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An Ordering of Measures of the Welfare Cost of Inflation in Economies with Interest-Bearing Deposits*

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Abstract

This paper builds on Lucas (2000) and on Cysne (2003) to derive and order six alternative measures of the welfare costs of inflation (five of which already existing in the literature) for any vector of opportunity costs. The ordering of the functions is carried out for economies with or without interest-bearing deposits. We provide examples and closed-form solutions for the log-log money demand both in the unidimensional and in the multidimensional setting (when interest-bearing monies are present). An estimate of the maximum relative error a researcher can incur when using any particular measure is also provided.

*Key Words: Inflation, Welfare, Interest-Bearing Assets, Money Demand, Divisia Index. JEL: C0, E40.

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1 Introduction

Measuring the welfare costs of inflation, even when restricted to a money-demand approach (e.g. [1], Lucas (2000), Simonsen and Cysne (2001), and Cysne (2003)), presents the researcher with at least four types of problems.

First, choosing among the several welfare measures existing in the literature and being reasonably aware of the consequences of this choice for the maximum possible relative error of the welfare figures to be obtained.

For instance, Lucas (2000) derives a measure of the welfare costs of inflation based on the Sidrauski model and another one based on the shopping-time model. He shows, using numerical simulations, that both Sidrauski's and the shopping-time measures are very close to Bailey's (1956) measure. However, he does not provide, either analytically or numerically, an ordering among such three functions of the nominal interest rate. A researcher might want to know how such measures compare to each other (if they can be ordered) and, moreover, what would be the consequences (the maximum relative error, for instance) for the welfare figures obtained using any particular measure.

A second problem a researcher might face is the necessity to take into consideration the existence in the economy of interest-bearing deposits performing monetary functions. This leads us, as conjectured by Lucas (2000, p. 270) and later shown by Cysne (2003), to welfare measures based on Divisia indices of monetary services. Using the unidimensional welfare formulas in this case can be misleading, particularly because the demand for non-interest-bearing money can be very sensitive to variations in the opportunity costs of other assets providing monetary services.

The same way as different unidimensional formulas can be defined, different Divisia indices can be used to calculate welfare losses in the multidimensional setting. Once again, the question of how such welfare measures relate to each other emerges, but now in a more complicated fashion. Knowing in advance that some measures always lead to welfare figures that are higher (or lower) than others, for any vector of opportunity costs, is again valuable information for those interested in investigating the losses generated by inflation.

Answers to such questions have been provided in the literature by

Simonsen and Cysne (2001) and by Cysne (2003), but only in relation to four measures of the welfare costs of inflation: Lucas' shopping-time measure, two (easier-to-calculate) approximations of Lucas' measure introduced by Simonsen and Cysne (2001) and Bailey's measure.

This paper extends such contributions, both in the unidimensional and in the multidimensional case, by reconsidering the previous ordering with respect to two additional measures: Lucas' measure based on Sidrauski's model, which he derives taking as reference the interest rate currently prevailing in the economy; and a measure which is new in the literature, the one which emerges from Sidrauski's model when the reference for income compensation is taken to be an interest rate equal to zero.

The paper also contributes to the related literature by providing with respect to the usual log-log money demand:

i) closed-form solutions to all six measures, both in the unidimensional and in the multidimensional setting;

ii) an estimate of the maximum relative error a researcher can incur when using

any of the six different measures;

iii) numerical examples illustrating the main results of the paper.

The remainder of the paper is organized as follows. In section 2 we present the results concerning the unidimensional case worked out by Lucas (2000). Starting with the unidimensional case is didactically important because it provides the reader, in a more simplified setting, a reference for the steps which are to be taken in the multidimensional case. Section 3 exemplifies the results of the unidimensional case and provides an order of magnitude of the maximum relative error. Sections 4 and 5 extend the previous results of Sections 2 and 3 to the case in which the existence of interest-bearing deposits performing monetary function is taken into consideration. Section 6 summarizes the paper.

2 Six alternative measures of the welfare costs of inflation

In this section we analyze the case of an economy with only one type of (non interest-bearing) money. We present the six alternative measures of the welfare cost of inflation we shall deal with and show how they can be ordered. By "show how they can be ordered" we mean that we shall demonstrate that these six different functions (here defined on a subset of $\mathbb{R}$ but, from section 4 on, defined on a subset of $\mathbb{R}^n$) can be pairwise compared and do not cross or touch each other, except when identical.

2.1 The shopping-time measure and its approximations

Lucas (2000, p. 265) shows that in the shopping-time model the welfare cost of inflation s is given as a solution to the non-separable differential equation:

$$\begin{cases} s'(r) = \frac{-rm'(r)}{1-s(r)+rm(r)}(1-s(r)) \\ s(0) = 0 \end{cases}, \tag{1}$$

where $r \in \mathbb{R}_+$ stands for the nominal interest rate and $m : [0, +\infty] \to [0, +\infty]$ is a money-demand function that arises from the model itself.

Since the non-separable differential equation given by (1) does not have any obvious solution in the general case,[1] it is natural to look for approximations to it. Simonsen and Cysne (2001) have shown that reasonable approximations to s in (1) are given by the upper bound $A : \mathbb{R}_+ \to [0, +\infty]$:

$$A(r) := \int_0^r -\frac{\rho m'(\rho)}{1+\rho m(\rho)} d\rho. \tag{2}$$

and by the lower bound $1 - e^{-A}$.

Let $B : \mathbb{R}_+ \to [0, +\infty]$ stand for Bailey's (1956) measure:

$$B(r) := \int_0^r -\rho m'(\rho) d\rho. \tag{3}$$

Through the transformation $\mu := m(\rho)$, (2) and (3) can be rewritten as

$$A(r) = \int_{m(r)}^{m(0)} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu \tag{4}$$

[1] Cysne (2005) has provided a solution to it in the case of a log-log money demand. This solution will be helpful in the calculations to be carried out in this paper.

and

$$B(r) = \int_{m(r)}^{m(0)} \psi(\mu) d\mu, \tag{5}$$

where this last formula is the traditional one for the area under the inverted money-demand curve.

Simonsen and Cysne (2001) have also shown that

$$1 - e^{-A} < s < A < B, \tag{6}$$

where we write "$f < g$" for "$f(r) < g(r), \forall r \in \mathbb{R}_{++}$", since all these measures coincide only at 0. This notation will be used for the ordering of functions.

2.2 The Sidrauski model

In the next subsection we will introduce two measures of the welfare cost of inflation which emerge from the Sidrauski model. Both are based on Lucas' version of this model, and since one of them is new in the literature, we present the model here in detail. Another reason for doing so is that later on we will want to generalize this model, so that it also accounts for the possibility of existence of other types of monies in the economy.

Let's assume a forever-living perfectly-foresighted representative agent maximizing a time-separable constant-relative-risk-aversion utility function, the arguments of which are the flows of real consumption of a single non-monetary nonstorable good and the holdings of real cash balances.

For every $t \in [0, +\infty)$, let $B_t \in \mathbb{R}_+$, $M_t \in \mathbb{R}_+$, $H_t \in \mathbb{R}$, $Y_t \in \mathbb{R}_{++}$ and $C_t \in \mathbb{R}_+$ represent the nominal values of, respectively, holdings of government

bonds and cash, a lump-sum tax (if negative, a transfer from the government to the individual), nominal output and consumption at time t. The budget constraint faced by our representative agent is:

$$\dot{B}_t + \dot{M}_t = Y_t - H_t - C_t + r_t B_t,$$

where the dots mean time-derivatives and $r_t \in \mathbb{R}_+$ stands for the nominal interest rate bonds yield at time t (by definition, cash is a monetary asset always yielding a nominal interest rate of 0).

Let $P_t \in \mathbb{R}_{++}$ be the (both expected and realized) price level, $\pi_t := \frac{\dot{P}_t}{P_t}$ be the inflation rate at time t, and γ stand for the constant rate of output growth:

$$y(t) = y_0 e^{\gamma t}. \tag{7}$$

Lower-case variables are real counterparts of the above nominal variables as fractions of output (that is, $b_t := \frac{B_t}{Y_t}$ etc.).

As in Lucas (2000), we assume the utility function to be homogeneous of degree $1-\sigma$, in which case we can use (7) (normalizing y_0 to be equal to one) to write our agent's problem ($\mathrm{P_S}$) as

$$\max_{c_t, m_t \geq 0} \int_0^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(c_t, m_t) dt \tag{$\mathrm{P_S}$}$$

subject to

$$\dot{b}_t + \dot{m}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma) b_t - (\pi_t + \gamma) m_t$$
$$b_0 > 0 \text{ and } m_0 > 0 \text{ given.}$$

Note that, in (P_S), the representative agent is supposed to perform her maximization over real consumption and real cash balances, and not over these variables as fractions of the output. Following Lucas (2000), we use an instantaneous utility function $U : \mathbb{R}_+ \times \mathbb{R}_+ \to [-\infty, +\infty]$ with the functional form:

$$U(c, m) = \frac{1}{1-\sigma} \left(c\varphi\left(\frac{m}{c}\right)\right)^{1-\sigma},$$

where $\sigma > 0$, $\sigma \neq 1$, extended to the ray $\{0\} \times \mathbb{R}_+$ by continuity. $\varphi : [0, +\infty] \to [0, +\infty]$ is a twice-differentiable function. We assume that there is an $\bar{m} \in (0, +\infty]$ such that $\varphi\,|_{[0,\bar{m})}$ is strictly increasing and strictly concave and $\varphi\,|_{[\bar{m},+\infty]}$ is constant.[2]

Remark 1 *The expression $\varphi(m) - m\varphi'(m)$ will appear many times throughout this work, so it should be noted from the start that, for positive m, this expression is positive. Indeed, for $m < \bar{m}$, the strict concavity of φ gives $\varphi(0) - \varphi(m) < \varphi'(m)(0 - m)$, and since $\varphi(0) \geq 0$, we have $\varphi(m) - m\varphi'(m) > 0$. For $m \geq \bar{m}$, $\varphi(m) - m\varphi'(m) = \varphi(\bar{m}) > 0$.*

Notice that U is strictly increasing in each of its variables and also strictly concave.[3] Therefore, assuming that (P_S) has a solution, it will be unique. Using

[2] It will turn out to be the maximum value the money-demand function arising from φ and (P_S) attains. For example, for the log-log money-demand specification $m = Kr^{-\alpha}$, we would have $\bar{m} = +\infty$, whereas for the semi-log specification $m = Ke^{-\alpha r}$, $\overline{m} = K$.

[3] All we need to check is that, for $(c, m) \in \mathbb{R}^2_{++}$,
$U_c(c,m) = \left(c\varphi\left(\frac{m}{c}\right)\right)^{-\sigma}\left(\varphi\left(\frac{m}{c}\right) - \frac{m}{c}\varphi'\left(\frac{m}{c}\right)\right) > 0$,
$U_m(c,m) = \left(c\varphi\left(\frac{m}{c}\right)\right)^{-\sigma}\varphi'\left(\frac{m}{c}\right) > 0$,
$U_{cc}(c,m) = \left(c\varphi\left(\frac{m}{c}\right)\right)^{-\sigma-1}\left(-\sigma\left(\varphi\left(\frac{m}{c}\right) - \frac{m}{c}\varphi'\left(\frac{m}{c}\right)\right)^2 + \frac{m^2}{c^2}\varphi\left(\frac{m}{c}\right)\varphi''\left(\frac{m}{c}\right)\right) < 0$,
$U_{mm}(c,m) = \left(c\varphi\left(\frac{m}{c}\right)\right)^{-\sigma-1}\left(-\sigma\varphi'\left(\frac{m}{c}\right)^2 + \varphi\left(\frac{m}{c}\right)\varphi''\left(\frac{m}{c}\right)\right) < 0$,
$U_{cm}(c,m) = \left(c\varphi\left(\frac{m}{c}\right)\right)^{-\sigma-1}\left(\sigma\frac{m}{c}\varphi'\left(\frac{m}{c}\right)^2 - \varphi\left(\frac{m}{c}\right)\left(\sigma\varphi'\left(\frac{m}{c}\right) + \frac{m}{c}\varphi''\left(\frac{m}{c}\right)\right)\right)$, and
$U_{cc}(c,m)U_{mm}(c,m) - U_{cm}(c,m)^2 = \frac{-\sigma}{c^3}\left(c\varphi\left(\frac{m}{c}\right)\right)^{1-2\sigma}\varphi''\left(\frac{m}{c}\right) > 0$.

the budget constraint to substitute for c, we get a standard calculus of variations problem in the variables b and m. The Euler equations are:

$$\begin{cases} (r-\pi)U_c = (-\rho+\gamma(1-\sigma))U + \frac{d}{dt}(-U_c) \\ -\pi U_c + U_m = (-\rho+\gamma(1-\sigma))U + \frac{d}{dt}(-U_c) \end{cases}.$$

Since the right-hand sides of these equations are the same, we at once obtain the very intuitive equilibrium relation

$$r = \frac{U_m}{U_c}. \tag{8}$$

In equilibrium, since c is taken as a fraction of output, $c = 1$. Equation (8) can then be rewritten as:

$$r = \frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)}, \tag{9}$$

which corresponds to equation 3.7 in Lucas (2000) (which he derives using Bellman's Optimality Principle instead). This equation gives us r as a non-negative differentiable function of m, for which we shall write $r = \psi(m)$, where $\psi : [0, \bar{m}] \to [0, +\infty]$.

Note that equation (9) may be rewritten as:

$$\varphi'(m) = \frac{\psi(m)}{1 + m\psi(m)}\varphi(m), \tag{10}$$

which is separable and yields the general solution

$$\varphi(m) = Ce^{\int_1^m \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}, \tag{11}$$

for some constant $C > 0$.

In the steady state the Euler equation relative to b gives

$$r = \rho + \pi + \gamma\sigma, \tag{12}$$

a fact that justifies our taking of the welfare cost of inflation as a function of the nominal interest rate, instead of inflation itself.

Since

$$\psi'(m) = \frac{\varphi(m)\varphi''(m)}{\left(\varphi(m) - m\varphi'(m)\right)^2} < 0, \tag{13}$$

ψ is strictly decreasing, therefore one-to-one. From its continuity (since φ is twice-differentiable, φ' is continuous), its image is also connected, that is, an interval $[\psi(\bar{m}), \psi(0)]$. From (9), we have $\psi(\bar{m}) = 0$. Let $R := \psi(0)$.

We shall call ψ's inverse function $m : [0, R] \to [0, \bar{m}]$ a "money-demand function". This function is strictly decreasing ($m'(r) = \frac{1}{\psi'(m(r))} < 0$) and surjective by construction. As a practical matter, since the economist does not know the function φ, he ends up using a money-demand function estimated by the econometric practice, implying restrictions that have to be satisfied by φ (the so-called "integrability problem" in economics).

2.3 The welfare cost of inflation in the Sidrauski model: two different approaches

In Sidrauski's framework, Lucas (2000, p. 257) "define[s] the welfare cost $w(r)$ of a nominal rate r to be the percentage income compensation needed to leave the household indifferent between r and 0". There are two diametrical ways of

interpreting this definition. The first one, employed by Lucas, uses the initial interest rate as reference and measures the percentage rise in income necessary to make people as well off *as* they would be if the nominal interest rate were to fall to zero. Given the 1-degree homogeneity of U one can simply write:

$$U(1+\overline{w}(r), m(r)) = U(1, \bar{m}). \tag{14}$$

Let $\varphi^* := \sup_{m>0} \varphi(m) = \varphi(\bar{m})$. From (11), we then have

$$\varphi^* = Ce^{\int_1^{\bar{m}} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu}$$

so that $\varphi^* < +\infty$ if and only if the integral $\int_1^{\bar{m}} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu$ converges (this is an issue if $\bar{m} = +\infty$).

In our framework, (14) implies:

$$(1+\overline{w}(r))\varphi\left(\frac{m(r)}{1+\overline{w}(r)}\right) = \varphi(\bar{m}) = \varphi^*. \tag{15}$$

Differentiating with respect to r, dividing through by $\varphi'\left(\frac{m(r)}{1+\overline{w}(r)}\right)$ and using (10):

$$\overline{w}'(r) = -\psi\left(\frac{m(r)}{1+\overline{w}(r)}\right) m'(r). \tag{16}$$

This is equation 3.11 in Lucas' paper which, together with the condition $\overline{w}(0) = 0$, enables us to find $\overline{w}$.

We will now turn our attention to another natural way of interpreting Lucas' definition for the welfare cost of inflation in Sidrauski's model: the one which takes as reference an interest rate equal to zero. That is, we will take as the

welfare cost of inflation the percentage fall in the representative agent's income that would make her as well off as she would have been, had no increase in the nominal interest rate taken place:

$$U(1, m(r)) = U(1 - \underline{w}(r), \bar{m}). \tag{17}$$

We shall see below that the welfare cost of inflation which takes as reference an interest rate equal to zero ($\underline{w}$) is lower than the one which takes as reference the prevailing (supposedly positive) interest rate ($\overline{w}$). Bailey's measure lies somewhere in between[4]. This will be proved in Proposition 1 below.

In our model, definition (17) implies:

$$\frac{\varphi(m(r))}{1 - \underline{w}(r)} = \varphi\left(\frac{\bar{m}}{1 - \underline{w}(r)}\right). \tag{18}$$

Now, since, by construction, $\underline{w}(r) \in [0, 1)$, we have $\frac{\bar{m}}{1-\underline{w}(r))} \geq \bar{m}$, so that

$$\varphi\left(\frac{\bar{m}}{1 - \underline{w}(r)}\right) = \varphi^*,$$

and (18) ends up yielding the very simple formula for the welfare cost of inflation $\underline{w}$:

$$\underline{w}(r) = \frac{\varphi^* - \varphi(m(r))}{\varphi^*}. \tag{19}$$

[4]Note that the inequalities presented in (6) as well as all others obtained throughout this paper relate measures of deadweight loss, rather than measures of welfare changes (as the consumers' surplus).

In either case ($\varphi^* < +\infty$ or $\varphi^* = +\infty$), using (4) we may write

$$\frac{\varphi(m(r))}{\varphi^*} = \frac{Ce^{\int_1^{m(r)} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}}{Ce^{\int_1^{+\infty} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}} = e^{\int_{+\infty}^{m(r)} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu} = e^{-A(r)}. \tag{20}$$

Taking into account (19), we obtain:

$$\underline{w}(r) = 1 - e^{-A(r)}. \tag{21}$$

In Simonsen and Cysne (2001) and in Cysne (2003), the measure $1 - e^{-A}$ has been derived in a completely different context (the shopping-time one) as an easier-to-calculate approximation to s, the welfare measure derived by Lucas, given in (1). Here, we've just seen that a different model, the money-in-the-utility-function model, can provide a sensible explanation to this measure.

Comparing (21) with (16), we see that $\underline{w}$ has at least one advantage over Lucas' measure $\overline{w}$: computational ease. We no longer have to solve a non-separable nonlinear differential equation – although we do still need to evaluate a possibly very difficult integral.

Proposition 1 below offers the main result of this Section.

Proposition 1 *Let $m : [0, R] \to [0, \bar{m}]$ be a differentiable and strictly decreasing money-demand function, so that we can use it to calculate s, A, B, $\overline{w}$, and $\underline{w}$ (by simply plugging it into (1), (2), (3), (16) and (21)). Then we have the following inequality chain:*

$$\underline{w} = 1 - e^{-A} < s < A < B < \overline{w}.$$

Proof. The equality has just been shown. The first, second and third inequalities have already been shown in Simonsen and Cysne (2001, prop. 1), and the reader

may notice that their proof that $1-e^{-A} < s < A < B$ draws only on the strict decreasingness of m, not on other possible characteristics enjoined by money-demand functions arising from the shopping-time model. As to the fourth inequality, (16) gives us, for $\rho \in (0, r]$, $\overline{w}'(\rho) > -\psi(m(\rho))m'(\rho) = -\rho m'(\rho)$ (remember that $m' < 0$ and that ψ is a strictly decreasing function), so all that is left to do is integrate both sides of this last inequality. ■

3 Some calculations for the unidimensional log-log money demand

3.1 Formulas for the unidimensional measures of the welfare cost of inflation

Let's say an estimated unidimensional log-log money demand specification,

$$m = Kr^{-\alpha}, \tag{22}$$

with $K > 0$ and $\alpha \in (0, 1)$, has been handed to us by the econometrician. How can we calculate the different measures of the welfare cost of inflation associated to a nominal interest rate of r?

Fortunately, s has already been calculated in the literature (see Cysne 2005) for this particular case, being given in an implicit manner by the formula:

$$(1 - s(r)) \left(1 - (1 - s(r))^{-1/\alpha}\right) + \frac{K}{1-\alpha} r^{1-\alpha} = 0. \tag{23}$$

Both Bailey's measure and the proxy measure A are straightforward:

$$B(r) = \int_0^r -\rho(-\alpha K \rho^{-\alpha-1}) d\rho = \frac{\alpha K}{1-\alpha} r^{1-\alpha} \tag{24}$$

and

$$\begin{aligned} A(r) &= \int_0^r \frac{-\rho(-\alpha K \rho^{-\alpha-1})}{1+\rho K \rho^{-\alpha}} d\rho = \alpha \int_0^r \frac{K\rho^{-\alpha}}{1+K\rho^{1-\alpha}} d\rho \\ &= \frac{\alpha}{1-\alpha} \int_1^{1+Kr^{1-\alpha}} \frac{du}{u} = \frac{\alpha}{1-\alpha} \log\left(1+Kr^{1-\alpha}\right). \end{aligned} \tag{25}$$

Regarding $\overline{w}$, since $\psi(m) = \left(\frac{K}{m}\right)^{1/\alpha}$, (16) takes the form

$$\overline{w}'(r) = -\left(\frac{K(1+\overline{w}(r))}{Kr^{-\alpha}}\right)^{1/\alpha} (-\alpha K r^{-\alpha-1}) = \alpha K (1+\overline{w}(r))^{1/\alpha} r^{-\alpha},$$

leading to:

$$\overline{w}(r) = -1 + \left(1 - Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{26}$$

You may notice that in order for $\overline{w}$ to be real, it's important that $r \in [0, K^{\frac{1}{\alpha-1}}]$.

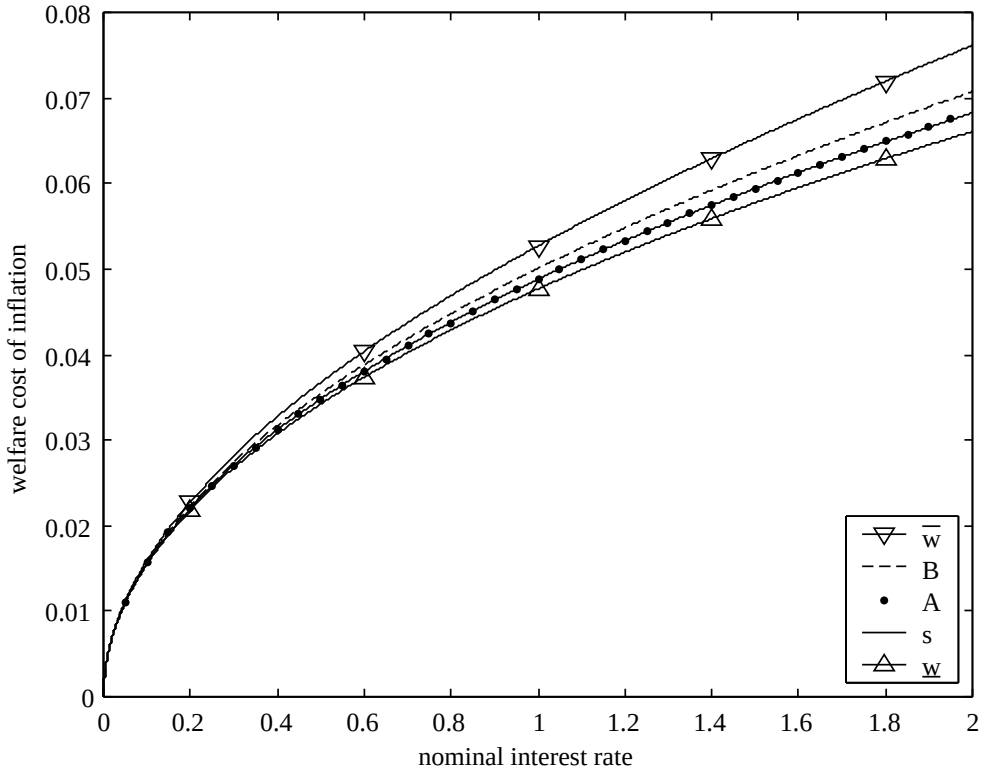
Finally, for the welfare measure associated to the compensating variation notion, we simply have:

$$\underline{w}(r) = 1 - e^{-A(r)} = 1 - \left(1 + Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{27}$$

In Figure 1 we plot these five measures for the estimated money-demand function $m(r) = 0.05 r^{-0.5}$.[5] Note how s and A are indistinguishable to the naked eye.

[5]These parameters were calibrated so as to fit the American economy. See Lucas (2000, pp. 258-9).

Figure 1: Welfare Costs in the Unidimensional Case



Another way of finding $\underline{w}$ would be to first ask what is the φ underlying the money-demand handed out to us, and then use (20). Looking at equation (11), we first calculate the integral that appears there:

$$\int_1^m \frac{\left(\frac{K}{\mu}\right)^{1/\alpha}}{1+\mu\left(\frac{K}{\mu}\right)^{1/\alpha}}d\mu = \int_1^m \frac{K^{1/\alpha}}{\mu^{1/\alpha}+K^{1/\alpha}\mu}d\mu = \int_1^m \frac{d\mu}{\mu} - \int_1^m \frac{\mu^{\frac{1}{\alpha}-2}}{K^{1/\alpha}+\mu^{\frac{1}{\alpha}-1}}d\mu$$
$$= \log m - \frac{\alpha}{1-\alpha}\int_{K^{1/\alpha}+1}^{K^{1/\alpha}+m^{\frac{1}{\alpha}-1}} \frac{du}{u} = \log\left(m \Big/ \left(\frac{K^{1/\alpha}+m^{\frac{1}{\alpha}-1}}{K^{1/\alpha}+1}\right)^{\frac{\alpha}{1-\alpha}}\right).$$

Therefore that equation gives us

$$\varphi(m) = \frac{Cm}{\left(K^{1/\alpha}+m^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\alpha}{1-\alpha}}}, \tag{28}$$

for a positive constant C.[6] So $\varphi^* = C$, and

$$\underline{w}(r) = 1 - \frac{\varphi(m(r))}{\varphi^*} = 1 - \frac{m(r)}{\left(K^{1/\alpha}+m(r)^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\alpha}{1-\alpha}}}$$
$$= 1 - \frac{Kr^{-\alpha}}{\left(K^{\frac{1-\alpha}{\alpha}}K + K^{\frac{1-\alpha}{\alpha}}r^{\alpha-1}\right)^{\frac{\alpha}{1-\alpha}}} = 1 - \frac{1}{(Kr^{1-\alpha}+1)^{\frac{\alpha}{1-\alpha}}},$$

which coincides with the previous result.

3.2 The maximum relative error

A question that could naturally arise from looking at Figure 1 is: *"How much greater can the Sidrauski-Lucas upper bound $\overline{w}$ be relatively to the lower bound*

[6] So that $\bar{m} = +\infty$, exactly as expected.

$\underline{w}$*?"*. For reasonable interest rates, not much greater indeed.

From (26) and (27):

$$\Delta(r) := \frac{\overline{w}(r) - \underline{w}(r)}{\underline{w}(r)} = \begin{cases} -1 + \frac{-1+\left(1-Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-(1+Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}}} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

Write

$$f(x) = \begin{cases} -1 + \frac{-1+(1-x)^{\frac{\alpha}{\alpha-1}}}{1-(1+x)^{\frac{\alpha}{\alpha-1}}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases},$$

so that

$$\Delta(r) = f(Kr^{1-\alpha}).$$

With the aid of a mathematics software, the following third-order Maclaurin's expansion for f can be obtained:

$$f(x) = 0 + \frac{1}{1-\alpha}x + \frac{1}{2(1-\alpha)^2}x^2 + \frac{-2\alpha^2+5\alpha-5}{12(1-\alpha)^3}x^3 + O(x^4),$$

where $O(x^4)$ means a function whose absolute value is less than a constant times x^4, for small enough values of x (we're only interested in reasonable interest rates, which correspond to low x's).

We want to have $\Delta(r)$ written as a function of $\overline{w}(r)$. In order to do this, take $g : \mathbb{R}_+ \to \mathbb{R}$ given by

$$g(x) = \frac{1}{\alpha}\left(-1 + (1-x)^{\frac{\alpha}{\alpha-1}}\right)$$

as a guess for an approximation to f (insight gained by comparing their graphs).

Its third-order Maclaurin's expansion is:

$$g(x) = 0 + \frac{1}{1-\alpha}x + \frac{1}{2(1-\alpha)^2}x^2 + \frac{2-\alpha}{6(1-\alpha)^3}x^3 + O(x^4).$$

Therefore, for small x, we have

$$f(x) = g(x) + O(x^3).$$

Note that

$$g\left(Kr^{1-\alpha}\right) = \frac{1}{\alpha}\overline{w}(r),$$

so that, by putting $x = Kr^{1-\alpha}$, we get:

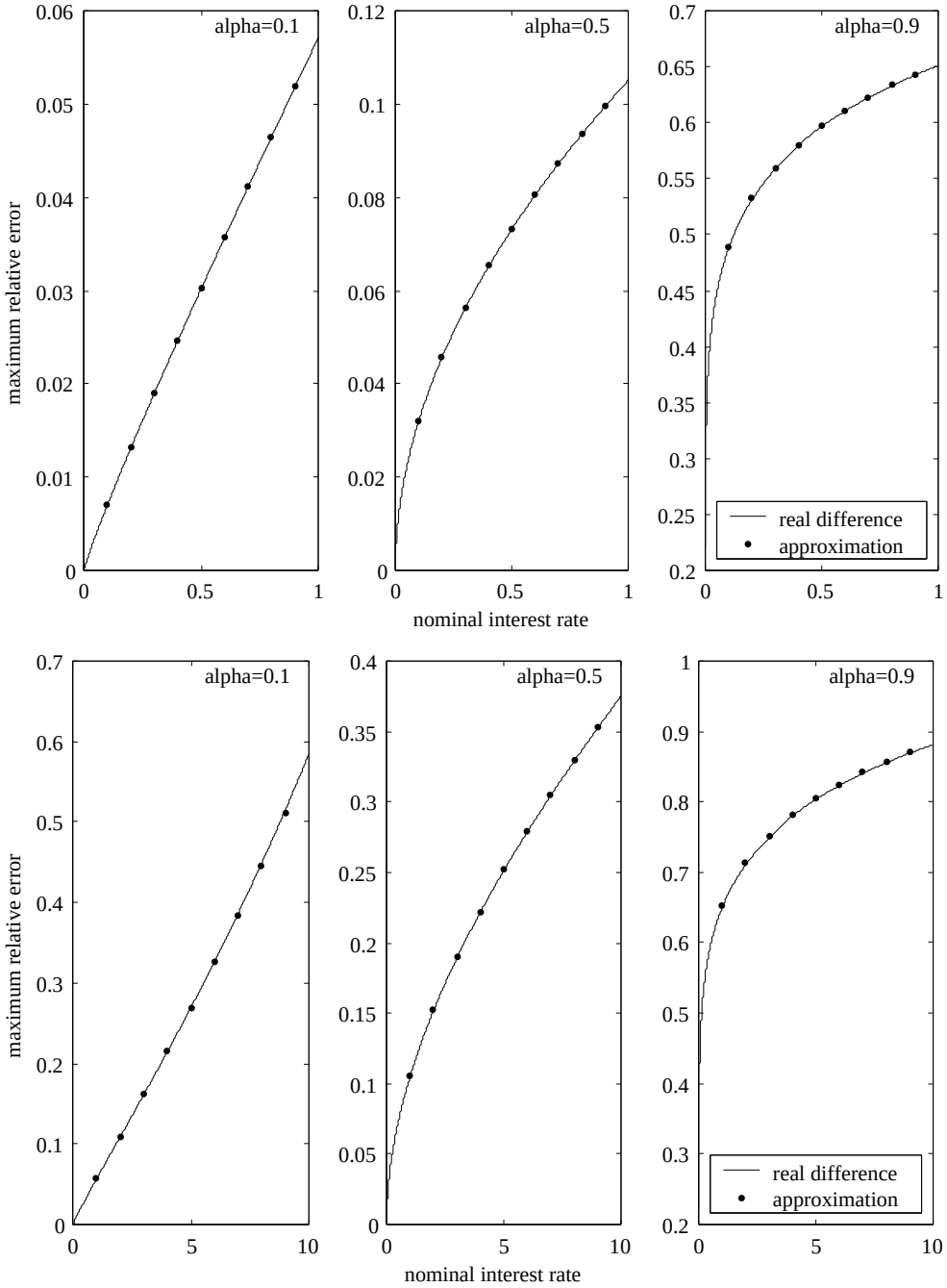
$$\Delta(r) = \frac{1}{\alpha}\overline{w}(r) + O(r^{3-3\alpha}), \tag{29}$$

which is a good approximation to this relative difference formula.

By comparing the dotted and the full line in Figure 2, which cannot be distinguished by the naked eye, one can see that the approximation of $\Delta(r)$ by $\frac{1}{\alpha}\overline{w}(r)$ is very good indeed. Figure 2 also allows us (again, using a value of $K = 0.05$), in relation to three possible alternative values of the interest-rate elasticity α, to enter an interest rate in the abscissa and to have an idea of the maximum relative error regarding the use of one or another of the six welfare measures we deal with in this paper. The upper part of Figure 2 has r ranging from 0 to 1, and is more appropriate for low-inflation economies. The bottom of Figure 2 gives the same data when r varies between 0 and 10.

For instance, for $\alpha = 0.5$ and $r = 15\%$ we obtain $\Delta(r) \approx 4.0\%$. This value

Figure 2: Maximum Relative Error



of the nominal interest rate could account for an inflation rate around 11% plus a long-term real interest rate around 4%. It could be understood as an upper bound for most industrialized economies. Thus, if one reports having found a welfare loss of, say, 1% of output associated with a money demand having a 0.5 elasticity and an 11% inflation rate, our results allow us to say that, regardless of which particular measure among the six was chosen, the estimate could vary at most between 0.96% and 1.04%, a very reasonable confidence interval. So one can feel secure about which measure to take, when considering low-inflation countries.

However, the preceding calculations tell only one side of the story. Consider now a country where the annual inflation rate has reached 400% (in Brazil, for instance, yearly inflation reached 1783% in 1989). For the same parameters, the relative measuring difference Δ reaches 22% (considering $r = 4$, since the long-term real interest rate becomes negligible). For example, estimates of the welfare costs reported as being of 8% of output in high-inflation countries could actually be measured as low as 6.5% or as high as 9.8%, depending on the formula being used. In such high-inflation cases, therefore, one has to be careful about which measuring strategy to pursue.

4 The presence of interest-bearing monies in the economy

In this section we extend all results obtained in section 2 to a framework in which n types of monies are available. This is important because, as we argued in the introduction, welfare formulas based on only one money can be misleading when

there are different assets in the economy performing monetary functions. The variables defined below will be used both in the case of the Sidrauski and the shopping-time models.

Let

$$\mathbf{m} = (m_1, \ldots, m_n) \in [0, +\infty]^n$$

represent the vector of real quantities of each type of money being demanded, as a fraction of nominal GDP (where m_1 is chosen to be m, real currency per output). Each m_i yields a nominal interest rate of r_i, and

$$\mathbf{r} := (r_1, \ldots, r_n) \in \mathbb{R}^n_+$$

(with $r_1 = 0$, by definition). We shall write

$$\mathbf{u} := (r, r - r_2, \ldots, r - r_n) \in \mathbb{R}^n_+$$

for the vector of opportunity costs of holding each type of money instead of government bonds.

The steady-state interest rate r in this extended model is determined by (12), as in the unidimensional case. The other interest rates $r_2, ..., r_n$ (as well as the respective opportunity costs $u_2, ..., u_n$) can be assumed to be determined by a competitive and costless banking system subject to $n-1$ exogenous non-interest-bearing reserve requirements $k_2, ...k_n$ (in which case $r_2 = (1 - k_2)r, ..., r_n = (1 - k_n)r$).

The reader may verify that:

$$r = 0 \Longleftrightarrow \mathbf{u} = \mathbf{0}. \tag{30}$$

Expressions (30) and (12) show how each of the opportunity costs u_i relates to the rate of inflation. When the nominal interest rate is equal to zero, all the opportunity costs are also equal to zero and inflation generates no welfare loss at all. The converse is also true.

The first-order conditions of the money-in-the-utility-function model, as well as of the shopping-time model, will imply a function ψ taking $\mathbf{m}$ into $\mathbf{u}$, analogous to the function ψ from section 2.

But it is important to keep in mind that from this section on all measures of the welfare cost of inflation will be evaluated at $\mathbf{m}$, rather than at $\mathbf{u}$. To keep the notation uniform, though, we will use the same symbols we've been using so far to denote each of the six n-dimensional welfare measures.

4.1 The shopping-time measure and its approximations in the multidimensional case

The multidimensional McCallum-Goodfriend (1987) model is introduced and solved in Cysne (2003). There, a transacting-technology constraint in the form

$$c = G(\mathbf{m})\, f(s)$$

is assumed, where c and s stand for the real level of consumption and the portion of time spent transacting rather than producing. The money aggregator function

$G : [0, +\infty]^n \to [0, +\infty]$ is a twice-differentiable 1-degree homogeneous concave function such that $G_{x_i} > 0$, $\lim_{m_i \to 0} G_{x_i}(\mathbf{m}) = +\infty$, $\lim_{m_i \to +\infty} G_{x_i}(\mathbf{m}) = 0$, $G_{x_i x_i} < 0$ for all $i \in \{1, \ldots, n\}$,[7] and $f : [0, 1] \to \mathbb{R}_+$ is a twice-differentiable function such that $f' > 0$ and $f'^2 - f\ f'' \geq 0$.[8]

This model's first-order conditions imply the equilibrium relations (Cysne 2003, p. 224):

$$\begin{cases} u_i = \frac{f(s)}{G(\mathbf{m}) f'(s)} G_{x_i}(\mathbf{m}), \forall i \in \{1, \ldots, n\} \\ G(\mathbf{m})\, f(s) = 1 - s \end{cases}.$$

This last equation, $Gf(s) = 1 - s$, can be seen to give us s as a function of G alone. For this purpose, we only need to check that the function $H : (0, 1] \to \mathbb{R}_+$ defined by

$$H(s) = \frac{1 - s}{f(s)}$$

is invertible. Since $H'(s) = \frac{-f(s) - (1-s) f'(s)}{f(s)^2} < 0$, $H(1) = 0$ and $\lim_{s \to 0_+} H(s) = +\infty$, we're done. By writing τ for its inverse function, the preceding set of equations becomes a money-demand specification of n variables and n equations:

$$u_i = \psi_i(\mathbf{m}) = \frac{f(\tau(G(\mathbf{m})))}{G(\mathbf{m}) f'(\tau(G(\mathbf{m})))} G_{x_i}(\mathbf{m}), \forall i \in \{1, \ldots, n\}, \tag{31}$$

where ψ_i is the i-th component of the function $\psi : [0, +\infty]^n \to [0, +\infty]^n$ taking $\mathbf{m}$

[7] If $n = 1$, G would have to be linear, whence $G'' = 0$. Therefore, our analysis in this section is restricted to the case $n > 1$. Even so, it yields exactly the same results as the $n = 1$ framework analyzed in section 2.

[8] This implies $\left(\frac{f}{f'}\right)$ to be a decreasing function, a fact we shall use soon. In the unidimensional case, this is a sufficient condition (weaker than $f'' \leq 0$) for the money demand which emerges from the shopping-time model to be a decreasing function; in the multidimensional case, a sufficient condition for each u_i to decrease along rays starting at the origin (see Remark 2).

into $\mathbf{u}$. In vector notation, we have

$$\mathbf{u} = \psi(\mathbf{m}) = \frac{f(\tau(G(\mathbf{m})))}{G(\mathbf{m}) f'(\tau(G(\mathbf{m})))} \nabla G(\mathbf{m}). \tag{32}$$

Remark 2 *Along rays starting at the origin, each ψ_i is strictly decreasing. In fact, if $k > 1$, we get, by G's strict increasingness in each variable, $G(k\mathbf{m}) > G(\mathbf{m})$. Therefore, since τ is strictly decreasing (like H), $\tau(G(k\mathbf{m})) < \tau(G(\mathbf{m}))$, and $\left(\frac{f}{f'}\right)(\tau(G(k\mathbf{m}))) < \left(\frac{f}{f'}\right)(\tau(G(\mathbf{m})))$. From G's homogeneity, we also get $G_{x_i}(k\mathbf{m}) = G_{x_i}(\mathbf{m})$.*[9] *Therefore*

$$\begin{aligned}\psi_i(k\mathbf{m}) &= \frac{1}{G(k\mathbf{m})}\left(\frac{f}{f'}\right)(\tau(G(k\mathbf{m})))\, G_{x_i}(k\mathbf{m}) \\ &< \frac{1}{G(\mathbf{m})}\left(\frac{f}{f'}\right)(\tau(G(\mathbf{m})))\, G_{x_i}(\mathbf{m}) = \psi_i(\mathbf{m}),\end{aligned}$$

as we wished.

The multidimensional analog of (1) was derived in Cysne (2003, eq. 14):

$$s_{x_i}(\mathbf{m}) = -\frac{\psi_i(\mathbf{m})}{1 - s(\mathbf{m}) + \psi(\mathbf{m}) \cdot \mathbf{m}}(1 - s(\mathbf{m})). \tag{33}$$

Consider a C^1 path $\chi : [0,1] \to [0,+\infty]^n$ such that $\chi(0) = +\infty := +\infty(1,\ldots,1)$ and $\chi(1) = \mathbf{m}$, and the following 1-forms in $[0,+\infty]^n$:

$$dA := -\frac{1}{1 + \mathbf{u} \cdot \mathbf{m}} \mathbf{u} \cdot d\mathbf{m} \tag{34}$$

and

$$dB := -\mathbf{u} \cdot d\mathbf{m}. \tag{35}$$

[9] Any partial derivative of a g-homogeneous function is a $(g-1)$-homogeneous function.

The line integrals

$$A(\mathbf{m}) := \int_{\chi} dA \tag{36}$$

and

$$B(\mathbf{m}) := \int_{\chi} dB \tag{37}$$

extend Simonsen and Cysne's (2001) proxy measure to s (2) and Bailey's measure (3) to the present framework.[10] In Cysne (2003, prop. 1), it is shown that when ψ arises from the shopping-time model, these integrals are path-independent, so that A and B are indeed well-defined.

From Remark 2, we see that the initial condition that a welfare-cost function w (such as s, A or B) has to satisfy in this multidimensional framework is $w(d) = 0, \forall d \in [0, +\infty]^n \setminus \mathbb{R}^n_+$ (as mentioned in the beginning of this section, now our welfare-cost functions are functions of the vector of monetary aggregates, rather than the interest rates). This is how we get the lowest possible values for the u_i's, which is the benchmark used for measuring the welfare cost of inflation (see (30)).

In proposition 2 of Cysne (2003), it is demonstrated that, exactly as happens in the unidimensional case, we have

$$1 - e^{-A} < s < A < B\,.$$

In the next subsection, we shall introduce the multidimensional analogs of $\underline{w}$ and $\overline{w}$, and prove that they relate to the above welfare measures in exactly the same way as in the unidimensional case.

[10] These are, in order, the additive inverses of the Divisia indices $DE(\chi)$ and $DG(\chi)$ presented in Cysne (2003).

4.2 The extended Sidrauski model

Our representative agent's instantaneous utility will now have the form:

$$U(c, \mathbf{m}) = \frac{1}{1-\sigma}\left(c\varphi\left(\frac{G(\mathbf{m})}{c}\right)\right)^{1-\sigma},$$

where σ and φ are exactly as in subsection 2.2 (so that φ has an $\bar{m} \in (0, +\infty]$ associated to it), and G is exactly as in the previous subsection. Assuming again real output growth to be equal to γ, her maximization problem will be:

$$\max_{c_t>0, \mathbf{m}_t \geq 0} \int_0^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(c_t, \mathbf{m}_t) dt \tag{$\mathrm{P}_{\mathrm{S}}^n$}$$

subject to

$$\dot{b}_t + \mathbf{1} \cdot \dot{\mathbf{m}}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma) b_t + (\mathbf{r}_t - (\pi_t + \gamma) \cdot \mathbf{1}) \cdot \mathbf{m}_t$$
$$b_0 > 0 \text{ and } \mathbf{m}_0 > 0 \text{ given.}$$

where we write $\mathbf{1}$ for the vector $(1, \ldots, 1) \in \mathbb{R}^n$, '$\cdot$' for the canonical inner product of $\mathbb{R}^n$, and all the non-bold letters have the same meaning as in the model introduced in subsection 2.2.

Considering only regular solutions and substituting for c as we've done before, we may rewrite our maximization problem as

$$\max_{b>0, \mathbf{m} \gg \mathbf{0}} \int_0^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(y - h + (r - \pi - \gamma) b + (\mathbf{r} - (\pi + \gamma) \cdot \mathbf{1}) \cdot \mathbf{m} - \dot{b} - \mathbf{1} \cdot \dot{\mathbf{m}}, \mathbf{m}) dt. \tag{$\mathrm{P}_{\mathrm{S}}^n$}$$

The Euler equations are:

$$\begin{cases} (r-\pi-\gamma))U_c = (-\rho+\gamma(1-\sigma))U + \frac{d}{dt}(-U_c) \\ (r_i-\pi-\gamma)U_c + U_{x_i} = (-\rho+\gamma(1-\sigma))U + \frac{d}{dt}(-U_c), \forall i \in \{1,\ldots,n\} \end{cases},$$

which really correspond to the optimum of $(\mathrm{P}_{\mathrm{S}}^n)$, by U's concavity.[11] As before, these equations immediately imply

$$r - r_i = \frac{U_{x_i}}{U_c}, \forall i \in \{1,\ldots,n\}. \tag{38}$$

In equilibrium, we have $c = 1$, so that (38) gives

$$u_i = \psi_i(\mathbf{m}) = \frac{\varphi'(G(\mathbf{m}))}{\varphi(G(\mathbf{m})) - G(\mathbf{m})\,\varphi'(G(\mathbf{m}))} G_{x_i}(\mathbf{m}), \forall i \in \{1,\ldots,n\}, \tag{39}$$

where now $\psi : G^{-1}([0,\bar{m}]) \to [0,+\infty]^n$, G^{-1} denoting the inverse image of $[0,\bar{m}]$ under G. Equation (39) is analogous to (9), giving us a differentiable function ψ taking $\mathbf{m}$ into $\mathbf{u}$.

We can rewrite (39) as

$$\mathbf{u} = \psi(\mathbf{m}) = F(G(\mathbf{m}))\,\nabla G(\mathbf{m}), \tag{40}$$

where $F : [0,+\infty] \to [0,+\infty]$ is a differentiable and strictly decreasing function (as already calculated in (13)). The reader may want to notice that this general form for multidimensional money-demand specifications also encompasses specifications

[11] Let V stand for the function U of section 2 (of only two variables), so that $U(c,\mathbf{m}) = V(c,G(\mathbf{m}))$. Given $c,d \in \mathbb{R}_{++}$ and $\mathbf{x},\mathbf{y} \in \mathbb{R}_+^n$, we have
$U(d,\mathbf{y}) - U(c,\mathbf{x}) = V(d,G(\mathbf{y})) - V(c,G(\mathbf{x})) \leq D_c V_{(c,G(\mathbf{x}))}(d-c) + D_m V_{(c,G(\mathbf{x}))}(G(\mathbf{y}) - G(\mathbf{x})) \leq D_c V_{(c,G(\mathbf{x}))}(d-c) + D_m V_{(c,G(\mathbf{x}))} DG_{\mathbf{x}}(\mathbf{y}-\mathbf{x}) = DU_{(c,\mathbf{x})}(d-c,\mathbf{y}-\mathbf{x})$, where we've used the concavity of G and the fact that $D_m V \geq 0$.

originating from the shopping-time model, given by (32).

We shall now see that the welfare-measures A and B are well-defined also for money-demand functions arising from the multidimensional money-in-the-utility-function model. Here, the only difference to the last subsection is that the path χ should be such that $\chi(0) \in G^{-1}(\{\bar{m}\})$, and our boundary condition for a generic welfare-measure w is that $w(d) = 0, \forall d \in G^{-1}(\{\bar{m}\})$.[12]

Lemma 1 *If* $\mathbf{u}$ *is given by (40), the line integrals in (36) and (37) are path-independent .*

Proof. All we need to do is check that the 1-forms dA and dB are closed. We start by the latter.

From (40), we have

$$(u_i)_{x_j}(\mathbf{m}) = F'(G(\mathbf{m}))\, G_{x_j}(\mathbf{m})\, G_{x_i}(\mathbf{m}) + F(G(\mathbf{m}))\, G_{x_i x_j}(\mathbf{m}),$$

evidently symmetric in (i,j). Therefore, $-(u_i)_{x_j} = -(u_j)_{x_i}$, and dB is closed.

Now for the path-independence of A. Since

$$\left(\frac{1}{1+\mathbf{u}\cdot\mathbf{m}}u_i\right)_{x_j} = \frac{1}{(1+\mathbf{u}\cdot\mathbf{m})^2}\left[(1+\mathbf{u}\cdot\mathbf{m})(u_i)_{x_j} - u_i\left(u_j + \textstyle\sum_{k=1}^n m_k (u_k)_{x_j}\right)\right],$$

and we already know that $(u_i)_{x_j} = (u_j)_{x_i}$, it suffices to check that $\sum_{k=1}^n m_k u_i (u_k)_{x_j}$ is symmetric in (i,j). Using the above expression for $(u_i)_{x_j}$ and (40), we find

$$\textstyle\sum_{k=1}^n m_k u_i (u_k)_{x_j} = \sum_{k=1}^n m_k F G_{x_i}\left(F' G_{x_j} G_{x_k} + F G_{x_k x_j}\right),$$

[12]This generalizes perfectly the analysis carried out in the last subsection, where the money-demand function (32) implied that, if $\mathbf{u}$ were to be $\mathbf{0}$, $\mathbf{m}$ would have to be unbounded.

which is symmetric in (i,j) iff $G_{x_i}\sum_{k=1}^{n} m_k G_{x_j x_k}$ is symmetric in (i,j). And this is the case, since this expression actually equals zero, from G_{x_j}'s 0-homogeneity and Euler's formula for homogeneous functions. ■

4.3 The welfare cost of inflation in the extended Sidrauski model: two different approaches

Let us now write down the equations defining the measures $\overline{w}$ and $\underline{w}$ of the welfare cost of inflation:

$$\begin{cases} U(1+\overline{w}(\mathbf{m}),\ G(\mathbf{m})) = U(1,\bar{m}) \text{ or} \\ (1+\overline{w}(\mathbf{m}))\varphi\left(\frac{G(\mathbf{m})}{1+\overline{w}(\mathbf{m})}\right) = \varphi(\bar{m}) = \varphi^*; \end{cases} \tag{41}$$

$$\begin{cases} U(1,G(\mathbf{m})) = U(1-\underline{w}(\mathbf{m}),\bar{m}) \text{ or} \\ \varphi(G(\mathbf{m})) = (1-\underline{w}(\mathbf{m}))\varphi\left(\frac{\bar{m}}{1-\underline{w}(\mathbf{m})}\right) = (1-\underline{w}(\mathbf{m}))\varphi^*. \end{cases} \tag{42}$$

Partially differentiating the first of these equations with respect to x_i, and dividing through by $\varphi'\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)$, we obtain:

$$\overline{w}_{x_i}(\mathbf{m})\frac{\varphi\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)}{\varphi'\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)} + \left[G_{x_i}(\mathbf{m}) - \overline{w}_{x_i}(\mathbf{m})\,G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right] = 0.$$

But (39) gives us

$$\frac{\varphi\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)}{\varphi'\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)} = \frac{G_{x_i}(\mathbf{m})}{\psi_i\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)} + G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right),$$

so that we get the expression

$$\overline{w}_{x_i}(\mathbf{m}) = -\psi_i\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right), \tag{43}$$

analogous to (16).

From ψ_i's strict decreasingness along rays starting at the origin, we have $\overline{w}_{x_i}(\mathbf{m}) < -\psi_i(\mathbf{m})$. Besides this, because of G's increasingness in every direction, it is always possible to choose $d \in G^{-1}(\{\bar{m}\})$ such that $d \gg \mathbf{m}$. Consider then a path χ as in the previous subsection such that $\chi(0) = d$. Therefore:

$$\begin{aligned}\overline{w}(\mathbf{m}) &= \int_0^1 \frac{d}{d\lambda}\overline{w}(\chi(\lambda))\,d\lambda = \int_0^1 \left[-\psi\left(\frac{1}{1+\overline{w}(\chi(\lambda))}\chi(\lambda)\right)\cdot\nabla\chi(\lambda)\right]d\lambda \\ &> \int_0^1 \left[-\psi(\chi(\lambda))\cdot\nabla\chi(\lambda)\right]d\lambda = \int_\chi -\psi(\mu)\cdot d\mu = B(\mathbf{m}),\end{aligned} \tag{44}$$

where the first equality follows from the Fundamental Theorem of Calculus and the condition $\overline{w}(d) = 0$; and the inequality sign follows from realizing that since the leftmost and the rightmost extremities of the above inequality are independent of χ (Lemma 1), and we took $d \gg \mathbf{m}$, we can consider without loss of generality that χ is strictly decreasing in every direction.

Let us now turn our attention to $\underline{w}$. We have already seen in (42) that $\underline{w}(\mathbf{m}) = \frac{\varphi^* - \varphi(G(\mathbf{m}))}{\varphi^*}$. On the other hand, writing $dG(\mathbf{m}) = \nabla G(\mathbf{m})\cdot d\mathbf{m}$ and using the fact that $G(\chi(0)) = \bar{m}$, (34) gives

$$\begin{aligned}A(\mathbf{m}) &= \int_\chi -\frac{\varphi'(G(\mu))}{\varphi(G(\mu))}dG(\mu) = \int_{\bar{m}}^{G(\mathbf{m})} -\frac{\varphi'(\tilde{G})}{\varphi(\tilde{G})}d\tilde{G} \\ &= \int_{\varphi^*}^{\varphi(G(\mathbf{m}))} -\frac{d\tilde{\varphi}}{\tilde{\varphi}} = \log\left(\frac{\varphi^*}{\varphi(G(\mathbf{m}))}\right),\end{aligned}$$

so

$$\frac{\varphi\left(G\left(\mathbf{m}\right)\right)}{\varphi^{*}}=e^{-A(\mathbf{m})}. \tag{45}$$

Therefore we have obtained a perfect analog to (21):

$$\underline{w}\left(\mathbf{m}\right)=1-e^{-A(\mathbf{m})}. \tag{46}$$

The measures $\underline{w}$ and $\overline{w}$ were introduced for Sidrauski's model, and were shown to relate in a precise way with other measures. We are now ready to state and prove our main ordering result, extending Proposition 1 to an economy with many types of monies.

Proposition 2 *Let $\psi : G^{-1}\left([0,\bar{m}]\right) \to [0,+\infty]^{n}$ be a money-demand specification taking the general form (40) (for instance, one arising from the extended shopping-time or Sidrauski model), so that we can use it to calculate s, A, B, $\overline{w}$, and $\underline{w}$ (by simply plugging it into (33), (36), (37), (43) and (46)). Then we have the following inequality chain:*

$$\underline{w}=1-e^{-A}<s<A<B<\overline{w}.$$

Proof. It has been proved in Cysne (2003, p. 236) that $1-e^{-A}<s<A<B$, in case we take our path χ so that $\frac{d}{dt}\nabla\chi \ll \mathbf{0}$ – and, as we've seen above in the proof of inequality (44), this can always be done. The equality between $\underline{w}$ and $1-e^{-A}$ has been proved in (46). Now it's only a matter of gluing the inequalities and the equality together. ■

5 Formulas for the Multidimensional Welfare Measures

Repeating the procedure used in the unidimensional case, in this section we exemplify with a multidimensional extension of the log-log money-demand specification the ordering of the welfare measures given in Proposition 2.

By observing (22) and (40), it is natural enough to propose an extension of the unidimensional log-log money-demand specification of the form:

$$\psi(\mathbf{m}) = \left(\frac{K}{G(\mathbf{m})}\right)^{1/\alpha} \nabla G(\mathbf{m}),$$

where $K > 0$ and $\alpha \in (0,1)$. This is the demand that would emerge from $(\mathrm{P}_{\mathrm{S}}^{n})$ and (28).

It is worth noticing that, in the case of G being a weighted geometric mean, $G(m_1,\ldots,m_n) = \prod_{i=1}^{n} m_i^{\beta_i}$ (where $\beta_i \geq 0$, $\forall i \in \{1,\ldots,n\}$ and $\sum_{i=1}^{n}\beta_i = 1$), we obtain the system

$$\begin{cases} \psi_1(\mathbf{m}) = \frac{K^{1/\alpha}\beta_1}{m_1}\prod_{j=1}^{n} m_j^{\left(1-\frac{1}{\alpha}\right)\beta_j} \\ \qquad\qquad \vdots \\ \psi_n(\mathbf{m}) = \frac{K^{1/\alpha}\beta_n}{m_n}\prod_{j=1}^{n} m_j^{\left(1-\frac{1}{\alpha}\right)\beta_j} \end{cases},$$

which, inverted,[13] gives a demand in the format:

$$\begin{cases} m_1(\mathbf{u}) = L_1\prod_{j=1}^{n} u_j^{\alpha_{1j}} \\ \qquad\qquad \vdots \\ m_n(\mathbf{u}) = L_n\prod_{j=1}^{n} u_j^{\alpha_{nj}} \end{cases},$$

[13] The very name of this kind of demand is a hint to how to invert this system of equations: apply the logarithm function on each side. This yields a linear, easy-to-invert system.

where the α_{ij}'s are the demand elasticities of type i of money with relation to the opportunity cost of holding type j.

Again, B and A can be easily obtained. Taking χ as in the previous section, and such that $\chi(0) = +\infty := +\infty(1, \ldots, 1)$ (we may do that since $\bar{m} = +\infty$), we have:

$$\begin{aligned} B(\mathbf{m}) &= \int_{\chi} -\psi(\mu) \cdot d\mu = -K^{1/\alpha} \int_{\chi} \frac{\nabla G(\mathbf{m}) \cdot d\mathbf{m}}{G(\mathbf{m})^{1/\alpha}} \\ &= -K^{1/\alpha} \left. \frac{G(\mathbf{m})^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}} \right|_{+\infty}^{\mathbf{m}} = \frac{\alpha K^{1/\alpha}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}, \end{aligned} \tag{47}$$

and

$$A(\mathbf{m}) = \int_{\chi} \frac{-\psi(\mu) \cdot d\mu}{1 + \psi(\mu) \cdot \mu} = -\int_{\chi} \frac{K^{1/\alpha} dG(\mathbf{m})}{G(\mathbf{m})^{1/\alpha} + K^{1/\alpha} G(\mathbf{m})}.$$

Notice how this is exactly the same integrand that appeared in our second approach to finding $\underline{w}$ in the unidimensional case. With that knowledge behind us, we can immediately write:

$$\begin{aligned} A(\mathbf{m}) &= \lim_{G^* \to +\infty} \left[\int_{G(\mathbf{m})}^{G^*} \frac{d\tilde{G}}{\tilde{G}} - \frac{\alpha}{1-\alpha} \int_{K^{1/\alpha} + G(\mathbf{m})^{\frac{1}{\alpha}-1}}^{K^{1/\alpha} + G^{*\frac{1}{\alpha}-1}} \frac{d\tilde{G}}{\tilde{G}} \right] \\ &= \lim_{G^* \to +\infty} \log \left[\frac{G^*}{\left(K^{1/\alpha} + G^{*\frac{1}{\alpha}-1}\right)^{\frac{\alpha}{1-\alpha}}} \frac{\left(K^{1/\alpha} + G(\mathbf{m})^{\frac{1}{\alpha}-1}\right)^{\frac{\alpha}{1-\alpha}}}{G(\mathbf{m})} \right] \\ &= \log \left(\frac{\left(K^{1/\alpha} + G(\mathbf{m})^{\frac{1}{\alpha}-1}\right)^{\frac{\alpha}{1-\alpha}}}{G(\mathbf{m})} \right) = \frac{\alpha}{1-\alpha} \log \left(1 + K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right). \end{aligned} \tag{48}$$

Here we have made implicit use of the obvious fact that, since G is a positive,

increasing in each variable and 1-homogeneous function, it is unbounded:

$$\lim_{\mathbf{m}\to+\infty} G(\mathbf{m}) = +\infty.$$

Then (48) and (46) give at once:

$$\underline{w}(\mathbf{m}) = 1 - \left(1 + K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{-\frac{\alpha}{1-\alpha}}. \tag{49}$$

Now for $\overline{w}$. Equation (43) takes, in this case, the form:

$$\overline{w}_{x_i}(\mathbf{m}) = -(1 + \overline{w}(\mathbf{m}))^{1/\alpha} \left(\frac{K}{G(\mathbf{m})}\right)^{1/\alpha} G_{x_i}(\mathbf{m}),$$

which, as in the unidimensional case, gives

$$\int_0^{\overline{w}(\mathbf{m})} \frac{d\nu}{(1+\nu)^{1/\alpha}} = -K^{1/\alpha} \int_\chi \frac{\nabla G(\mathbf{m}) \cdot d\mathbf{m}}{G(\mathbf{m})^{1/\alpha}} = B(\mathbf{m}),$$

so that

$$\frac{1}{1-\frac{1}{\alpha}} \left((1 + \overline{w}(\mathbf{m}))^{1-\frac{1}{\alpha}} - 1\right) = \frac{\alpha K^{1/\alpha}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}},$$

and

$$\overline{w}(\mathbf{m}) = -1 + \left(1 - K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}. \tag{50}$$

Finally, for s, the easiest thing to do is just imitate (23) in the obvious way, and verify that it satisfies (33). That is, take s given implicitly by equation

$$D(\mathbf{m}, s(\mathbf{m})) = 0, \tag{51}$$

where

$$D(\mathbf{m}, s) := \frac{K^{1/\alpha}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}} + (1-s)\left(1-(1-s)^{-1/\alpha}\right).$$

The Implicit Function Theorem gives

$$\begin{aligned} s_{x_i}(\mathbf{m}) = -\frac{(\partial D/\partial x_i)(\mathbf{m}, s(\mathbf{m}))}{(\partial D/\partial s)(\mathbf{m}, s(\mathbf{m}))} &= \frac{\frac{1}{\alpha} K^{1/\alpha} G(\mathbf{m})^{-1/\alpha} G_{x_i}(\mathbf{m})}{-1 + \left(1-\frac{1}{\alpha}\right)(1-s(\mathbf{m}))^{-1/\alpha}} \\ &= \frac{\psi_i(\mathbf{m})}{-1 + (1-\alpha)\left(1-(1-s(\mathbf{m}))^{-1/\alpha}\right)}, \end{aligned}$$

whence

$$-\frac{1}{s_{x_i}(\mathbf{m})} = \frac{1}{\psi_i(\mathbf{m})} - \frac{(1-\alpha)\left(1-(1-s(\mathbf{m}))^{-1/\alpha}\right)}{K^{1/\alpha} G(\mathbf{m})^{-1/\alpha} G_{x_i}(\mathbf{m})}.$$

Substituting from (51), we get

$$\begin{aligned} -\frac{1}{s_{x_i}(\mathbf{m})} &= \frac{1}{\psi_i(\mathbf{m})} + \frac{K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}}{K^{1/\alpha} G(\mathbf{m})^{-1/\alpha} G_{x_i}(\mathbf{m})(1-s(\mathbf{m}))} \\ &= \frac{1}{\psi_i(\mathbf{m})} + \frac{G(\mathbf{m})}{G_{x_i}(\mathbf{m})(1-s(\mathbf{m}))}, \end{aligned}$$

an expression equivalent to (33). The reader may wish to check that these formulas really extend the unidimensional ones.

All these measures are plotted in Figures 3 and 4, where $n = 2$, $K = 0.05$ and $\alpha = 0.5$. In Figure 3, G is taken so that $G(m, x) = m^{0.5} x^{0.5}$, and in Figure 4, $G(m, x) = m^{0.7} x^{0.3}$. The ordering of the surfaces is the one implied by Proposition 2. Again, as in the unidimensional case, we see that A approximates s so well that it is almost impossible to visualize them separately.

Figure 3: Welfare Costs in the Multidimensional Case

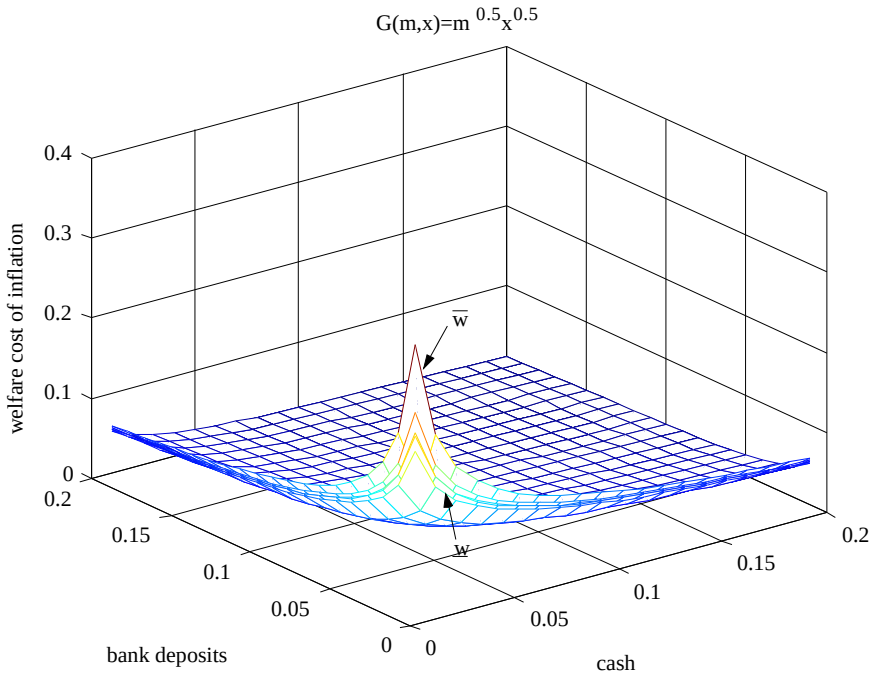
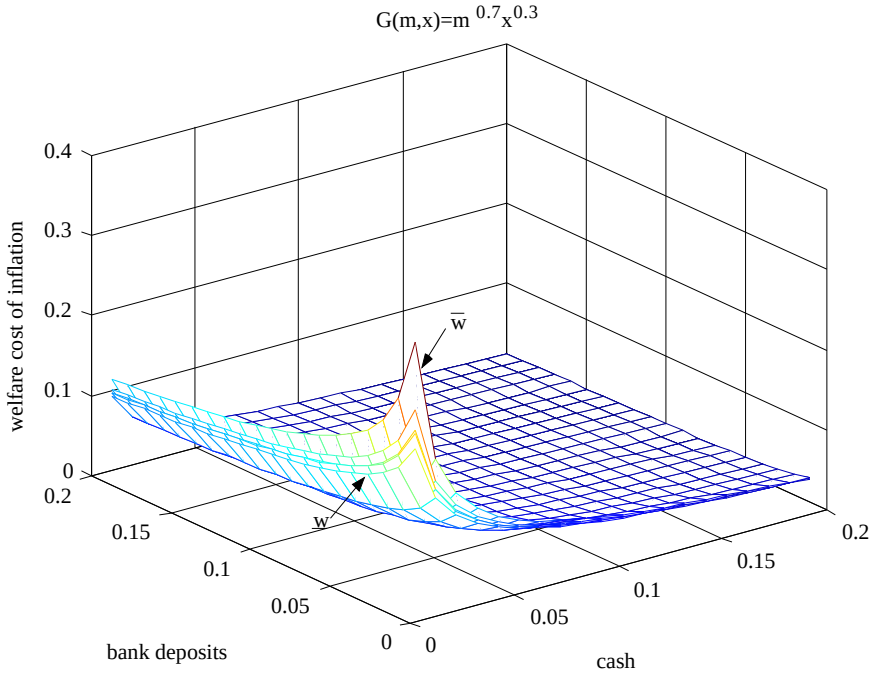


Figure 4: Welfare Costs in the Multidimensional Case



6 Conclusion

The present work has extended the ordering of measures of the welfare costs of inflation provided by Cysne (2003) to include two new measures derived from Sidrauski's money-in-the-utility-function framework. The first measure is provided by Lucas (though only in relation to the unidimensional case). The second measure is new in the literature (both for the unidimensional and the multidimensional case).

The main result of the paper, given in Proposition 2, is the ordering of six different measures of the welfare costs of inflation both when there is only one type of money and in the case where there are several interest-bearing assets performing monetary services.

We have also illustrated our results with the well-known log-log money-demand, both in the unidimensional and in the multidimensional case. In the presence of only one type of money we have shown that for parameter values as those usually found in the literature, the greatest possible relative error a researcher can incur by deciding to use a particular measure is extremely low in normal scenarios (only 4% for annual nominal interest rates as high as 15%), but may be high in hyperinflations (22% if this interest rate is around 400%).

Finally, we have provided closed-form solutions to all six measures of the welfare costs of inflation both for the unidimensional and the multidimensional framework assuming a log-log money demand.

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