

- *Policy and value $R(p_{-1}), V(p_{-1})$ solve*

$$V(p_{-1}) = \max_{R \geq -Z} -p^2 - \psi y^2 + \beta V(p)$$

subject to

$$\begin{aligned} p &= p_{-1} + \pi, \\ \pi &= \kappa y + \beta g_\pi(p), \\ R &= \sigma\left(g_y(p) - y\right) + g_\pi(p). \end{aligned}$$

- *Allocation functions $f_\pi(p_{-1}), f_y(p_{-1}), f_p(p_{-1})$ satisfy the equilibrium conditions,*

$$\begin{aligned} f_p(p_{-1}) &= p_{-1} + f_\pi(p_{-1}), \\ f_\pi(p_{-1}) &= \kappa f_y(p_{-1}) + \beta g_\pi\left(f_p(p_{-1})\right), \\ R(p_{-1}) &= \sigma\left(g_y\left(f_p(p_{-1})\right) - f_y(p_{-1})\right) + g_\pi\left(f_p(p_{-1})\right). \end{aligned}$$

- *Private-sector expectations $g_\pi(p_{-1}), g_y(p_{-1})$ satisfy the rational expectations hypothesis,*

$$\begin{aligned} g_\pi(p_{-1}) &= f_\pi(p_{-1}), \\ g_y(p_{-1}) &= f_y(p_{-1}). \end{aligned}$$

Proof of Proposition 6.1. We start with a partial conjecture regarding the equilibrium functions. For some $p^* > 0$, let

$$\begin{aligned} R(p_{-1}) &= -Z \\ f^p(p_{-1}) &= p_{-1} - Z \\ f^\pi(p_{-1}) &= -Z \\ f^y(p_{-1}) &= -(1-\beta)Z \end{aligned}$$

for all $p_{-1} \leq p^*$. For the conjectured law of motion of the price level, clearly if $p_{-1} \leq p^*$ then $f^p(p_{-1}) < p_{-1} \leq p^*$. Thus the set $\{p \leq p^*\}$ is absorbing. Since the price level is ever decreasing according to $f^p(p_{-1})$, I need to show that the loss function (14) is still well defined. Given $p_0 \leq p^*$, the price level in $t \geq 1$ periods will be $p_t = p_0 - tZ$. Thus the loss function will be

$$L(p_0) = \sum_{t=1}^{\infty} \beta^{t-1} \left\{ (p_0 - tZ)^2 + \frac{\psi}{\kappa^2}(1-\beta)^2 Z^2 \right\}$$

for $p_0 \leq p^*$. Expanding the price-level term inside the sum, I obtain

$$\sum_{t=1}^{\infty} \beta^{t-1}(p_0 - tZ)^2 = \frac{p_0^2}{1-\beta} + \beta^{-1} \sum_{t=1}^{\infty} \beta^t (t^2 Z^2 - 2tZp_0).$$

The sum $\sum_{t=1}^{\infty} \beta^{t-1} t^a$ converges for all a. Using the appropriate formulas,

$$\sum_{t=1}^{\infty} \beta^{t-1}(p_0 - tZ)^2 = \frac{p_0^2}{1-\beta} + Z^2 \frac{1+\beta}{(1-\beta)^3} - \frac{2Zp_0}{(1-\beta)^2}.$$

Thus $L(p_0)$ is finite. For later use, the derivative of L is

$$L'(p_0) = \frac{2p_0}{1-\beta} - \frac{2Z}{(1-\beta)^2}.$$

Note $L' < 0$ for all $p_0 \leq 0$.

Next I check that f^p, f^π, and f^y satisfy the Euler equation (2) with $R_t = -Z$ and the NKPC (1). The Euler equation is used to solve for the output gap

$$\kappa y_t = -\sigma^{-1}\kappa R_t + \kappa y_{t+1} + \sigma^{-1}\kappa \pi_{t+1}. \tag{17}$$

If $p_0 \leq p^*$, then $f^p(p_{-1}) < p_{-1} \leq p^*$ and $R_t = -Z$. Thus

$$\kappa y_t = \sigma^{-1}\kappa Z - (1-\beta)Z - \sigma^{-1}\kappa Z = -(1-\beta)Z$$

as conjectured. The NKPC (1) quite trivially confirms the inflation function,

$$\pi_t = \kappa y_t + \beta \pi_{t+1} = -Z.$$

The key step is to show that $R(p_{-1}) = -Z$ is indeed the choice of the monetary authority. If $R_t > -Z$, then $p < p_{-1} - Z \leq p^*$. The monetary authority problem can then be written as

$$\min_{p \leq p_{-1} - Z} p^2 + \frac{\psi}{\kappa^2} \left(p - p_{-1} + \beta Z \right)^2 + \beta L(p),$$

where $L(p)$ is as given earlier. The first-order condition evaluated at $p = p_{-1} - Z$ must satisfy

$$p_{-1} - Z - Z(1-\beta)\frac{\psi}{\kappa^2} + \beta \left(\frac{p_{-1} - Z}{1-\beta} - \frac{Z}{(1-\beta)^2} \right) \leq 0.$$

This is true if

$$p_{-1} \leq (1-\beta) Z \left(1 + (1-\beta)\frac{\psi}{\kappa^2} + \frac{\beta^2}{(1-\beta)^2} \right) \equiv p^*.$$

Clearly, $p^* > 0$.

To complete the proof, conjecture that

$$\begin{aligned} f^p(p_{-1}) &\leq p_{-1} - Z \\ f^\pi(p_{-1}) &\leq -Z \\ f^y(p_{-1}) &\leq -(1-\beta)Z, \end{aligned}$$

for all $p_{-1} > p^*$. Note this is only a partial characterization of the equilibrium functions. To check the conjecture, note that from (17) we have

$$\begin{aligned} \kappa y_t &= -\sigma^{-1}\kappa R_t + \kappa y_{t+1} + \sigma^{-1}\kappa \pi_{t+1} \\ &< \sigma^{-1}\kappa Z + \kappa y_{t+1} + \sigma^{-1}\kappa \pi_{t+1} \\ &\leq \sigma^{-1}\kappa Z - (1-\beta)Z - \sigma^{-1}\kappa Z = -(1-\beta)Z \end{aligned}$$

where I have first used $R_t \geq -Z$ and then the conjecture regarding f^π, f^y. Similarly, from the NKPC

$$\pi_t = \kappa y_t + \beta \pi_{t+1} \leq -Z.$$

It follows that the price level p_t is always below $p_{t-1} - Z$ and, after a finite time, $p_{t+d} < p^*$ □