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Apparent impact: the hidden cost of one-shot trades

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Abstract

We study the problem of the execution of a large order in an illiquid market within the framework of a solvable Markovian model. We suppose that in order to avoid impact costs, a trader decides to execute her order through a unique trade, waiting for enough liquidity to accumulate at the best quote. We find that despite the absence of a proper price impact, such trader faces an execution cost arising from a non-vanishing correlation among volume at the best quotes and price changes. We characterize analytically the statistics of the execution time and its cost by mapping the problem to the simpler one of calculating a set of first-passage probabilities on a semi-infinite strip. We finally argue that price impact cannot be completely avoided by conditioning the execution of an order to a more favorable liquidity scenario.

1 Introduction

Market impact is commonly measured as the expected price change after the sequential execution of a given volume of contracts in a financial market [1]. It refers to one of the most fundamental aspects of market microstructure, as it encompasses the information about how a financial market reacts to an incoming fluxes of trades, and ultimately allows prices to reflect fundamental information [2]. It is also a central concept for market practitioners, who need to minimize market impact of their own trades in order to minimize their execution costs.

Despite these strong motivations, no such a thing as a fully consistent theory of market impact has been devised yet. One of the main reasons behind this difficulty is the extremely small entity of the price change signal: while the fluctuations of daily returns are typically of the order of a few %, the impact generally consists of some basis points, implying a signal to noise ratio around 10^{-2} [2]. Moreover, proprietary data required in order to perform this analysis is rare and, except for a few cases, confidential.
A final problem which one needs to face when analyzing impact is the presence of biases in the dataset which is studied: which among the many features emerging from the empirical analysis depend upon the behavior of the institution providing the data, and which others genuinely reflect the market universal properties? Reassuringly, according to the literature

so far the most prominent features of market impact seem to be roughly independent by the institution performing the trades [3, 4, 5, 6, 7, 8, 9, 10].

Yet, even in the ideal scenario in which impact is very weakly dependent on the details of the trading strategy, it is of fundamental importance to control why such a weak dependence emerges, and what are the other effects that can in principle be entangled to the genuine one (which one could measure just in the extremely interesting – but rather costly – case of random trading experiments).

In this work we identify an extra component of the *slippage* (i.e., the expected price change between the *decision time* and the *execution time* of a trade) previously neglected in the literature, which is unrelated to the traditional notion of market impact and that is linked to the microstructural liquidity profile of the traded contracts. Yet, we dub it *apparent impact*, as in data it would appear as a component of market impact emerging even in absence of a trend and in the case in which no trade at all is performed. This point of view challenges the traditional idea that *trades cause price changes*: while it is common to assume that trades move prices, what we are going to show is a specific case in which future trading activity emerges mechanically due to past price moves[1].

We will focus our attention on a specific framework in which these ideas can be addressed analytically. In particular we will consider within the market model of [11, 12], the problem of a trader who needs to submit a buy order of size Q to the market (in an arbitrary amount of time), and who decides to wait until a sufficient amount of liquidity is available at the best quote before executing her order in a unique trade, in order not to incur impact costs. Within this framework, will be able to argue that:

- Between the *decision time* and the *execution time* the price will drift on average, even in absence of trend, by an amount $\mathcal{I}(Q)$, where $\mathcal{I}(\cdot)$ is a function of the executed volume, whose precise shape depends upon the details of market microstructure;
- Such a price change is not linked to genuine impact (neither mechanical or behavioral), as we take as the execution time the instant *right before* the submission of the order, in which the trader has not yet executed her order.

Our main conclusion is that past price changes can induce future trading activity simply for liquidity reasons: If less liquidity is available at the ask whenever prices have a bullish trend, then buyers need to be more patient while price moves up. Conversely, they can execute more quickly when prices decline. We believe that, irrespective of the particular model that we have chosen in order to illustrate this behavior, such phenomenon can exist is general, and that it generates additional execution costs with respect to the traditional market impact. Summarizing, if the amount of outstanding liquidity is correlated with

[1] While one could argue that trading can happen because market participants *forecast* future prices, we will study the problem in the even simpler setting in which even though traders do not try to anticipate prices, yet past price changes are correlated with future trading activity.

past price moves, waiting for sufficient volume to accumulate at the best quote can be expensive.

The plan of the paper is the following: In Sec. 2 we introduce the market model and discuss its behavior under a symmetric flux of orders. In Sec. 3 we introduce the order execution procedure for the extra trader and calculate the expected properties of the model at the moment of the execution. Finally, we draw our conclusions in Sec. 4. For the sake of clarity, the more technical parts are relegated to the appendices.

2 The model

2.1 A Markovian description of large tick contracts

We adopt a framework similar to the one considered in [11, 12] as a background model of the market. We consider a Markovian model in which all the information relative to the current state of the market is encoded in these variables:

- The mid-price x_t;
- The volume available at the bid queue V_t^b and the volume available at the ask queue V_t^a.

This choice is motivated by the strong concentration of market activity at the best quotes for large tick stocks [13], and by the observation that a large component of price variation can be accounted for by the dynamics at the best quotes [14, 15]. In order to model the mechanism of quote revision, we suppose that anytime either queue is exhausted, the volume of both queues is reset and prices are updated accordingly. In particular if the ask queue is emptied, we suppose the volumes at the best queues $\boldsymbol{V} = (V^b, V^a)$ to instantaneously revert to (V_{sml}, V_{lrg}), while in the opposite case we assume the new volumes to be (V_{lrg}, V_{sml}), which allows to model microscopic mean-reversion of price (see e.g. [15]). We suppose in the former case the price to jump up by one half of a tick (which we assume without loss of generality to be of size $w = 2$), while in the latter case we assume price to decrease by $w/2$. This should be appropriate in the case of large tick contracts, for which the spread is closed almost immediately after the exhaustion of the volume at either of the best queues [16].

While the framework considered in [11, 12] focuses on a simpler diffusion dynamics for the queues, in [16] it is shown that a less trivial Fokker-Planck equation needs to be employed in order to describe the rich behavior of empirical data. Nevertheless, we will restrict ourselves to an even more stylized scenario than the one considered in [11, 12], and adopt a simplified description in which the volumes at the best quotes evolve according to the

equation

$$\frac{\partial P(t,V^b,V^a)}{\partial t} = \frac{D}{2}\frac{\partial^2 P(t,V^b,V^a)}{\partial (V^b)^2} + \frac{D}{2}\frac{\partial^2 P(t,V^b,V^a)}{\partial (V^a)^2} + \mu \frac{\partial P(t,V^b,V^a)}{\partial V^b} + \mu \frac{\partial P(t,V^b,V^a)}{\partial V^b} \,, \tag{1}$$

where we take $D = 1$ without loss of generality and where we assume $\mu \geq 0$, so to model queues whose volume drifts towards zero.Under this dynamics no detailed description of the distinct type of orders hitting the best queues is provided[2]. Rather, this scenario builds on a coarse-grained description of the volume available at the best quotes which is appropriate when volumes sitting at the best queues are sufficiently large.

Finally, the procedure of one-shot execution of a buy order illustrated above implies in this context in the presence of a random variable T representing the time elapsed between the *decision time* $t = 0$ and the *execution time* $t = T$, the first time in which the ask volume V^a equals Q. Our goal is to provide a statistical characterization of T, together with the one of the price x_t (the number of positive jumps minus the negative ones) and the one for hitting number n_t (the number of jumps of either type) at the execution time T.

2.2 Properties of the free model

At first we will illustrate the properties of this model in the regime in which no trader is present, and the two queues evolve freely under the dynamics (1) and the boundary conditions specified above. In this case the coordinates of the system (V^b, V^a) diffuse freely in the positive orthant, reverting to the appropriate state (either (V_{lrg}, V_{sml}) or (V_{sml}, V_{lrg})) as soon as they touch either of the boundaries. In either case, the variables x_t and n_t are conveniently updated. We are interested in particular in showing the results for the statistics of the evolution of the price x_t and the hitting number n_t. The object that we are required to compute in order to obtain a solution for x_t and n_t is the probability that a system starting from coordinates (V^b, V^a) hits one of the the two boundaries (either $V^a = 0$ or $V^b = 0$) at a given time t. These quantities will be denoted respectively with $p_\uparrow(t, V^b, V^a)$ and $p_\downarrow(t, V^b, V^a)$. It is easy to show (see for instance Ref. [17]) that their Laplace transforms $\hat{p}_\uparrow(\omega, V^b, V^a)$ and $\hat{p}_\downarrow(\omega, V^b, V^a)$ satisfy the equations

$$\begin{aligned} \omega\, \hat{p}_\alpha(\omega, V^b, V^a) &= \frac{1}{2}\frac{\partial^2 \hat{p}_\alpha(\omega, V^b, V^a)}{\partial (V^b)^2} + \frac{1}{2}\frac{\partial^2 \hat{p}_\alpha(\omega, V^b, V^a)}{\partial (V^a)^2} \\ &- \mu \frac{\partial \hat{p}_\alpha(\omega, V^b, V^a)}{\partial V^b} - \mu \frac{\partial \hat{p}_\alpha(\omega, V^b, V^a)}{\partial V^a} \,, \end{aligned} \tag{2}$$

where $\alpha \in \{\uparrow, \downarrow\}$. The difference among $p_\uparrow$ and $p_\downarrow$ is encoded in the different boundary conditions: for $V^b \neq 0$ one has $p_\uparrow(t, V^b, 0) = \delta_t$, $p_\downarrow(t, V^b, 0) = 0$, while for $V^a \neq 0$

[2]Notice that in this framework market orders and cancellations are not distinguishable.

one has $p_\downarrow(t,0,V^a)=\delta_t$, $p_\uparrow(t,0,V^a)=0$. Equivalently $\hat{p}_\uparrow(\omega,V^b,0)=1-\hat{p}_\downarrow(\omega,V^b,0)=1$, $\hat{p}_\downarrow(\omega,0,V^a)=1-\hat{p}_\uparrow(\omega,0,V^a)=1$. The values $\bar{p}_\uparrow(V^b,V^a)=\hat{p}_\uparrow(0,V^b,V^a)$ and $\bar{p}_\downarrow(V^b,V^a)=\hat{p}_\downarrow(0,V^b,V^a)$ obtained for $\omega=0$ represent the probability of hitting the boundaries at any time between $t=0$ and $t=\infty$ starting from the configuration (V^b,V^a), and verify $\bar{p}_\uparrow(V^b,V^a)+\bar{p}_\downarrow(V^b,V^a)=1$, indicating that the boundaries are hit almost surely. Once that these functions have been calculated (the details about the derivation can be found in App. A), it is possible to obtain the generating function for the price x_t, denoted with $\Phi_x(\omega,s)$, as well as the one for the hitting number n_t, denoted with $\Phi_n(\omega,s)$. They are defined as

$$\Phi_x(\omega,s) = \sum_{x=-\infty}^{\infty} e^{-xs}\int_0^\infty dt\, e^{-\omega t}P_x(t,x) \tag{3}$$

$$\Phi_n(\omega,s) = \sum_{n=-\infty}^{\infty} e^{-ns}\int_0^\infty dt\, e^{-\omega t}P_n(t,n)\ , \tag{4}$$

where $P_x(t,x)$ and $P_n(t,x)$ are respectively the probability for the price and for the hitting number of taking value x (respectively n) at time t. By exploiting the Markov property of the model, one can prove (App. B) that

$$\begin{aligned}\Phi_x(\omega,s) &= \frac{1}{\omega}\left[\begin{pmatrix}1-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\uparrow}\\ 1-\hat{p}_{\uparrow\downarrow}-\hat{p}_{\downarrow\downarrow}\end{pmatrix}^T\sum_{n=0}^{\infty}\begin{pmatrix}\hat{p}_{\uparrow\uparrow}e^{-s} & \hat{p}_{\uparrow\downarrow}e^{-s}\\ \hat{p}_{\downarrow\uparrow}e^{s} & \hat{p}_{\downarrow\downarrow}e^{s}\end{pmatrix}^n\begin{pmatrix}\hat{p}_{\uparrow 0}e^{-s}\\ \hat{p}_{\downarrow 0}e^{s}\end{pmatrix}\right] \\ &+ \frac{1}{\omega}\left(1-\hat{p}_{\uparrow 0}-\hat{p}_{\downarrow 0}\right)\ ,\end{aligned} \tag{5}$$

$$\begin{aligned}\Phi_n(\omega,s) &= \frac{1}{\omega}\left[\begin{pmatrix}1-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\uparrow}\\ 1-\hat{p}_{\uparrow\downarrow}-\hat{p}_{\downarrow\downarrow}\end{pmatrix}^T\sum_{n=0}^{\infty}\begin{pmatrix}\hat{p}_{\uparrow\uparrow} & \hat{p}_{\uparrow\downarrow}\\ \hat{p}_{\downarrow\uparrow} & \hat{p}_{\downarrow\downarrow}\end{pmatrix}^n\begin{pmatrix}\hat{p}_{\uparrow 0}\\ \hat{p}_{\downarrow 0}\end{pmatrix}e^{-(n+1)s}\right] \\ &+ \frac{1}{\omega}\left(1-\hat{p}_{\uparrow 0}-\hat{p}_{\downarrow 0}\right)\ ,\end{aligned} \tag{6}$$

where we have introduced the notation

$$\hat{p}_{\uparrow 0}=\hat{p}_\uparrow(\omega,V_0^b,V_0^a) \qquad \hat{p}_{\downarrow 0}=\hat{p}_\downarrow(\omega,V_0^b,V_0^a) \tag{7}$$

$$\hat{p}_{\uparrow\uparrow}=\hat{p}_\uparrow(\omega,V_{sml},V_{lrg}) \qquad \hat{p}_{\downarrow\uparrow}=\hat{p}_\downarrow(\omega,V_{sml},V_{lrg}) \tag{8}$$

$$\hat{p}_{\uparrow\downarrow}=\hat{p}_\uparrow(\omega,V_{lrg},V_{sml}) \qquad \hat{p}_{\downarrow\downarrow}=\hat{p}_\downarrow(\omega,V_{lrg},V_{sml})\,, \tag{9}$$

and (V_0^b,V_0^a) are the coordinates from which the system starts at $t=0$. Equations (5) and (6) are readily solved by diagonalisation of their respective transition matrices, as shown in App. B. In the free case the *bid-ask symmetry* holds (i.e, $p_\uparrow=p_\downarrow$, $p_{\uparrow\uparrow}=p_{\downarrow\downarrow}$ and $p_{\uparrow\downarrow}=p_{\downarrow\uparrow}$). We will show in Sec. 3 that the presence of a extra agent passively waiting for a volume Q breaks this symmetry, leading to the emergence of the aforementioned apparent component of the impact.

As the the behavior of the model changes qualitatively in the cases $\mu=0$ and $\mu>0$, we will discuss the phenomenology of the model in the two cases separately.

2.3 The driftless case

In the driftless case the volumes at the best almost surely drop to zero for any starting condition (one can in fact verify that the sum $\bar{p}_\uparrow + \bar{p}_\downarrow$ is equal to one). The exhaustion of volume can indeed take a very large time, as the volume fluctuations are anomalously large. This leads to a logarithmic sub-diffusion of the price in the long time limit. The assumption $V_{sml} < V_{lrg}$ induces short-time mean reversion – in accordance with well-established empirical evidencece concerning the short-time behavior of the volatility function – partially damping the asymptotic value of the price fluctuations. We will illustrate this behavior in the following section.
The absorption probabilities in the driftless scenario can be calculated explicitly (App. A), and their values may be expressed as the integrals

$$\hat{p}_\uparrow(\omega, V^b, V^a) = \frac{2}{\pi}\int_0^\infty dk \frac{\sin(kV^a)}{k}\frac{1}{1+2\omega/k^2}(1 - e^{-\sqrt{k^2+2\omega}\,V^b}) \tag{10}$$

$$\hat{p}_\downarrow(\omega, V^b, V^a) = \frac{2}{\pi}\int_0^\infty dk \frac{\sin(kV^a)}{k} e^{-\sqrt{k^2+2\omega}\,V^b}\ . \tag{11}$$

The values for $\omega = 0$, expressing the probability of the sign of the next price change, are more conveniently written in terms of elementary functions, and result

$$\bar{p}_\uparrow(V^b, V^a) = \frac{2}{\pi}\arctan\left(\frac{V^b}{V^a}\right) \tag{12}$$

$$\bar{p}_\downarrow(V^b, V^a) = \frac{2}{\pi}\arctan\left(\frac{V^a}{V^b}\right)\ . \tag{13}$$

The mean absorption time by either of the boundary is infinite (equivalently, the derivatives of Eqs. (10) and (11) with respect to ω are divergent at $\omega = 0$). The explicit expression of the absorption probability then needs to be inserted into the generating functions (81) and (82) in order to estimate numerically the momenta of the price change and the hitting times. A small ω expansion (performed in App. A) allows to evaluate the large time behavior of these quantities, which results

$$\langle x_t \rangle = 0 \tag{14}$$

$$\langle x_t^2 \rangle - \langle x_t \rangle^2 = \frac{\chi\, t}{\log t}\left(\frac{\pi}{2V_{sml}V_{lrg}}\right) + o(t/\log t) \tag{15}$$

$$\langle n_t \rangle = \frac{t}{\log t}\left(\frac{\pi}{2V_{sml}V_{lrg}}\right) + o(t/\log t) \tag{16}$$

$$\langle n_t^2 \rangle - \langle n_t \rangle^2 = \frac{t^2}{\log^3(t)}\left(\frac{\pi}{2V_{sml}V_{lrg}}\right)^2 + o(t^2/\log^3 t)\ , \tag{17}$$

where we have introduced the asymmetry parameter $\chi = \hat{p}_{\uparrow\uparrow}/\hat{p}_{\downarrow\uparrow} = \hat{p}_{\downarrow\downarrow}/\hat{p}_{\uparrow\downarrow}$ accounting for the short-time mean-reversion of price. An inspection Eq. (15) reveals that the behavior of price is sub-diffusive, as the variance of price increases less then linearly. This behavior derives from the broad tails of the distribution for the absorption time, whose anomalously large fluctuations cause the mean absorption time to diverge. Fig. 1 summarizes these results by showing the evolution in time of the price change x_t and the hitting number n_t, and by comparing it with the results of numerical simulations.

2.4 The case $\mu > 0$

The phenomenology of a system in which the drift μ is non-zero is to some extent similar to the one discussed above, as the volumes at the best fall to zero almost surely. The most notable difference is that the average absorption time by either of the boundaries is finite. This leads to a diffusive behavior of the price for large times, while at short times the price has a mean-reverting behavior for $V_{sml} < V_{lrg}$ similar to the one observed in the driftless case.

The Laplace transforms of the absorption probabilities for $\mu > 0$ can be expressed as the integrals

$$\hat{p}_{\uparrow}(\omega, V^b, V^a) = \frac{2}{\pi}\int_0^\infty dk \frac{\sin(kV^a)}{k} \frac{e^{\mu V^a}}{1+(2\omega+\mu^2)/k^2}\left[1 - e^{-\left(\sqrt{k^2+2\omega+2\mu^2}-\mu\right)V^b}\right] \quad (18)$$

$$\hat{p}_{\downarrow}(\omega, V^b, V^a) = \frac{2}{\pi}\int_0^\infty dk \frac{\sin(kV^a)}{k} \frac{e^{\mu V^a}}{1+\mu^2/k^2} e^{-\left(\sqrt{k^2+2\omega+2\mu^2}-\mu\right)V^b} . \quad (19)$$

while the probabilities for the sign of the next price change are recovered by setting $\omega = 0$ in above formula.
The average absorption times can be obtained by differentiation of Eqs. (18) and (19), which lead to convergent integrals, as opposed to the driftless scenario discussed above in which the resulting expressions were divergent.
These results have been used in Fig. 1 in order to show the large time asymptotics for the mean values of x_t and n_t and their fluctuations in the case $\mu \neq 0$. In fact the large time behavior can be written explicitly by expanding the generating functions close to the point $\omega = 0$. We find at leading order in time:

$$\langle x_t \rangle = 0 \quad (20)$$

$$\langle x_t^2 \rangle - \langle x_t \rangle^2 = \frac{t}{\langle t_{hit} \rangle}\chi + O(1) \quad (21)$$

$$\langle n_t \rangle = \frac{t}{\langle t_{hit} \rangle} + O(1) \quad (22)$$

$$\langle n_t^2 \rangle - \langle n_t \rangle^2 = \left(\frac{\langle t_{hit}^2 \rangle - \langle t_{hit} \rangle^2}{\langle t_{hit} \rangle^3}\right) t + O(1) , \quad (23)$$

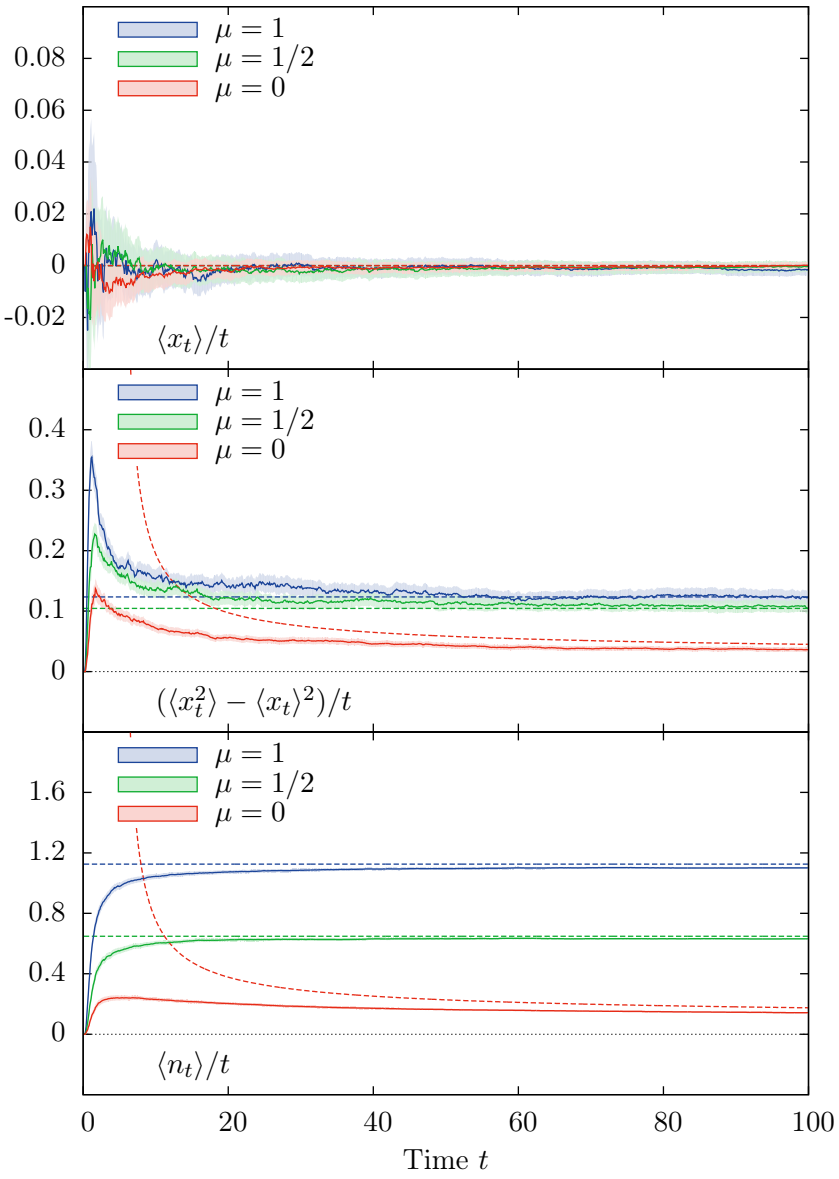

Figure 1: Evolution in time of the average price change $\langle x_t \rangle$, hitting number $\langle n_t \rangle$ and variogram $(\langle x_t^2 \rangle - \langle x_t \rangle^2)/t$ for the free diffusion problem. The solid lines indicate the result of simulations for which we used the set of paramters $V_0 = 2, V_{sml} = 1, V_{lrg} = 3$, while the shaded regions account for two-sigma statistical fluctuations around the mean values obtained by averaging over 1000 realizations. The different curves describe both the driftless case ($\mu = 0$, corresponding to the red line) and the drifted case ($\mu = 1/2$ and $\mu = 1$ are associated respectively with the green and the blue line). The dashed lines indicate the asymptotic predictions displayed in Sec. 2.

where for $q \in \mathbb{N}$ we have defined the momenta $\langle t^q_{hit} \rangle = \langle t^q_{hit,\uparrow} \rangle = \langle t^q_{hit,\downarrow} \rangle$ relative to the average time required to hit any of the boundaries starting from either initial condition. The above expression allows to relate the (asymptotic) *volatility* $\sigma^2 = \lim_{t\to\infty} \langle x_t^2 \rangle / t = \chi / \langle t_{hit} \rangle$ to the microstructural parameters governing the model.

3 One-shot execution

In order to illustrate the notion of apparent impact in our scenario, we consider a setting in which that the trader starts waiting for a volume Q to accumulate on the ask queue from an initial time $t = 0$ in which the system is characterized by coordinates (V_0^b, V_0^a), with $V_0^b < Q$. As we are interested in charcterizing the statistics of the price change x_t and the hitting number n_t as soon the volume at the ask queue reaches a volume Q, we define a stopping time T at which $V^a = Q$ at which the trader executes her order with a single trade.

The main change with respect to the free case discussed above is the fact that the diffusion of the coordinates (V^b, V^a) no longer takes place on the positive orthant $(V^b, V^a) \in (0, \infty) \times (0, \infty)$, but on the semi-infinite strip $(V^b, V^a) \in (0, \infty) \times (0, Q)$. We need correspondingly to define a third type of probability associated with the absorption by the $V^a = Q$ boundary, which we denote as $p_{ex}(t, V^b, V^a)$. Its Laplace transform evolves according to equation (2) with the boundary conditions $\hat{p}_{ex}(\omega, V^b, Q) = 1 - \hat{p}_{\uparrow}(\omega, V^b, Q) = 1 - \hat{p}_{\downarrow}(\omega, V^b, Q) = 1$. Also in this case it is possible to show that $\bar{p}_{\uparrow}(V^b, V^a) + \bar{p}_{\downarrow}(V^b, V^a) + \bar{p}_{ex}(V^b, V^a) = 1$. We remark that the bid-ask symmetry present in the free case is broken by the presence of the $V^a = Q$ boundary, so that an asymmetric evolution of the price is expected in this setting. This is, in a nutshell, the reason why the apparent component of the impact emerges in our model.

One can define the generating functions for the price x_t and the hitting number n_t which are relevant for this problem by supposing that, as soon as the boundary $V^a = Q$ is reached, the price and the hitting number are frozen at their respective values at time $t = T$. We denote the relevant probabilities for this problem as $P_x^{ex}(T, x)$ and $P_n^{ex}(T, x)$, respectively the probability for the price and the hitting number to assume the value x and n when at the stopping time T, and define their corresponding generating functions as

$$\Psi_x(\omega, s) = \sum_{x=-\infty}^{\infty} e^{-xs} \int_0^{\infty} dT\, e^{-\omega T} P_x^{ex}(T, x) \tag{24}$$

$$\Psi_n(\omega, s) = \sum_{n=-\infty}^{\infty} e^{-ns} \int_0^{\infty} dT\, e^{-\omega T} P_n^{ex}(T, n)\ . \tag{25}$$

These functions satisfy the set of equations

$$\Psi_x(\omega,s) = \hat{p}_{ex,0} + \begin{pmatrix} \hat{p}_{ex,\uparrow} \\ \hat{p}_{ex,\downarrow} \end{pmatrix}^T \sum_{n=0}^{\infty} \begin{pmatrix} \hat{p}_{\uparrow\uparrow}e^{-s} & \hat{p}_{\uparrow\downarrow}e^{-s} \\ \hat{p}_{\downarrow\uparrow}e^{s} & \hat{p}_{\downarrow\downarrow}e^{s} \end{pmatrix}^n \begin{pmatrix} \hat{p}_{\uparrow 0}e^{-s} \\ \hat{p}_{\downarrow 0}e^{s} \end{pmatrix} \tag{26}$$

$$\Psi_n(\omega,s) = \hat{p}_{ex,0} + \begin{pmatrix} \hat{p}_{ex,\uparrow} \\ \hat{p}_{ex,\downarrow} \end{pmatrix}^T \sum_{n=0}^{\infty} \begin{pmatrix} \hat{p}_{\uparrow\uparrow} & \hat{p}_{\uparrow\downarrow} \\ \hat{p}_{\downarrow\uparrow} & \hat{p}_{\downarrow\downarrow} \end{pmatrix}^n \begin{pmatrix} \hat{p}_{\uparrow 0} \\ \hat{p}_{\downarrow 0} \end{pmatrix} e^{-(n+1)s}, \tag{27}$$

and as in the previous case can be solved by diagonalization of their respective transition matrices (App. B).

Summarizing, in order to characterize analytically the price x_T and the hitting number n_T at the moment preceding the trade, we need to solve the diffusion problem (1) in the modified geometry of a semi-infintie strip, and successively plug the probabilities of hitting the boundaries into the generating functions $\Psi_x(\omega,s)$ and $\Psi_n(\omega,s)$, whose explicit expressions have been worked out in App. B.

3.1 Expected slippage and average hitting number

The qualitative behavior of the price change x_t at the time of execution $t = T$ is similar in the $\mu = 0$ and the $\mu > 0$ scenarios: In both cases the apparent impact, that we identify with the average price change $\mathcal{I}(Q) = \langle x_T \rangle$ increases – in general non-monotonically – with the executed volume Q from the initial value $\langle x_T \rangle = 0$ corresponding to $Q = V_0$ to an asymptotically constant value reached for $Q \to \infty$. The hitting number $\langle n_T \rangle$ always increases monotonically in Q, growing unbounded from the value $\langle n_T \rangle = 0$ at $Q = V_0$ up to infinity. The precise form of the asymptotic scaling in Q depends crucially on μ: while for $\mu = 0$ the growth of $\langle n_T \rangle$ is asymptotically quadratic in Q, for $\mu > 0$ the growth is exponential, signaling that if the volumes at the best queues are drifted towards zero, one needs to wait an exponentially long number of price changes in order to execute a large order.
A similar behavior is observed for the price fluctuations $\langle x_T \rangle^2 - \langle x_T \rangle^2$, implying that a patient trader who is willing to execute a large order needs to be ready to face a potentially large volatility risk. Notice that at large T one has $\langle n_T \rangle \sim (\langle x_T \rangle^2 - \langle x_T \rangle^2)^{1/2} \gg \langle x_T \rangle$, due to the fact that the asymmetry induced in the diffusion problem by the absorbing boundary $V^a = Q$ is a smaller order effect with respect to the dominating symmetric behavior associated with the $Q \to \infty$ limit. Practically, this implies that the average value of the apparent impact is at most of the order of one tick, while the variance of price is expected to grow linearly in time as in the free case. We also remark that the evolution of the momenta of x_t and n_t display a point of non-analyticity for $Q = V_{lrg}$. This is explained with the change of regime of the hitting probabilities for $V_{lrg} = Q$: we have in fact assumed that $\bar{p}_\uparrow = \bar{p}_\downarrow = 1 - \bar{p}_{ex} = 0$ independently of Q as long as $Q < V_{lrg}$ (in such

case the trader can in fact execute all the volume instantly after a positive price change), while for $Q > V_{lrg}$ we assume the hitting probabilities to satisfy Eq. (2).

Fig. 2 show the result of a numerical simulation of the execution process, comparing it with the semi-analytical results obtained by integrating numerically the expressions for the average slippage and the hitting number.

The next part of this section will be devoted to a more detailed description the qualitative behavior sketched above. In particular we will use the results proved in the appendices in order to extract analytically the leading behavior of x_T and n_T at small and large Q.

By exploting the results of the $1/Q$ expansion showed in App. A, we find in fact that for large executed volumes the apparent impact function tends to

$$\mathcal{I}(Q) = \langle x_T \rangle \xrightarrow[Q\to\infty]{} \begin{cases} \dfrac{1+\chi}{6} & \text{for} \quad \mu = 0 \\ \dfrac{1+\chi}{2} & \text{for} \quad \mu > 0 \end{cases} , \tag{28}$$

which indicates that for a market with short-time mean revertion of price ($\chi < 1$) the mean price change is asymptotically smaller than $\langle x_T \rangle = 1/3$ (for $\mu = 0$) or $\langle x_T \rangle = 1$ (for $\mu > 0$). The asymptotics of the average hitting number can also be extracted by means of a $1/Q$ expansion, and reads

$$\langle n_T \rangle \xrightarrow[Q\to\infty]{} \begin{cases} \dfrac{2Q^2}{\pi V_{sml} V_{lrg}} & \text{for} \quad \mu = 0 \\ 2\sqrt{\pi}\mu^2 \left(\dfrac{Q}{2\sqrt{2}\mu}\right)^{3/2} \dfrac{e^{\mu((1+\sqrt{2})Q - V_{sml} - V_{lrg})}}{V_{sml}\sinh\left(\sqrt{2}\mu V_{lrg}\right) + V_{lrg}\sinh\left(\sqrt{2}\mu V_{sml}\right)} & \text{for} \quad \mu > 0 \end{cases} , \tag{29}$$

so that the total number of price changes asymptotically grows either as Q^2 (in the case $\mu = 0$) or with the exponential law $Q^{3/2}\exp(\mu(1+\sqrt{2})Q)$ (when $\mu > 0$). Finally, the price fluctuations obey the expected asymptotic scaling

$$\langle x_T^2 \rangle - \langle x_T^2 \rangle \xrightarrow[Q\to\infty]{} \chi \langle n_T \rangle , \tag{30}$$

which indicates that at leading order the asymptotic volatility of the free problem is not affected by the conditioning effect induced by the execution.

The small Q asymptotics of the process (provided $V_{sml} < V_0 < V_{lrg}$) leads to a linear behavior in the vicinity of the point $Q = V_0$ for all the observables. In particular can use

the relation $\hat{p}_{\uparrow\uparrow} = \hat{p}_{\downarrow\uparrow} = 0$ valid for $Q < V_{lrg}$ in order to show that

$$\langle x_T \rangle \xrightarrow[Q \to V_0]{} (Q - V_0) \left(\delta_\uparrow - \delta_\downarrow \frac{1 - \hat{p}_{\uparrow\downarrow}}{1 - \hat{p}_{\downarrow\downarrow}} \right) \tag{31}$$

$$\langle n_T \rangle \xrightarrow[Q \to V_0]{} (Q - V_0) \left(\delta_\uparrow + \delta_\downarrow \frac{1 + \hat{p}_{\uparrow\downarrow}}{1 - \hat{p}_{\downarrow\downarrow}} \right) \tag{32}$$

$$\langle x_T^2 \rangle - \langle x_T^2 \rangle \xrightarrow[Q \to V_0]{} (Q - V_0) \left(\delta_\uparrow + \delta_\downarrow \frac{(1 + \hat{p}_{\downarrow\downarrow})(1 - \hat{p}_{\uparrow\downarrow})}{(1 - \hat{p}_{\downarrow\downarrow})^2} \right) , \tag{33}$$

where we have defined

$$\delta_\uparrow = \begin{cases} \dfrac{1}{V_0} \left(\dfrac{e^\pi - 1}{e^\pi + 1} \right) & \text{for} \quad \mu = 0 \\ \dfrac{2}{\pi} \displaystyle\int_0^\infty dp \frac{\beta(p) p \sin(p V_0)}{p^2 + \mu^2} \left(\frac{e^{2\mu V_0}}{\sinh(\beta(p) V_0)} \right) & \text{for} \quad \mu > 0 \end{cases} \tag{34}$$

$$\delta_\downarrow = \begin{cases} \dfrac{1}{V_0} \left(\dfrac{2}{\sinh \pi} \right) & \text{for} \quad \mu = 0 \\ \mu + \dfrac{2}{\pi} \displaystyle\int_0^\infty dp \frac{\beta(p) p \sin(p V_0)}{p^2 + \mu^2} \left(\frac{\cosh(\beta(p) V_0) - e^{\mu V_0}}{\sinh(\beta(p) V_0)} \right) e^{\mu V_0} & \text{for} \quad \mu > 0 \end{cases} \tag{35}$$

with $\beta(p) = \sqrt{p^2 + 2\mu^2}$. Hence, the small volume behavior of the apparent impact is analytical close to the starting volume V_0, in contrast with the ordinary price impact, which for small executed volumes grows roughly as Q^η, being η an exponent between 0.6 and 0.8 [3, 4, 5, 6, 7, 8, 9, 10].

3.2 Average execution time

The execution time T follows a statistics similar to the one of the hitting number n_T, as showed in Fig. 2 where we display the evolution of the execution time as a function of the executed volume V. As for the average price change and the hitting number, we have compared with the simulations the semi-analytical result obtained by integrating numerically Eqs. (57), (58) and (59) for the hitting probabilities and inserting them into the Eqs. (97) and (98) for the momenta of T. We find that the average execution time increases monotonically from $\langle T \rangle = 0$ for $Q = V_0$ up to an asymptotic regime whose scaling depends on the value of μ.

In particular by performing a $1/Q$ expansion of the execution time, we find the asymptotic scaling

$$\langle T \rangle \xrightarrow[Q \to \infty]{} \langle n_T \rangle \langle t_{hit} \rangle , \tag{36}$$

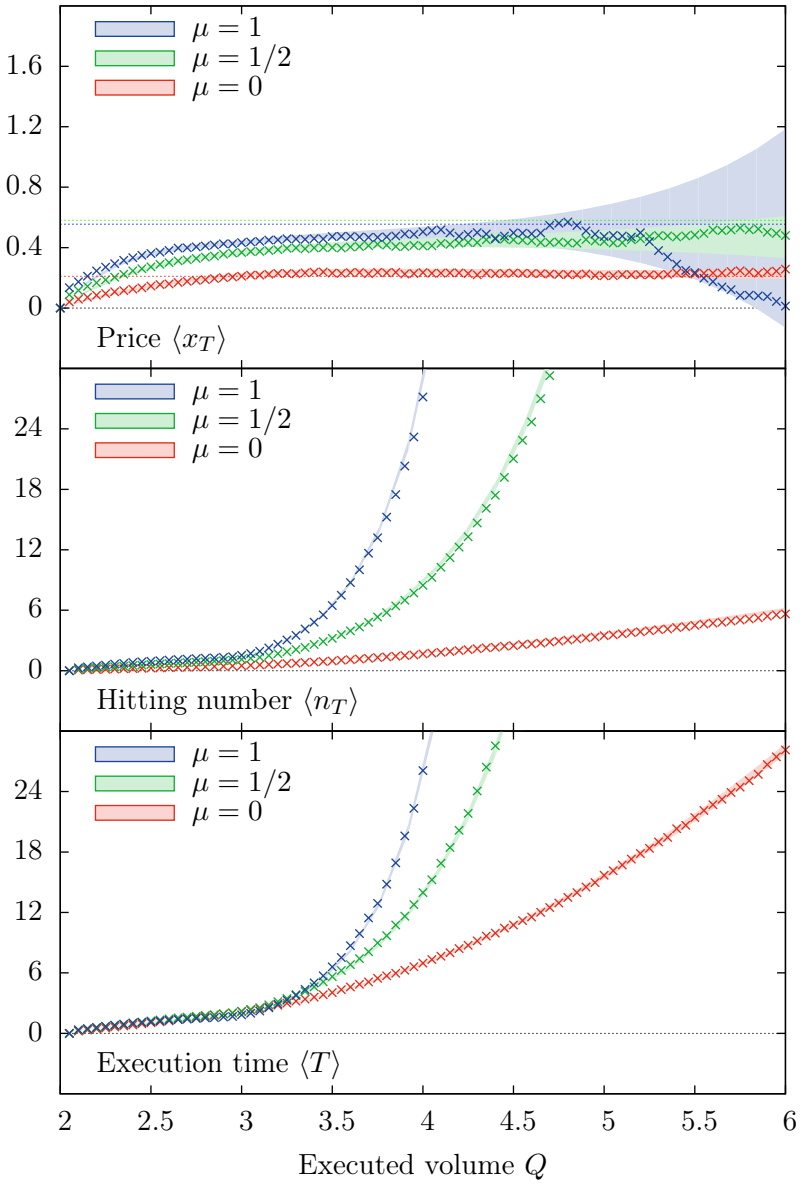

Figure 2: Statistics for the one-shot execution problem. We represent the averages of the price change $\langle x_t \rangle$, hitting number $\langle n_t \rangle$ and execution time $\langle T \rangle$ as functions of the executed volume Q, for different values of the drift μ. We have simulated 8000 realization of a process defined by the set of parameters $V_0 = 2, V_{sml} = 1$ and $V_{lrg} = 3$. The shaded regions correspond to the theoretical predictions, accounting for two-sigma regions, while the crosses are the results of numerical simulations.

where we have used the results of App. A.4 in order to obtain the scaling of the hitting times

$$\langle t_{hit,\alpha}\rangle \xrightarrow[Q\to\infty]{} \begin{cases} \dfrac{4V^bV^a}{\pi}\log(Q) & \text{for} \quad \mu = 0 \\ \dfrac{V^b}{\mu} - \dfrac{4}{\pi}\displaystyle\int_0^\infty dp \frac{p\sin(pV^b)}{(p^2+\mu^2)^2} e^{-(\sqrt{p^2+2\mu^2}-\mu)V^a+\mu V^b} & \text{for} \quad \mu > 0 \end{cases} , \tag{37}$$

with $(V^b, V^a) = (V_0, V_0)$ for $\alpha = 0$ and $(V^b, V^a) = (V_{lrg}, V_{sml})$ for $\alpha \in \{\uparrow, \downarrow\}$. In particular, the integral above is invariant under the exchange $V^b \leftrightarrow V^a$, so that

$$\langle t_{hit,\uparrow}\rangle, \langle t_{hit,\downarrow}\rangle \xrightarrow[Q\to\infty]{} \langle t_{hit}\rangle \tag{38}$$

We remark that the $Q \to \infty$ limit of $\langle t_{hit}\rangle$ for $\mu > 0$ is independent of Q, so that the asymptotic scaling of the average execution time is dominated by the exponential divergence of the hitting number. Also notice that, even though the time required to hit the $V^a = Q$ boundary is also diverging at a speed proportional to Q, its contribution is subleading: The barrier $V^a = Q$ needs to be hit just once, as opposed to the $V^a, V^b = 0$ boundaries which are hit a large number of times.
The phenomenology of the case $\mu = 0$ is extremely different: while the hitting number scales like Q^2, the average $\langle t_{hit,\alpha}\rangle$ is proportional to $\log Q$, implying that the average execution time scales as $Q^2 \log Q$. Indeed, in both cases the increase of the expectation time with the volume justifies the large volatility risk which a trader faces in the large Q regime. In limit of small executed volumes, the average execution time is dominated by

$$\langle T\rangle \xrightarrow[Q\to V_0]{} \langle t_{hit,0}\rangle \, , \tag{39}$$

where $t_{hit,0}$ can be expanded around $Q = V_0$ by using the results of App. A.4, allowing to express the average execution time as an increasing function of the difference $Q - V_0$.

4 Conclusions

This work shows that even in a risk-neutral setting and in absence of a price trend, the market impact of an order cannot be completely avoided by delaying the transaction: even if a one-shot strategy is adopted, thus completely removing the traditional price impact from the cost of the execution strategy, such a cost reappears due to an asymmetric conditioning effect induced by the execution strategy. Moreover, the volatility risk associated with such a passive strategy can be (exponentially) large if the volume to execute is too ingent. This indicates that the access to liquidity has a price which is to some extent unavoidable. Within our analysis we have characterized such cost, its fluctuations, and provided the

probability of complely executing a one-shot order in a stylized market model, relating these quantities to microstructural parameters. Even though we have expressed our ideas in a rather ad-hoc setting, we want to underline that even in a more general scenario these qualitative predictions are expected to be observed: any type of correlation among price changes and liquidity may cause asymmetric effects which can induce an apparent impact effect for passive traders.

This model also constitues a concrete example of a setting in which past price moves indirectly cause future trades (in line with the empirical results shown in Refs. [14, 18, 19]), a perspective clashing with the paradigm according to which only past trades are responsable of future price changes.

As a future research perspective, we look forward for the possibility of validating the qualitative predictions of this model through an inspection of proprietary data describing the execution of one-shot orders.

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A Diffusion in a semi-infinite strip

In this appendix we will be interested in computing the probabilities $p_{ex}(t, V^b, V^a)$, $p_\uparrow(t, V^b, V^a)$, $p_\downarrow(t, V^b, V^a)$ introduced in Sec. 2, denoting respectively the probability of executing a trade of volume Q, emptying the ask queue and emptying the bid queue in a time t starting from bid and ask volumes respectively equal to V^b and V^a. We will focus exclusively on the case in which a trader is present, reminding that free regime discussed in Sec. 2.2 can be recovered by taking the limit $Q \to \infty$, in which the execution probability $p_{ex}(t, V^b, V^a)$ becomes zero. Finally, we remind that this problem corresponds to the one of finding the first-passage probabilities of a diffusing particle through any of the three boundaries of the semi-infinite strip $(0, \infty) \times (0, Q)$.

Such first-passage probabilities $\hat{p}_\alpha$ satisfy the set of the independent equations

$$\frac{1}{2}\boldsymbol{\nabla}^2 \hat{p}_\alpha(\omega, \boldsymbol{V}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\mu})\, \hat{p}_\alpha(\omega, \boldsymbol{V}) \cdot = \omega \hat{p}_\alpha(\omega, \boldsymbol{V}) \tag{40}$$

where $\boldsymbol{\nabla} = (\partial_x, \partial_y)$, $\boldsymbol{\mu} = (\mu, \mu)$, $\boldsymbol{V} = (V^b, V^a)$ and $\alpha \in \{ex, \uparrow, \downarrow\}$. These equations differ for the boundary conditions as specified in Sec. 2. Specifically, $p_\alpha(\omega, \boldsymbol{V})$ is equal to zero on all the boundaries of the region $(0, \infty) \times (0, Q)$ except for the one corresponding to the event labeled by α (i.e., $\hat{p}_{ex} = 1$ for $V^a = Q$, $\hat{p}_\uparrow = 1$ for $V^a = 0$, $\hat{p}_\downarrow = 1$ for $V^b = 0$). The choice $\omega = 0$ is associated to the probability of hitting a specific boundary in any point in time, in whose case the problem reduces to a simpler problem, for which we will be able to provide explicit solutions.
In order to solve this problem with elementary methods, we define the functions $\phi_\alpha(\omega, \boldsymbol{V}) = \exp(-\mu(V^a + V^b))\hat{p}_\alpha(\omega, \boldsymbol{V})$, which satisfy the simpler set of Helmholtz equations

$$\boldsymbol{\nabla}^2 \phi_\alpha(\omega, \boldsymbol{V}) = 2(\omega + \mu^2)\phi_\alpha \tag{41}$$

subject to the modified boundary conditions

$$\begin{aligned} \phi_{ex}(\omega, V^b, Q) &= e^{-\mu(V^a+Q)} && (42)\\ \phi_\uparrow(\omega, V^b, 0) &= e^{-\mu V^b} && (43)\\ \phi_\downarrow(\omega, 0, V^a) &= e^{-\mu V^a} \ . && (44) \end{aligned}$$

The problem of determining ϕ_α is more conveniently handled by treating the semi-infinite strip $(0, \infty) \times (0, Q)$ as the $P \to \infty$ limit of the rectangle $(0, P) \times (0, Q)$, in whose geometry the general solution of the problem (40) can be written as follows:

$$\begin{aligned} \phi_{ex}(\omega, \boldsymbol{V}) &= -\int_0^P d\zeta \left[\frac{\partial}{\partial \eta} G(\boldsymbol{V}, \zeta, \eta)\right]_{\eta=Q} \phi_{ex}(\omega, \zeta, Q) && (45)\\ \phi_\uparrow(\omega, \boldsymbol{V}) &= \int_0^P d\zeta \left[\frac{\partial}{\partial \eta} G(\boldsymbol{V}, \zeta, \eta)\right]_{\eta=0} \phi_\uparrow(\omega, \zeta, 0) && (46)\\ \phi_\downarrow(\omega, \boldsymbol{V}) &= \int_0^Q d\eta \left[\frac{\partial}{\partial \zeta} G(\boldsymbol{V}, \zeta, \eta)\right]_{\zeta=0} \phi_\downarrow(\omega, 0, \eta) && (47) \end{aligned}$$

where the Green function $G(\boldsymbol{V}, \zeta, \eta)$ admits the two forms

$$\begin{aligned} G(\boldsymbol{V}, \zeta, \eta) &= \frac{2}{P} \sum_{n=1}^{\infty} \frac{\sin(p_n V^b) \sin(p_n \zeta)}{\beta_n \sinh(\beta_n Q)} H_n(V^a, \eta) && (48)\\ &= \frac{2}{Q} \sum_{n=1}^{\infty} \frac{\sin(q_n V^b) \sin(q_n \zeta)}{\mu_n \sinh(\mu_n a)} Q_n(V^b, \zeta) \ , && (49) \end{aligned}$$

and where we have defined

$$\begin{aligned} p_n = \frac{\pi n}{P} \quad , \quad \beta_n = \sqrt{p_n^2 + 2\omega + 2\mu^2} \ , && (50)\\ q_n = \frac{\pi n}{Q} \quad , \quad \mu_n = \sqrt{q_n^2 + 2\omega + 2\mu^2} \ . && (51) \end{aligned}$$

Finally,

$$H_n(V^a,\eta) = \begin{cases} \sinh(\beta_n\eta)\sinh(\beta_n(Q-V^a)) & \text{if} \quad V^a>\eta \\ \sinh(\beta_n V^a)\sinh(\beta_n(Q-\eta)) & \text{if} \quad \eta>V^a \end{cases} \tag{52}$$

$$Q_n(V^b,\zeta) = \begin{cases} \sinh(\mu_n\zeta)\sinh(\mu_n(P-V^b)) & \text{if} \quad V^b>\zeta \\ \sinh(\mu_n V^b)\sinh(\mu_n(P-\zeta)) & \text{if} \quad \zeta>V^b \end{cases} \tag{53}$$

The explicit form of the solution for the ϕ_α can be written after performing the above integrals and taking the $P\to\infty$ limit. A further transformation back to the original functions $\hat{p}_\alpha$ finally allows to express the solution as the series

$$\hat{p}_{ex}(\omega,\boldsymbol{V}) = \frac{2}{\pi}\sum_n \frac{\sin(q_nV^a)}{n}\frac{(-1)^{n+1}e^{-\mu(Q-V^a)}}{1+(2\omega+\mu^2)Q^2/\pi^2n^2}(1-e^{-(\mu_n-\mu)\,V^b}) \tag{54}$$

$$\hat{p}_{\uparrow}(\omega,\boldsymbol{V}) = \frac{2}{\pi}\sum_n \frac{\sin(q_nV^a)}{n}\frac{e^{\mu V^a}}{1+(2\omega+\mu^2)Q^2/\pi^2n^2}(1-e^{-(\mu_n-\mu)\,V^b}) \tag{55}$$

$$\hat{p}_{\downarrow}(\omega,\boldsymbol{V}) = \frac{2}{\pi}\sum_n \frac{\sin(q_nV^a)}{n}\frac{e^{\mu V^a}}{1+\mu^2Q^2/\pi^2n^2}(1-(-1)^ne^{-\mu Q})e^{-(\mu_n-\mu)\,V^b}\,, \tag{56}$$

or equivalently as the integral

$$\hat{p}_{ex}(\omega,\boldsymbol{V}) = \frac{2}{\pi}\int_0^\infty dp\frac{p}{p^2+\mu^2}\frac{\sin(pV^b)\sinh(\beta(p)V^a)}{\sinh(\beta(p)Q)}e^{\mu V^b-\mu(Q-V^a)} \tag{57}$$

$$\hat{p}_{\uparrow}(\omega,\boldsymbol{V}) = \frac{2}{\pi}\int_0^\infty dp\frac{p}{p^2+\mu^2}\frac{\sin(pV^b)\sinh(\beta(p)(Q-V^a))}{\sinh(\beta(p)Q)}e^{\mu V^b+\mu V^a} \tag{58}$$

$$\begin{aligned}\hat{p}_{\downarrow}(\omega,\boldsymbol{V}) &= \frac{2}{\pi}\int_0^\infty dp\frac{p}{p^2+\mu^2+2\omega}\left(\frac{\sin(pV^b)}{\sinh(\beta(p)Q)}\right)e^{\mu V^b} \\ &\times \left(\sinh(\beta(p)Q)-e^{-\mu(Q-V^a)}\sinh(\beta(p)V^a)-e^{\mu V^a}\sinh(\beta(p)(Q-V^a))\right).\end{aligned} \tag{59}$$

A.1 Hitting probabilities in the free, driftless case

In the special case $\mu=\omega=0$ it is possible to sum analytically the above series so to express the result in term of elementary functions. It is sufficient to remind that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{zn}}{n} = \log(1+e^{zn}) \tag{60}$$

and to expand the trigonometric functions in term of exponentials in order to reduce the above series to sums of logarithms. Then, by exploiting the identities

$$\frac{i}{2}\left(\log(1\pm e^{i\pi V^a/Q}e^{-\pi V^b/Q})-\log(1\pm e^{-i\pi V^a/Q}e^{-\pi V^b/Q})\right) = \arctan\left(\frac{\sin(\pi V^a/Q)}{e^{\pi V^b/Q}\pm\cos(\pi V^a/Q)}\right) \tag{61}$$

and

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\pi V^a/Q)}{n} = 1 - \frac{V^a}{Q} \tag{62}$$

one can obtain the following expression for the hitting probabilities:

$$\bar{p}_{ex}(\boldsymbol{V}) = \frac{V^a}{Q} - \frac{2}{\pi}\arctan\left(\frac{\sin(\pi V^a/Q)}{e^{\pi V^b/Q}+\cos(\pi V^a/Q)}\right) \tag{63}$$

$$\bar{p}_{\uparrow}(\boldsymbol{V}) = 1 - \frac{V^a}{Q} - \frac{2}{\pi}\arctan\left(\frac{\sin(\pi V^a/Q)}{e^{\pi V^b/Q}-\cos(\pi V^a/Q)}\right) \tag{64}$$

$$\begin{aligned}\bar{p}_{\downarrow}(\boldsymbol{V}) &= \frac{2}{\pi}\arctan\left(\frac{\sin(\pi V^a/Q)}{e^{\pi V^b/Q}-\cos(\pi V^a/Q)}\right) \\ &+ \frac{2}{\pi}\arctan\left(\frac{\sin(\pi V^a/Q)}{e^{\pi V^b/Q}+\cos(\pi V^a/Q)}\right) .\end{aligned} \tag{65}$$

Interestingly enough, the dependence on ω of the absorption probabilities close to the point $\omega = 0$ is regular enough to lead to finite mean hitting times for any of the boundaries, as opposed to the $\mu = 0$ case of the free diffusion problem, in which all these quantities were divergent.

A.2 Small ω expansion in the free case

The asymptotic analysis of Eqs. (57), (58) and (59) is particularly interesting, and has been used in order to determine the large time behavior of the free diffusion problem (corresponding to the $Q = \infty$, $\omega \to 0$ regime) in Sec. 2. Such behavior depends crucially upon μ: for $\mu > 0$ the hitting probabilities are analytic around $\omega = 0$, and thus can be expanded as

$$\hat{p}_\alpha(\omega, \boldsymbol{V}) = \bar{p}_\alpha(\boldsymbol{V})\left(1 - \omega\langle t_\alpha(\boldsymbol{V})\rangle + \frac{1}{2}\omega^2\langle t_\alpha^2(\boldsymbol{V})\rangle + O(\omega^3)\right) \tag{66}$$

where the symbols $\langle t_\alpha(\boldsymbol{V})\rangle$ and $\langle t_\alpha^2(\boldsymbol{V})\rangle$ refer to the average hitting times conditional to the initial condition $\boldsymbol{V} = (V^b, V^a)$ and a first absorption though a boundary of type α. An integral representation these terms can be obtained staightforwardly by differentiation with respect to ω under the integral sign.
For $\mu = 0$ the hitting probabilities are non-analytic around $\omega = 0$, and admit in particular the expansion

$$\hat{p}_\alpha(\omega, \boldsymbol{V}) = \sum_n a_\alpha^{(n)}(\boldsymbol{V})\omega^n + \log(\omega)\sum_n b_\alpha^{(n)}(\boldsymbol{V})\omega^n , \tag{67}$$

with $a_\alpha^{(0)} = \bar{p}_\alpha(0)$ and $b_\alpha^{(0)} = 0$. The subleading terms in the expansion are

$$a^{(1)}(\boldsymbol{V}) = \frac{V^b V^a}{\pi}\left((2\gamma-3) + 2\frac{V^a}{V^b}\arctan\left(\frac{V^b}{V^a}\right) + \log\left(\frac{(V^b)^2+(V^a)^2}{2}\right)\right) \quad (68)$$

$$b^{(1)}(\boldsymbol{V}) = \frac{V^b V^a}{\pi}\,, \quad (69)$$

where γ denotes the Euler-Mascheroni constant. These terms can be obtained by using the series representation for the absorption probabilities

$$\hat{p}_\uparrow(\omega, \boldsymbol{V}) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{V^b}{V^a}\right)^{2n+1}\int_0^\infty z^{2n}e^{-\sqrt{z^2+2\omega(V^a)^2}}\,. \quad (70)$$

The correction terms (68) and (69) are finally recovered after exploiting the identity $\int_0^\infty z^{2n}e^{-\sqrt{z^2+\alpha^2}} = (2n-1)!!\alpha^{n+1}K_{n+1}(\alpha)$, with $K_{n+1}(\alpha)$ denoting the modified Bessel function of the second type of order $n+1$ calculated in α. Notice in particular that the divergence of the Bessel function close to $\alpha = 0$ is canceled by the factor α^{n+1}.

A.3 Hitting probabilities in the large Q regime

The regime of large Q and $\omega = 0$ has been analyzed in order to find the asymptotic scaling of the price change x_t and of the hitting number n_t in the execution problem illustrated in Sec. 3. In the case $\mu > 0$, Eqs. (57), (58) and (59) can be expanded as

$$\bar{p}_{ex}(\boldsymbol{V}, Q) \xrightarrow[Q\to\infty]{} \bar{p}_{ex}(\boldsymbol{V}, \infty) + \frac{x}{\sqrt{\pi}\mu^2}\left(\frac{2\sqrt{2}\mu}{Q}\right)^{3/2}\sinh\left(\sqrt{2}\mu y\right)e^{\mu(x+y-(1+\sqrt{2})Q)} \quad (71)$$

$$\bar{p}_\uparrow(\boldsymbol{V}, Q) \xrightarrow[Q\to\infty]{} \bar{p}_\uparrow(\boldsymbol{V}, \infty) - \frac{x}{\sqrt{\pi}\mu^2}\left(\frac{\sqrt{2}\mu}{Q}\right)^{3/2}\sinh\left(\sqrt{2}\mu y\right)e^{\mu(x+y-2\sqrt{2}Q)} \quad (72)$$

$$\bar{p}_\downarrow(\boldsymbol{V}, Q) \xrightarrow[Q\to\infty]{} \bar{p}_\downarrow(\boldsymbol{V}, \infty) - \frac{x}{\sqrt{\pi}\mu^2}\left(\frac{2\sqrt{2}\mu}{Q}\right)^{3/2}\sinh\left(\sqrt{2}\mu y\right)e^{\mu(x+y-(1+\sqrt{2})Q)}, \quad (73)$$

while for $\mu = 0$ one can simply differentiate Eqs. (63), (64) and (65) and obtain

$$\bar{p}_{ex}(\boldsymbol{V}, Q) \xrightarrow[Q\to\infty]{} \bar{p}_{ex}(\boldsymbol{V}, \infty) + \frac{\pi V^a V^b}{2Q^2} \quad (74)$$

$$\bar{p}_\uparrow(\boldsymbol{V}, Q) \xrightarrow[Q\to\infty]{} \bar{p}_\uparrow(\boldsymbol{V}, \infty) - \frac{\pi V^a V^b}{6Q^2} \quad (75)$$

$$\bar{p}_\downarrow(\boldsymbol{V}, Q) \xrightarrow[Q\to\infty]{} \bar{p}_\downarrow(\boldsymbol{V}, \infty) - \frac{\pi V^a V^b}{3Q^2}\,. \quad (76)$$

Above expressions can be inserted in the formulae for $\langle x_T\rangle$ and $\langle n_T\rangle$ calculated in App. B so to estimate their large Q scaling.

A.4 Total hitting time

The total hitting time $\langle t_{hit}(\boldsymbol{V})\rangle$ is defined in App.B as

$$\langle t_{hit}(\boldsymbol{V})\rangle = -\frac{d}{d\omega}(\hat{p}_{ex}+\hat{p}_{\uparrow}+\hat{p}_{\downarrow})\Big|_{\omega=0} . \tag{77}$$

We want to show that it can be conveniently computed by first summing Eqs. (57), (58) and (59), and successively performing the derivative with respect to ω, in order to exploit the cancellations among the summands. We obtain in particular

$$\begin{aligned}\langle t_{hit}(\boldsymbol{V})\rangle &= \frac{4}{\pi}\int_0^\infty dp \frac{p}{(p^2+\mu^2)^2}\sin(pV^b) \\ &\times \left(e^{\mu V^b} - \frac{\sinh(\beta(p)V^a)}{\sinh(\beta(p)Q)}e^{\mu(V^b+V^a-Q)} - \frac{\sinh(\beta(p)(Q-V^a))}{\sinh(\beta(p)Q)}e^{\mu(V^b+V^a)}\right) ,\end{aligned} \tag{78}$$

which in the driftless case $\mu = 0$ reduces to

$$\langle t_{hit}(\boldsymbol{V})\rangle = \frac{4Q^2}{\pi}\int_0^\infty dp \frac{\sin(pV^b/Q)}{p^3}\left[1-\cosh(pV^a/Q)+\tanh(p/2)\sinh(pV^a/Q)\right] . \tag{79}$$

B Solution of a Markovian market model

B.1 Generating functions for the free case

In Sec. 2 we have considered the problem of calculating the quantities $P_x(t,x)$ and $P_n(t,n)$ describing the probability for the price and the hitting number to take values respectively of x and n at time t. The former one can be written as

$$\begin{aligned}P_x(t,x) &= \sum_{n=0}^{\infty}\sum_{\alpha_1}\cdots\sum_{\alpha_{n+1}}\int_0^\infty \left(\prod_{i=0}^{n+1}dt_i\right)\delta\left(t-\sum_{i=0}^{n+1}t_i\right)\delta\left(x-\sum_{i=1}^{n+1}X(\alpha_i)\right) \\ &\times \left(\int_0^{t_{n+1}}dt'\,(1-p_{\uparrow\alpha_{n+1}}(t')-p_{\downarrow\alpha_{n+1}}(t'))\right)\left(\prod_{i=1}^{n}p_{\alpha_{i+1}\alpha_i}(t_i)\right)p_{\alpha_1 0} \\ &+ \int_0^t dt_0\,(1-p_{\uparrow 0}(t)-p_{\downarrow 0}(t))\,\delta(x) ,\end{aligned} \tag{80}$$

where $X(\uparrow) = -X(\downarrow) = 1$. Each of the terms in above equation can be represented schematically as in Fig. 3, in which we show that the n-th term of the sum comprises $n+1$ absorptions from the starting time to the last one t, while the succession of indices $\{\alpha_i\}_{i=1}^{n+1} \in \{\uparrow,\downarrow\}^{n+1}$ labels the type of boundary hit during absorption number i. The

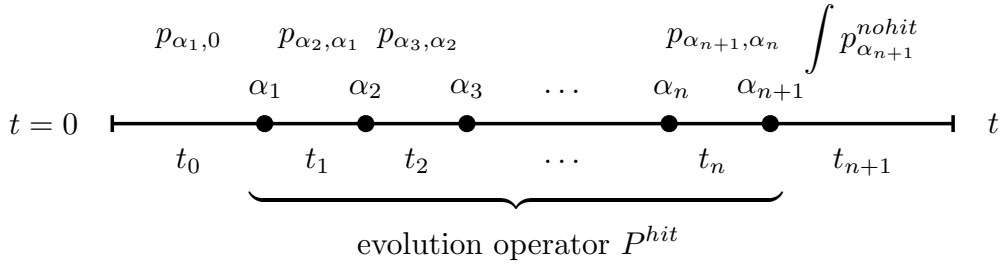

Figure 3: Schematic representation of a generic term of the evolution equation (80) for the problem of the free evolution of the price.

probability $P_n(n,t)$ can be obtained from the previous expression after the substitution $X(\alpha_i) \to 1$. The generating function associated to the probability distribution (80) is readily found by integrating in the t and x coordinates, and can be expressed compactly in matrix form as in Eq. (3). The generating function for the hitting number is found analogously, leading to Eq. (4). We can rewrite Eqs. (3) and (4) more succintly as

$$\Phi_x(\omega,s) = \frac{1}{\omega}\left[\hat{p}_0^{nohit} + (\hat{\boldsymbol{p}}^{nohit})^T \sum_{n=0}^{\infty} (\hat{P}^{hit}K_x)^n K_x \hat{\boldsymbol{p}}_0^{hit}\right] \tag{81}$$

$$\Phi_n(\omega,s) = \frac{1}{\omega}\left[\hat{p}_0^{nohit} + (\hat{\boldsymbol{p}}^{nohit})^T \sum_{n=0}^{\infty} (\hat{P}^{hit}K_n)^n K_n \hat{\boldsymbol{p}}_0^{hit}\right] , \tag{82}$$

where we have defined the matrices

$$\hat{P}^{hit} = \begin{pmatrix} \hat{p}_{\uparrow\uparrow} & \hat{p}_{\uparrow\downarrow} \\ \hat{p}_{\downarrow\uparrow} & \hat{p}_{\downarrow\downarrow} \end{pmatrix} \quad K_x = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \quad K_n = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-s} \end{pmatrix} , \tag{83}$$

together with the vectors

$$\hat{\boldsymbol{p}}_0^{\,hit} = \begin{pmatrix} \hat{p}_{\uparrow 0} \\ \hat{p}_{\downarrow 0} \end{pmatrix} \quad \hat{\boldsymbol{p}}^{\,nohit} = \begin{pmatrix} 1-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\uparrow} \\ 1-\hat{p}_{\uparrow\downarrow}-\hat{p}_{\downarrow\downarrow} \end{pmatrix} \tag{84}$$

and the scalar $\hat{p}_0^{nohit} = 1-\hat{p}_{\uparrow 0}-\hat{p}_{\downarrow 0}$. Notice the factor ω^{-1} appearing in Eqs. (81) and (82), which arises from the integration by parts of the last time t_{n+1}. The infinite sum appearing in the expression for the generating functions can be performed explicitly by passing to principal component. One obtains

$$\Phi_x(\omega,s) = \frac{1}{\omega}\left[\hat{p}_0^{nohit} + (\hat{\boldsymbol{p}}^{nohit})^T U_x (\mathbb{I}-\Lambda_x)^{-1} U_x^{-1}\, \hat{\boldsymbol{p}}_0^{hit}\right] \tag{85}$$

$$\Phi_n(\omega,s) = \frac{1}{\omega}\left[\hat{p}_0^{nohit} + (\hat{\boldsymbol{p}}^{nohit})^T U_n (\mathbb{I}-\Lambda_n)^{-1} U_n^{-1}\, \hat{\boldsymbol{p}}_0^{hit}\right] , \tag{86}$$

where $\mathbb{I}$ denotes the identity matrix, and where we have introduced the eigenvalue decompositions $\hat{P}^{hit}K_x = U_x\Lambda_x U_x^{-1}$ and $\hat{P}^{hit}K_n = U_n\Lambda_n U_n^{-1}$. The matrix product appearing in the first of above equations results

$$U_x(\mathbb{I}-\Lambda_x)^{-1}U_x^{-1} = \frac{1}{1-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow}-e^{-s}\hat{p}_{\uparrow\uparrow}-e^{s}\hat{p}_{\downarrow\downarrow}}\begin{pmatrix} 1-e^{s}\hat{p}_{\downarrow\downarrow} & e^{-s}\hat{p}_{\uparrow\downarrow} \\ e^{s}\hat{p}_{\downarrow\uparrow} & 1-e^{-s}\hat{p}_{\uparrow\uparrow}\end{pmatrix}, \tag{87}$$

while the one relative to the second equation can be obtained through the substitution $\hat{p}_{\downarrow\alpha} \to e^{-2s}\hat{p}_{\downarrow\alpha}$.

The differentiation of the generating functions with respect to s leads finally to the Laplace transforms of mean price change langle $\langle\hat{x}_\omega\rangle$ and hitting number langle $\langle\hat{n}_\omega\rangle$, together with their squares $\langle\hat{x}_\omega^2\rangle$ and $\langle\hat{n}_\omega^2\rangle$:

$$\langle\hat{x}_\omega\rangle = \frac{1}{\omega}\left(\frac{\hat{p}_{\uparrow 0}(1-\hat{p}_{\downarrow\downarrow}-\hat{p}_{\downarrow\uparrow})-\hat{p}_{\downarrow 0}(1-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\uparrow\downarrow})}{1-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\downarrow}}\right) \tag{88}$$

$$\begin{aligned}\langle\hat{x}_\omega^2\rangle = {} & \frac{\hat{p}_{\uparrow 0}}{\omega}\left[\frac{(1-2\hat{p}_{\downarrow\downarrow}+\hat{p}_{\downarrow\downarrow}^2-\hat{p}_{\downarrow\uparrow}+3\hat{p}_{\downarrow\downarrow}\hat{p}_{\downarrow\uparrow}-\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\downarrow}-\hat{p}_{\downarrow\downarrow}\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\downarrow})}{(1-\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow})^2}\right. \\ & + \left.\frac{\hat{p}_{\downarrow\uparrow}^2\hat{p}_{\uparrow\downarrow}+\hat{p}_{\uparrow\uparrow}-2\hat{p}_{\downarrow\downarrow}\hat{p}_{\uparrow\uparrow}+\hat{p}_{\downarrow\downarrow}^2\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\downarrow}\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\uparrow}}{(1-\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow})^2}\right] \\ & + (\uparrow \text{ exch. } \downarrow)\end{aligned} \tag{89}$$

$$\langle\hat{n}_\omega\rangle = \frac{1}{\omega}\left(\frac{\hat{p}_{\uparrow 0}(1-\hat{p}_{\uparrow\uparrow}+\hat{p}_{\downarrow\uparrow})+\hat{p}_{\downarrow 0}(1-\hat{p}_{\uparrow\uparrow}+\hat{p}_{\uparrow\downarrow})}{1-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\downarrow}}\right) \tag{90}$$

$$\begin{aligned}\langle\hat{n}_\omega^2\rangle = {} & \frac{\hat{p}_{\uparrow}}{\omega}\left[\frac{(1-2\hat{p}_{\downarrow\downarrow}+\hat{p}_{\downarrow\downarrow}^2+3\hat{p}_{\downarrow\uparrow}-\hat{p}_{\downarrow\downarrow}\hat{p}_{\downarrow\uparrow}+3\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\downarrow}-\hat{p}_{\downarrow\downarrow}\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\downarrow})}{(1-\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow})^2}\right. \\ & + \left.\frac{\hat{p}_{\downarrow\uparrow}^2\hat{p}_{\uparrow\downarrow}+\hat{p}_{\uparrow\uparrow}-2\hat{p}_{\downarrow\downarrow}\hat{p}_{\uparrow\uparrow}+\hat{p}_{\downarrow\downarrow}^2\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\uparrow}-\hat{p}_{\downarrow\downarrow}\hat{p}_{\downarrow\uparrow}\hat{p}_{\uparrow\uparrow}}{(1-\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow})^2}\right] \\ & + (\uparrow \text{ exch. } \downarrow)\,.\end{aligned} \tag{91}$$

B.2 Generating functions for the one-shot execution problem

The problem which we are required to solve during the execution of a one-shot order requires finding the generating functions for the probabilities $P_x^{ex}(T,x)$ and $P_n^{ex}(T,x)$ that the price and the hitting number assume respectively the values of x and n at the stopping

time T. Those probabilities follow a law extremely similar to Eq. (80), we have in fact

$$\begin{aligned} P_x^{ex}(t,x) &= \sum_{n=0}^{\infty}\sum_{\alpha_1}\cdots\sum_{\alpha_{n+1}}\int_0^{\infty}\left(\prod_{i=0}^{n+1}dt_i\right)\delta\left(t-\sum_{i=0}^{n+1}t_i\right)\delta\left(x-\sum_{i=1}^{n+1}X(\alpha_i)\right) \\ &\times\ p_{ex,\alpha_{n+1}}(t_{n+1})\left(\prod_{i=1}^{n}p_{\alpha_{i+1}\alpha_i}(t_i)\right)p_{\alpha_1 0}+p_{ex,0}(t)\,\delta(x)\ , \end{aligned}$$

while an analogous expression holds for $P_n^{ex}(T,n)$. In particular, the above equation can be recovered from Eq. (80) by exploiting the different terminal condition

$$\int_0^t dt'(1-p_{\uparrow\alpha}(t')-p_{\downarrow\alpha}(t'))\to p_{ex,\alpha}(t)\ , \tag{92}$$

which in Laplace space reads

$$\frac{1}{\omega}[1-\hat{p}_{\uparrow\alpha}(\omega)-\hat{p}_{\downarrow\alpha}(\omega)]\to\hat{p}_{ex,\alpha}(\omega)\ . \tag{93}$$

The generating functions for this modified problem can then be found as shown in the previous section by taking into account the substitution (93).

We are interested in particular in calculating the statistics of the price and the hitting number at the stopping time T, which can recovered by differentiation with respect to s of the generating function evaluated at the point $\omega=0$. We notice in particular that by substituting $\hat{p}_{\alpha,\alpha'}\to\bar{p}_{\alpha,\alpha'}$, and by exploiting the relation $\bar{p}_{ex,\alpha}=1-\bar{p}_{\uparrow,\alpha}-\bar{p}_{\downarrow,\alpha}$ in Eqs. (88), (89), (90) and (91), it is possible to use these same formulae even for the execution problem by simply multiplying by ω, thus removing the overall ω^{-1} factor.

Finally, we would like to obtain the statistics of the stopping time T itself, which can be obtained from the $\omega=0$ value of the generating functions $\Psi(\omega)=\Psi_x(\omega,0)=\Psi_n(\omega,0)$. It reads

$$\Psi(\omega)=\hat{p}_{ex,0}+\frac{\hat{p}_{\uparrow 0}(\hat{p}_{\downarrow\downarrow}\hat{p}_{ex,\uparrow}-\hat{p}_{\downarrow\uparrow}\hat{p}_{ex,\downarrow}-\hat{p}_{ex,\uparrow})+\hat{p}_{\downarrow 0}(\hat{p}_{\uparrow\uparrow}\hat{p}_{ex,\downarrow}-\hat{p}_{\uparrow\downarrow}\hat{p}_{ex,\uparrow}-\hat{p}_{ex,\downarrow})}{(1-\hat{p}_{\downarrow\downarrow}-\hat{p}_{\uparrow\uparrow}-\hat{p}_{\uparrow\downarrow}\hat{p}_{\downarrow\uparrow}+\hat{p}_{\uparrow\uparrow}\hat{p}_{\downarrow\downarrow})}\ . \tag{94}$$

In the hypothesis in which those functions are at least q times differentiable at $\omega=0$ (valid for $\mu>0$) one can write

$$\partial_\omega^q\hat{p}_{\alpha,\alpha'}|_{\omega=0} = (-1)^q\bar{p}_{\alpha,\alpha'}\langle t^q_{\alpha,\alpha'}\rangle\ . \tag{95}$$

It is also convenient to remind the definition of the unconditional hitting times $t_{hit,\alpha'}$, whose momenta of order q are related to the conditional ones through the relation

$$\langle t^q_{hit,\alpha'}\rangle=\sum_{\alpha}\bar{p}_{\alpha,\alpha'}\langle t^q_{\alpha,\alpha'}\rangle\ . \tag{96}$$

By employing these definitions, one can prove that

$$\langle T \rangle = \frac{1}{B} \left(A_\downarrow \langle t_{hit,\downarrow} \rangle + A_\uparrow \langle t_{hit,\uparrow} \rangle + B \langle t_{hit,0} \rangle \right) \ , \tag{97}$$

and

$$\begin{aligned} \langle T^2 \rangle - \langle T \rangle^2 &= \frac{1}{B} \left(A_\downarrow [\langle t_{hit,\downarrow}^2 \rangle - \langle t_{hit,\downarrow} \rangle^2] + \frac{B}{2} [\langle t_{hit,0}^2 \rangle - \langle t_{hit,0} \rangle^2] \right) \\ &+ \frac{1}{B^2} \left(- \frac{(A_\downarrow \langle t_{hit,\downarrow} \rangle + A_\uparrow \langle t_{hit,\uparrow} \rangle)^2}{2} + A_\downarrow B (\langle t_{hit,\downarrow} \rangle - \langle t_{hit,0} \rangle)^2 - A_\downarrow B \langle t_{hit,0} \rangle^2 \right) \\ &+ \frac{2}{B^2} \left(A_\downarrow \bar{p}_{\uparrow\downarrow} \langle t_{\uparrow\downarrow} \rangle + A_\uparrow \bar{p}_{\uparrow\uparrow} \langle t_{\uparrow\uparrow} \rangle + B \hat{p}_{\uparrow 0} \langle t_{\uparrow 0} \rangle \right) \left((1 - \bar{p}_{\downarrow\downarrow}) \langle t_{hit,\uparrow} \rangle + \bar{p}_{\downarrow\uparrow} \langle t_{hit,\downarrow} \rangle \right) \\ &+ (\uparrow \text{ exch. } \downarrow) \ , \end{aligned} \tag{98}$$

where we have defined

$$A_\alpha = \bar{p}_{-\alpha} \bar{p}_{\alpha,-\alpha} + \bar{p}_\alpha - \bar{p}_\alpha \bar{p}_{-\alpha,-\alpha} \tag{99}$$

$$B = 1 - \bar{p}_{\downarrow\downarrow} - \bar{p}_{\uparrow\uparrow} - \bar{p}_{\downarrow\uparrow} \bar{p}_{\uparrow\downarrow} + \bar{p}_{\uparrow\uparrow} \bar{p}_{\downarrow\downarrow} \ , \tag{100}$$

and introduced the convention $\pm(\uparrow) = \mp(\downarrow)$.

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