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Optimal split of orders across liquidity pools: a stochastic algorithm approach

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Abstract

Evolutions of the trading landscape lead to the capability to exchange the same financial instrument on different venues. Because liquidity issues the trading firms split large orders across trading destinations to optimize their execution. To solve this problem we devised two stochastic recursive learning procedures which adjust the proportions of the order to be sent to the different venues, one based on an optimization principle, the other on reinforcement ideas. We investigate both procedures from a theoretical point of view. In particular, we prove convergence results for the optimization algorithm when the innovations are supposed to be Markov stationary and ergodic (and speed with some mixing properties or when they are i.i.d.). Finally, we compare the behaviours of both algorithms on several simulations results (with simulated data and real data). We evaluate their performances with respect to an "oracle" strategy of an "insider" who could know *a priori* the executed quantities by every venues.

Keywords *Asset allocation, Stochastic Lagrangian algorithm, reinforcement principle, monotone dynamic system*

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1 Introduction

The trading landscape have seen a large number of evolutions following two regulations: Reg NMS in the US and MiFID in Europe. One of their consequences is the capability to exchange the same financial instrument on different trading venues. New trading destinations appeared to complement the trading capability of primary markets as the NASDAQ and the NYSE in the US, or EURONEXT, the London Stock Exchange and Xetra in Europe. Such alternative venues are called "Electronic Communication Network" (ECN) in the US and Multilateral Trading Facilities

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(MTF) in Europe. Each trading venue differentiates from the others at any time because of the fees or rebate it demands to trade and the liquidity it offers.

As the concerns about consuming liquidity increased with the financial crisis, trading firms use Smart Order Routers (SOR) as a key element in the process of optimizing their execution of large orders. Such devices are dedicated to split orders across trading destinations as a complement to the temporal slicing coming from the well known balance between the need to trade rapidly (to minimize market risk) and trading slow (to avoid market impact).

If the temporal slicing has been studied since the end of the nineties [1] with recent advances to adapt it to sophisticated investment strategies [15], this kind of spatial slicing (across trading destinations) has been mainly studied by economists from the point of view of its global market efficiency [7] rather than from one investor's point of view.

The complexity of spreading an order between N trading destinations comes from the fact that you never knows the quantity D_i available on the i^{th} trading venue to execute your order of size V during a time interval δt at your given price. If the fraction $r_i V$ of your order that you sent to the i^{th} liquidity pool is higher than D_i: you will loose time and may loose opportunity to execute $r_i V - D_i$ in an another pool; on another hand if $r_i V$ is lower than D_i: you will loose money if this pool fees are cheap, and an opportunity to execute more quantity here. The only way to optimise such a split on real time is to adjust on the fly the proportions $(r_i)_i$ according to the result of your previous executions.

This paper is an in depth analysis of the optimal split of orders. The illustrations and most of the vocabulary come from the "Dark pool" case, where the price S is not chosen by the trader (it is the market "mid point" price) and the answer of the pool is immediate (i.e. $\delta t = 0$). Dark pools are MTFs that do not publish pre trade informations, so an efficient use of the results of the previous executions (namely the realizations of the $\min(D_i^t, r_i^t V^t)$ for any i and all t in the past) is crucial. The results exposed here solve the problem of simultaneously splitting orders and using the information coming back from the pools to adjust the proportions to send for the next order, according to a criteria linked to the overall quantity executed (i.e. a linear combination of the $\min(D_i, r_i V)$).

The resulting trading strategy (which optimality is proven here) can be considered as an extension of the one conjectured by Almgren in [2]. Globally, our approach is a stochastic recursive learning procedure that belong to the wide family of "stochastic algorithms" also known as "stochastic approximation" and extensively investigated in the applied probability literature (see [12], [3], [6], etc).

The paper is organized as follows: in Section 2, we make precise the modeling of splitting orders among several venues in the framework of *Dark pools*, first in static then in a dynamic way. This leads to an optimization problem under constraints. In Section 3, we study the execution function of one dark pool and introduce the recursive stochasic algorithm resulting from the optimization problem. In Section 4 we analyze in depth this algorithm ($a.s.$ convergence and speed) when the "innovations" (data related to the orders, the executed quantities and the market price) are assumed i.i.d.. In Section 5 we extend our convergence result to the more realistic framework where these innovations are supposed to be stationary and ergodic (with some mixing properties). In Section 6, we investigate a second learning procedure based this time on reinforcement ideas introduced in [4] making a connexion with the theory of monotonous dynamical systems. Finally, in Section 7, we present several simulations results on simulated and real data to evaluate the performances of both procedures with respect to an "oracle" strategy of an "insider" who could know *a priori* the

executed quantities by every dark pool.

NOTATIONS: • For every $N \geq 1$, set $\mathcal{I}_N := \{1, 2, \ldots, N\}$. We denote by $\mathcal{P}_N := \{r = (r_i)_{1 \leq i \leq n} \in \mathbb{R}_+^N \mid \sum_{i=1}^N r_i = 1\}$ and by $\mathbf{1}^\perp := \{u \in \mathbb{R}^d \mid \sum_{i \in \mathcal{I}_N} u_i = 0\}$.
• δ_{ij} denotes the Kronecker symbol.
• $\langle . | . \rangle$ denotes the canonical inner product on $\mathbb{R}^d$ and $|\,.\,|$ the derived Euclidean norm.
• $int(A)$ denotes the interior of a subset A of $\mathbb{R}^d$.

2 A simple model for the execution of orders by dark pools

2.1 Static modelling

As mentioned in the introduction, we will focus in this paper on the splitting order problem in the case of (competing) *dark pools*. The execution policy of a dark pool differs from a primary market: thus a dark pool proposes bid/ask prices with no guarantee of executed quantity at the occasion of an over the counter transaction. Usually its bid price is lower than the bid price offered on the regular market (and the ask price is higher). Let us temporarily focus on a buying order sent to several dark pools. One can model the impact of the existence of N dark pools ($N \geq 2$) on a given transaction as follows: let $V > 0$ be the random volume to be executed and let $\theta_i \in (0, 1)$ be the *discount factor* proposed by the dark pool $i \in \{1, \ldots, N\}$. We will make the assumption that this discount factor is deterministic or at least known prior to the execution. Let r_i denote the percentage of V sent to the dark pool i for execution and! let $D_i \geq 0$ be the quantity of securities that can be delivered (or made available) by the dark pool i at price $\theta_i S$ where S denotes the bid price on the primary market (this is clearly an approximation since on the primary market, the order will be decomposed into slices executed at higher and higher prices following the order book). The rest of the order has to be executed on the primary market, at price S. Then the cost C of the executed order is given by

$$\begin{aligned} C &= S \sum_{i=1}^N \theta_i \min(r_i V, D_i) + S \left(V - \sum_{i=1}^N \min(r_i V, D_i) \right) \\ &= S \left(V - \sum_{i=1}^N \rho_i \min(r_i V, D_i) \right) \end{aligned}$$

where $\rho_i = 1 - \theta_i > 0$, $i = 1, \ldots, N$. At this stage, one may wish to minimize the mean execution cost, *given the price* S. This amounts to solving the following (conditional) maximization problem

$$\max \left\{ \sum_{i=1}^N \rho_i \, \mathbb{E} \left(\min(r_i V, D_i) \,|\, S \right), \; r \in \mathcal{P}_N \right\}. \tag{2.1}$$

However, none of the agents being insiders, they do not know the price S when the agent decides to buy the security and when the dark pools answer to their request. This means that one may assume that $(V, D_1, \ldots, D_n)$ and S are independent so that the maximization problem finally reads

$$\max \left\{ \sum_{i=1}^N \rho_i \mathbb{E} \left(\min(r_i V, D_i) \right), \; r \in \mathcal{P}_N \right\} \tag{2.2}$$

where we assume that all the random variables $\min(V, D_1), \ldots, \min(V, D_N)$ are integrable (otherwise the problem is meaningless).

An alternative choice could be to include the price S of the security into the optimization which leads to the mean maximization problem

$$\max \left\{ \sum_{i=1}^{N} \rho_i \, \mathbb{E}\left(S \min(r_i V, D_i)\right), \; r \in \mathcal{P}_N \right\} \tag{2.3}$$

(with the appropriate integrability assumption to make the problem consistent). It is then convenient to *include the price S into both random variables V and D_i* by considering $\widetilde{V} := V\,S$ and $\widetilde{D}_i := D_i S$ instead of V and D_i which leads again to the maximization problem (2.2).

If one considers symmetrically a selling order to be executed, the dark pool is supposed to propose a higher ask price $\theta_i S$, $\theta_i > 1$, than the order book. The seller aims at maximizing the execution global (mean) price of the transaction. This yields to the same formal optimization problem, this time with $\rho_i = \theta_i - 1$, $i = 1, \ldots, N$.

All these considerations lead us to focus on the abstract optimal allocation problem (2.2) which explains why the price variable S will no longer appear explicitly in what follows.

2.2 The dynamical aspect

In practice, there is no *a priori* assumption – or information available – on the joint distribution of $(V, D_1, \ldots, D_N)$ under $\mathbb{P}$. So the only reasonable way to provide a procedure to solve this allocation problem is to devise an *on-line learning algorithm based on historical data*, namely the results of former transactions with the dark pools on this security executed in the past. This underlines that our agent dealing with the dark pools is a financial institution like a bank, a broker or possibly a large investor which often – that means at least daily – faces some large scale execution problems on the same securities.

This means that we will have to make some assumptions on the dynamics of these transactions *i.e.* on a sequence $(V^n, D_1^n, \ldots, D_N^n)_{n\geq 1}$ (all the random variables are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$). Our basic assumption will be to make a *stationarity* assumption for every "couple" of sequences $(V^n, D_i^n)_{n\geq 1}$ (supposed to be Markov). This can be criticized since one may imagine models based on a general Markov dynamics with or without switching regimes. But we wish to provide a first natural solution to the problem and this stationarity assumption can be considered as likely for executions of orders on a daily basis : we mean that the historical sample data is made of transactions executed every day at the same approximative hour within a medium period of time (say a few months).

We will need an additional assumption of statistical nature – namely of ergodicity type – on every couple $(D_i^n, V^n)_{n\geq 1}$. This leads to the following formal assumption: for every $i = 1, \ldots, N$,

$$(ERG)_i \equiv \begin{cases} (i) & \text{the sequence } (V^n, D_i^n)_{n\geq 1} \text{ is a stationary Feller homogeneous Markov chain with distribution } \mathcal{L}(V, D_i), \\ (ii) & \text{the sequence } (V^n, D_i^n)_{n\geq 1} \text{ is ergodic } i.e. \\ & \mathbb{P}\text{-}a.s. \qquad \frac{1}{n}\sum_{k=1}^{n} \delta_{(V^k, D_i^k)} \overset{(\mathbb{R}_+^2)}{\Longrightarrow} \nu_i = \mathcal{L}(V, D_i), \\ (iii) & V \in L^{2+\eta}(\mathbb{P}) \text{ for some } \eta > 0. \end{cases}$$

where $\overset{(\mathbb{R}_+^2)}{\Longrightarrow}$ denotes the weak convergence of probability measures on $\mathbb{R}_+^2$ ([1]).

Assumption (iii) on the marginal distribution of the sequence $(V^n)_{n\geq 1}$ is mainly technical. In fact standard arguments form weak convergence theory show that combining (ii) and (iii) implies

$$\frac{1}{n}\sum_{k=1}^{n} V^k \longrightarrow \mathbb{E}\, V \quad \text{as} \quad n \to \infty.$$

As a special case of this ergodic setting, one may consider the (IID) setting

$$(IID) \equiv \begin{cases} (i) & \text{the sequence } (V^n, D_1^n, \ldots, D_N^n)_{n\geq 1} \text{ is i.i.d. with distribution } \nu = \mathcal{L}(V, D_1, \ldots, D_N) \\ & \text{on } (\mathbb{R}_+^{N+1}, \mathcal{B}or(\mathbb{R}_+^{N+1})) \\ (ii) & V \in L^{2+\eta}(\mathbb{P}) \text{ for some } \eta > 0. \end{cases}$$

This setting is undoubtedly less realistic in term of modeling. However it is a usual and reasonably elementary framework to apply the standard machinery of Stochastic Approximation ($a.s.$ convergence, asymptotically normal fluctuations, etc). So, its interest may be considered at least as pedagogical and we will provide our first results in that case. The (ERG) setting will be investigated as a second step, using some recent results established in [14] which are well suited to the specificities of our problem (in particular we will not need to assume the existence of a solution to the Poisson equation related to the procedure like in the reference book [3]).

3 Optimal allocation under constraints: a stochastic Lagrangian algorithm

3.1 The mean execution function of a dark pool

In view of the modeling section, we need to briefly describe the precise behaviour of the mean execution function $\varphi : [0, 1] \to \mathbb{R}_+$ of a single dark pool.

Let (V, D) be an $\mathbb{R}_+^2$-valued random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ representing the *global volume* to be executed and the *deliverable quantity* (by the dark pool) respectively. Throughout this paper we will assume the following consistency assumption

$$V > 0 \quad \mathbb{P}\text{-}a.s., \; V \in L^1(\mathbb{P}) \quad \text{and} \quad \mathbb{P}(D > 0) > 0. \tag{3.1}$$

[1] A homogeneous Markov chain $(X_n)_{n\geq 0}$ taking values in $\mathbb{R}^d$ is Feller if its transition operator $P(x, dy) = \mathbb{P}(X_{n+1} \in dy \,|\, X_n = x)$ leaves the set $\mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$ of bounded continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ stable.

The *a.s.* positivity of V means that we only consider true orders. The fact that D is not identically 0 means that the dark pool does exist in practice. The "rebate" coefficient ρ is specific to the dark pool.

Here the *mean execution function* $\varphi : [0,1] \to \mathbb{R}_+$ of the dark pool is defined by

$$\forall\, r \in [0,1], \quad \varphi(r) = \rho\, \mathbb{E}(\min(rV, D)) \tag{3.2}$$

where $\rho > 0$.The function φ is finite, non-identically 0. It is clearly a concave non-decreasing bounded function. Furthermore, one easily checks that its right and left derivatives are given at every $r \in [0,1]$ by

$$\varphi_l'(r) = \rho\, \mathbb{E}\left(\mathbf{1}_{\{rV \le D\}} V\right) \qquad \text{and} \qquad \varphi_r'(r) = \rho\, \mathbb{E}\left(\mathbf{1}_{\{rV < D\}} V\right). \tag{3.3}$$

In particular,

$$\varphi'(0) = \rho\, \mathbb{E}(V \mathbf{1}_{\{D>0\}}) > 0$$

and if

$$\textit{the (right continuous) distribution function of } \tfrac{D}{V} \textit{ is continuous on } \mathbb{R}_+, \tag{3.4}$$

then

$$\varphi \textit{ is everywhere differentiable on the unit interval } [0,1] \textit{ with } \varphi'(r) = \varphi_r'(r).$$

Assumption (3.4) means that the distribution of $\frac{D}{V}$ has no atom except possibly at 0. It can be interpreted as the fact that a dark pool has no "quantized" answer to an order.

More general models of execution functions in which the rebate ρ and the deliverable quantity D may depend upon the quantity to be executed rV are briefly discussed further on.

3.2 Design of the stochastic Lagrangian algorithm

Let V be the quantity to be executed by N dark pools. For every dark pool $i \in \mathcal{I}_N$ the available quantity D_i is defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ as V. We assume that all couples (V, D_i) satisfy the consistency assumption (3.1).

To each dark pool $i \in \mathcal{I}_N$ is attached a (bounded concave) mean execution function φ_i of type (3.2), (8.1) or (8.3) introduced in Section 2.1.

Then for every $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$,

$$\Phi(r_1, \ldots, r_N) := \sum_{i=1}^{N} \varphi_i(r_i). \tag{3.5}$$

In order to design the algorithm we will need to extend the mean execution function φ (whatever its form is) as a concave function on the whole real line by setting

$$\varphi(r) = \left(r - \frac{(-r)^{1+\vartheta}}{1+\vartheta}\right) \varphi'(0) \quad \text{if } r < 0 \quad \text{and} \quad \varphi(r) = \varphi(1) + \varphi'(1) \log r \quad \text{if } r > 1 \tag{3.6}$$

where $\vartheta \in (0,1]$ is a fixed parameter (one may always set $\vartheta = 1$ in the IID setting).

Based on the extension of the functions φ_i defined by (3.6), we can formally extend Φ on the whole affine hyperplane spanned by $\mathcal{P}_N$ *i.e.*

$$\mathcal{H}_N := \{r \in \mathbb{R}^N \mid \sum_i r_i = 1\}.$$

As announced, we aim at solving the following maximization problem

$$\max_{r \in \mathcal{P}_N} \Phi(r)$$

but we will also have to deal for algorithmic purpose with the same maximization problem when r runs over $\mathcal{H}_N$.

Before stating a rigorous result, let us a have a look at a Lagrangian approach that only takes into account the affine constraint that is $\max_r \Phi(r) - \lambda \sum_i r_i$. Straightforward formal computations suggest that

$$r^* \in \operatorname{argmax}_{\mathcal{P}_N} \Phi \text{ iff } \varphi_i'(r_i^*) \text{ is constant when } i \text{ runs over } \mathcal{I}_N$$

or equivalently if

$$\forall\, i \in \mathcal{I}_N, \qquad \varphi_i'(r_i^*) = \frac{1}{N} \sum_{j=1}^{N} \varphi_j'(r_j^*). \tag{3.7}$$

In fact this statement is not correct in full generality because the Lagrangian method does not provide a necessary and sufficient condition for a point to be a maximum of a (concave) function; thus, it does not take into account the case where Φ reaches its maximum on the boundary $\partial\mathcal{P}_N$ where the above condition on the derivatives may fail. So, an additional assumption is necessary to make it true as established in the proposition below.

Proposition 3.1 *Assume that (V, D_i) satisfies the consistency assumptions (3.1) and (3.4) for every $i \in \mathcal{I}_N$.*

(a) Assume that the functions φ_i defined by (3.2) satisfy the following assumption

$$(\mathcal{C}) \quad \equiv \quad \min_{i \in \mathcal{I}_N} \varphi_i'(0) \geq \max_{i \in \mathcal{I}_N} \varphi_i' \left(\frac{1}{N-1} \right).$$

Then $\operatorname{argmax}_{\mathcal{P}_N} \Phi$ *is a compact convex set and*

$$\operatorname{argmax}_{\mathcal{P}_N} \Phi = \{r \in \mathcal{P}_N, \mid \varphi_i'(r_i) = \varphi_1'(r_1),\ i = 1, \ldots, N\}.$$

Furthermore $\operatorname{argmax}_{\mathcal{H}_N} \Phi = \operatorname{argmax}_{\mathcal{P}_N} \Phi$.

(b) If the functions φ_i satisfy the slightly more stringent assumption,

$$(\mathcal{C}_<) \quad \equiv \quad \min_{i \in \mathcal{I}_N} \varphi_i'(0) > \max_{i \in \mathcal{I}_N} \varphi_i' \left(\frac{1}{N-1} \right).$$

then

$$\operatorname{argmax}_{\mathcal{H}_N} \Phi = \operatorname{argmax}_{\mathcal{P}_N} \Phi \subset int(\mathcal{P}_N).$$

Remarks. • If $N = 2$, one checks that Assumption $(\mathcal{C})$ is also necessary to derive the conclusion of item (a).
• As a by-product of the proof below we have the following more precise result on the optimal allocation r^*: if $r^* \in \operatorname{argmax}_{\mathcal{P}_N}$ and $\mathcal{I}_0(r^*) := \{i \in \mathcal{I}_N \,|\, r_i^* = 0\}$, then

$$\max_{i \in \mathcal{I}_0(r^*)} \varphi_i'(0) \le \min_{i \in \mathcal{I}_0(r^*)^c} \varphi_i'(0).$$

Interpretation and comments: • In the case of a "regular" mean execution function, Assumption $(\mathcal{C})$ is a kind of homogeneity assumption on the rebates made by the involved dark pools. If we assume that $\mathbb{P}(D_i = 0) = 0$ for every $i \in \mathcal{I}_N$ (all dark pools buy or sell at least one security with the announced rebate!), then $(\mathcal{C})$ reads

$$\min_{i \in \mathcal{I}_N} \rho_i \ge \max_{i \in \mathcal{I}_N} \left(\rho_i \frac{\mathbb{E}\, V \mathbf{1}_{\{\frac{V}{N-1} \le D_i\}}}{\mathbb{E}\, V} \right)$$

since $\varphi_i'(0) = \rho_i \, \mathbb{E}\, V$. In particular,

Assumption $(\mathcal{C})$ is always satisfied when all the ρ_i's are equal

(all dark pools propose the same rebates).

• Assumption $(\mathcal{C})$ is in fact our main assumption in terms of modeling. It may look somewhat difficult to satisfy when the rebates are not equal. But the crucial fact in order to preserve the generality of what follows is that it contains *no assumption about the dependence between the volume V and the "answers" D_i form the dark pools.*

Proof. (a) The function Φ is continuous on a compact set hence $\operatorname{argmax}_{\mathcal{P}_N} \Phi$ is not empty. Let $r \in \operatorname{argmax}_{\mathcal{P}_N} \Phi$ and $\mathcal{I}_0(r) := \{i \in \mathcal{I}_N \,|\, r_i = 0\}$. Clearly $\mathcal{I}_0(r) \ne \mathcal{I}_N$ so that $\operatorname{card} \mathcal{I}_0(r) \le N - 1$. Let $u \in \mathbf{1}^\perp$ such that $u_i > 0$, $i \in \mathcal{I}_0(r)$. Then $t \mapsto \Phi(r + tu)$ defined on the right neighbourhood of 0 reaches its maximum at 0 so that its derivative at 0 is non-positive. Specifying the vector u yields

$$\forall\, i \in \mathcal{I}_0(r),\ \forall\, j \in \mathcal{I}_0(r)^c, \quad \varphi_i'(0) \le \varphi_j'(r_j).$$

Now if $u \in \mathbf{1}^\perp$ with $u_i = 0$, $i \in \mathcal{I}_0(r)$, then the $t \mapsto \Phi(r + tu)$ is defined on a neighbourhood of 0 and reaches its maximum at $t = 0$ so that its derivative is 0 at 0; specifying the vector u yields

$$\forall\, i,\, j \in \mathcal{I}_0(r)^c, \quad \varphi_i'(r_i) = \varphi_j'(r_j).$$

Now, there exists at least one index $i_1 \in \mathcal{I}_0(r)^c$ such that $r_{i_1} \ge \frac{1}{|I_0(r)^c|} \ge \frac{1}{N-1}$. Hence $\varphi_{i_1}'(r_{i_1}) \le \varphi_{i_1}'(\frac{1}{N-1})$ which implies in turn that for every $i_0 \in \mathcal{I}_0(r)$, $\varphi_{i_0}'(0) \le \varphi_{i_1}'(r_{i_1}) \le \varphi_{i_1}'(\frac{1}{N-1})$. Finally Assumption $(\mathcal{C})$ implies that these inequalities hold as equalities so that

$$\forall\, i \in \mathcal{I}_N, \quad \varphi_i'(r_i) = \varphi_1'(r_1).$$

Conversely, let $r \in \mathcal{P}_N$ satisfying the above equalities. Then, for every $r' \in \mathcal{P}_N$, the function $t \mapsto \Phi(tr' + (1-t)r)$ is concave on $[0, 1]$ with a right derivative equal to 0 at $t = 0$. So it is maximum at $t = 0$ *i.e.* $\Phi(r) \ge \Phi(r')$.

Now we pass to the the maximization over $\mathcal{H}_N$. Since it is an affine space and Φ is concave, it is clear, *e.g.* by considering Φ as a function of $(r_1, \ldots, r_{N-1})$, that

$$\text{argmax}_{\mathcal{H}_N}\Phi = \{r \in \mathcal{H}_N, \, | \, \varphi_i'(r_i) = \varphi_1'(r_1), \; i = 1, \ldots, N\}$$

(which is non-empty since it contains at least $\text{argmax}_{\mathcal{P}_N}$). Now let $r \in \mathcal{H}_N \backslash \mathcal{P}_N$. Assume there exists $i_0 \in \mathcal{I}_N$ such that $r_{i_0} < 0$. Then there always exists an index $i_1 \in \mathcal{I}_N$ such that $r_{i_1} \geq \frac{1-r_{i_0}}{N-1} > \frac{1}{N-1}$. Consequently

$$\varphi_{i_0}'(r_{i_0}) = (1 + (-r_{i_0})^{\vartheta})\varphi_{i_0}'(0) > \varphi_{i_0}'(0) \geq \min_i \varphi_i'(0) \geq \max_i \varphi_i' \left(\frac{1}{N-1}\right) \geq \varphi_{i_1}' \left(\frac{1}{N-1}\right) \geq \varphi_{i_1}'(r_{i_1})$$

which contradicts the equality of these two derivatives. Consequently all r_i's are non-negative so that $r \in \mathcal{P}_N$.

(*b*) If $\mathcal{C}_<$ holds, the above proof shows that $\mathcal{I}_0(r) = \emptyset$ so that $\text{argmax}_{\mathcal{P}_N}\Phi_N \subset int(\mathcal{P}_N)$. □

3.3 Design of the stochastic algorithm

Now we are in position to devise the stochastic allocation algorithm, taking advantage of the characterization of $\text{argmax}_{\mathcal{P}_N}\Phi$. In fact we will simply use the obvious remark that N numbers $a_1, \ldots, a_N$ are equal if and only if they are all equal to their arithmetic mean $\frac{a_1+\cdots+a_N}{N}$.

In order to alleviate notations we mainly focus in this section on the (simplest) case of the regular mean execution function (3.2) and the representation (3.3) of its derivative as an expectation.

Using the representation (3.3) of the derivatives φ_i' yields that, if Assumption $(\mathcal{C})$ is satisfied, then $\text{argmax}_{\mathcal{H}_N}\Phi = \text{argmax}_{\mathcal{P}_N}\Phi$ and

$$r^* \in \text{argmax}_{\mathcal{P}_N}\Phi \Longleftrightarrow \forall \, i \in \{1, \ldots, N\}, \; \mathbb{E}\left(V\left(\rho_i \mathbf{1}_{\{r_i^* V < D_i\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbf{1}_{\{r_j^* V < D_j\}}\right)\right) = 0.$$

However, the set $\mathcal{P}_N$ is not stable for the "naive" zero search algorithm naturally derived from the above characterization, we are led to devise the procedure on the hyperplane $\mathcal{H}_N$.

Consequently, this leads to devise the following zero search procedure

$$r^{n+1} \;\; = \;\; r^n + \gamma_{n+1} H(r^n, V^{n+1}, D_1^{n+1}, \ldots, D_N^{n+1}), \quad r^0 \in \mathcal{P}_N,$$

where, for every $i \in \mathcal{I}_N$, every $r \in \mathcal{H}_N$, every $V > 0$ and every $D_1, \ldots, D_N \geq 0$,

$$\begin{aligned} H_i(r, V, D_1, \ldots, D_N) \;\; = \;\; & V\Big(\rho_i \mathbf{1}_{\{r_i V < D_i\} \cap \{r_i \in [0,1]\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbf{1}_{\{r_j V < D_j\} \cap \{r_j \in [0,1]\}} \\ & + R_i(r, V, D_1, \ldots, D_N)\Big) \end{aligned} \tag{3.8}$$

where $(V^n, D_1^n, \ldots, D_N^n)_{n \geq 1}$ is a sequence of random vectors with non negative components such that, for every $n \geq 1$ and $i \in \mathcal{I}_N$, $(V^n, D_i^n) \stackrel{d}{=} (V, D_i)$ and the remainder terms R_i have a mean-reverting effect to pull back the algorithm into $\mathcal{P}_N$. They are designed form the extension (3.6) of the derivative functions φ_i' outside the unit interval $[0, 1]$; to be precise, for every $i \in \mathcal{I}_N$,

$$R_i(r,V,D_1,\dots,D_{N}) = \left(\rho_i\left((1+|r_i|^{\vartheta})\mathbf{1}_{\{D_i>0\}\cap\{r_i<0\}}+\frac{1}{r_i}\mathbf{1}_{\{V<D_i\}\cap\{r_i>1\}}\right)\right.$$
$$\left.-\frac{1}{N}\sum_{j=1}^{N}\rho_j\left((1+|r_j|^{\vartheta})\mathbf{1}_{\{D_j>0\}\cap\{r_j<0\}}+\frac{1}{r_j}\mathbf{1}_{\{V<D_j\}\cap\{r_j>1\}}\right)\right).$$

4 The (IID) setting: $a.s$ convergence and CLT

Theorem 4.1 *Assume that $V\in L^2(\mathbb{P})$ and that Assumption $(\mathcal{C})$ holds. Let $\gamma:=(\gamma_n)_{n\geq 1}$ be a step sequence satisfying the usual decreasing step assumption*

$$\sum_{n\geq 1}\gamma_n=+\infty \quad \text{and} \quad \sum_{n\geq 1}\gamma_n^2<+\infty.$$

Let $(V^n,D_1^n,\dots,D_{N}^n)_{n\geq 1}$ be an i.d.d. sequence defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$. Then, there exists an $\operatorname{argmax}_{\mathcal{P}_{N}}\Phi$-valued random variable r^∞ such that

$$r^n\longrightarrow r^\infty \quad a.s.$$

If the functions φ_i satisfy $(\mathcal{C}_<)$ then $\operatorname{argmax}_{\mathcal{P}_{N}}\Phi\subset int(\mathcal{P}_{N})$.

Proof of the theorem. In this setting, the algorithm is (non homogenous) Markov discrete time process with respect to the natural filtration $\mathcal{F}_n:=\sigma(r^0,(V^k,D_1^k,\dots,D_{N}^k),\,k=1,\dots,n)$ with the following canonical representation

$$\begin{aligned} r^{n+1} &= r^n+\gamma_{n+1}H(r^n,V^{n+1},D_1^{n+1},\dots,D_{N}^{n+1}),\ r^0\in\mathcal{P}_{N}\\ &= r^n+\gamma_{n+1}h(r^n)+\gamma_{n+1}\Delta M_{n+1}\end{aligned}$$

where, for every $r\in\mathcal{H}_{N}$,

$$h(r):=\mathbb{E}\,H(r,V,D_1,\dots,D_{N})=\left(\varphi_i'(r_i)-\frac{1}{N}\sum_{j=1}^{N}\varphi_j'(r_j)\right)_{1\leq i\leq N}$$

is the so-called *mean* function of the algorithm, and

$$\begin{aligned}\Delta M_{n+1} &= H(r^n,V^{n+1},D_1^{n+1},\dots,D_{N}^{n+1})-\mathbb{E}(H(r^n,V^{n+1},D_1^{n+1},\dots,D_{N}^{n+1})\,|\,\mathcal{F}_n)\\ &= H(r^n,V^{n+1},D_1^{n+1},\dots,D_{N}^{n+1})-h(r^n)\end{aligned}$$

since $(V^{n+1},D_1^{n+1},\dots,D_{N}^{n+1})$ is independent of $\mathcal{F}_n$.

One derives from Proposition 3.1(a) that the mean function h of the algorithm satisfies $\{h=0\}=\operatorname{argmax}_{\mathcal{P}_{N}}$ and that, for every $r\in\mathcal{H}_{N}\setminus\{h=0\}$ and every $r^*\in\{h=0\}$,

$$\langle h(r)\,|\,r-r^*\rangle=\langle h(r)-h(r^*)\,|\,r-r^*\rangle=\sum_{i=1}^{N}\underbrace{(\varphi_i'(r_i)-\varphi_i'(r_i^*))(r_i-r_i^*)}_{\leq 0}<0 \tag{4.1}$$

simply because each function φ_i' is non-increasing which implies that each term of the sum is non-positive. The sum is not zero otherwise $\varphi'(r_i) = \varphi'(r_i^*)$ as soon as $r_i \neq r_i^*$ which would imply $h(r) = 0$.

The random vector V being square integrable, it is clear that $H_i(r, V, D_1, \ldots, D_{_N})$ satisfies the linear growth assumption

$$\forall\, i \in \mathcal{I}_{_N},\ \forall\, r \in \mathcal{H}_{_N}, \quad \|H_i(r, V, D_1, \ldots, D_{_N})\|_2 \leq 2\,(\max_j \rho_j)\|V\|_2(N + |r|^{\vartheta})$$

At this stage one may conclude using a simple variant of the standard Robbins-Monro Theorem (like that established in [16]): there exists a random variable r^{∞} taking values in $\{h = 0\}$ such that $r^n \to r^{\infty}$.

Otherwise, simple variant of the theorem (applied on the affine space $\mathcal{H}_{_N}$) like that established in [16] completes the proof. □

4.1 Rate of convergence

Our aim in this section is to show that the assumptions of the regular Central Limit Theorem (CLT) for stochastic approximation procedures are fulfilled. For a precise statement, we refer (among others to) to [3] (Theorem 13 p.332) For the sake of simplicity, we will assume that the mean function h has a single zero denoted r^{∞}. The following lemma provides a simple criterion to ensure this uniqueness.

Lemma 4.1 *Assume that all the functions φ_i, $i \in \mathcal{I}_{_N}$, are decreasing (strictly). Then*

$$\{h = 0\} = \mathrm{argmax}_{\mathcal{P}_{_N}}\Phi = r^{\infty} \in int(\mathcal{P}_{_N}).$$

Proof. In particular $(\mathcal{C}_<)$ is satisfied so that $\mathrm{argmax}_{\mathcal{P}_{_N}}\Phi \subset r^* \in int(\mathcal{P}_{_N})$. If $r,\, r' \in \{h = 0\}$, $r \neq r'$, it follows from (4.1) that $\varphi_i'(r_i) = \varphi_i'(r_i')$ for some index i such that $r_i \neq r_i'$. □

The second ingredient needed to establish a CLT will be the Hessian of function Φ. To ensure its existence we will make one further assumption on a generic random couple (V, D), keeping in mind that $\mathbb{P}(D > 0) > 0$. Namely, we assume that the conditional distribution function of D_i given V *i.e.*

$$F_{_D}(u \,|\, V = v, D > 0) := \mathbb{P}(D \leq u \,|\, V = v, D > 0),\ u \geq 0,\ v > 0,$$

admits a density $f_{_D}(u, v)$ such that and satisfying

$$\begin{cases} (i) & F_{_D}(u \,|\, V = v, D > 0) = \int_0^u f_{_D}(u', v)du' \\ (ii) & \text{for every } v > 0,\ u \mapsto f_{_D}(u, v) \text{ is continuous and positive,} \\ (iii) & \forall\, \varepsilon \in (0, 1), \quad \sup_{\varepsilon V \leq u \leq V/\varepsilon} f_{_D}(u, V)V^2 \in L^1(\mathbb{P}). \end{cases} \tag{4.2}$$

Note that (iii) is clearly satisfied when $V \in L^2(\mathbb{P})$ and f_D is bounded.

Lemma 4.2 *(a) Assume (V, D) satisfies the above assumption (4.2). Then the mean execution function $\varphi(u) := \rho\,\mathbb{E}(\min(uV, D))$ is concave, twice differentiable on $\mathbb{R}_+$ and for every $u > 0$,*

$$\varphi''(u) = -\rho\,\mathbb{P}(D > 0)\mathbb{E}\left(f_D(uV, V)\right) < 0.$$

(b) If (V, D_i) satisfies the above assumption (4.2) for every $i \in I_N$, then the function $\tilde{h}$ defined on $\mathbb{R}_+^N$ by $\tilde{h}(u_1, \ldots, u_N) = \left(\varphi_i'(u_i) - \frac{1}{N}\sum_{1\le j\le N}\varphi_j'(u_j)\right)_{1\le i\le N}$ is differentiable on $(0,\infty)^N$ and admits a continuous extension on $\mathbb{R}_+^N$ given by

$$D\tilde{h}(u) = -\frac{1}{N}\Big[-a_j(u_j) + Na_i(u_i)\,\delta_{ij}\Big]_{1\le i,j\le N} \qquad \textit{with} \qquad a_i(u) = -\varphi_i''(u) > 0.$$

(c) Let $A := [-a_j + Na_i\delta_{ij}]_{1\le i,j\le N}$, $a_1, \ldots, a_N > 0$ and let $\underline{a} = \min_i a_i$. Its kernel $\mathrm{Ker}(A)$ is one dimensional, $A(\mathbb{R}^N) = \mathbf{1}^\perp$ and $A_{|\mathbf{1}^\perp}$ is bijective. Every non-zero eigenvalue λ (with eigenspace E_λ) satisfies

$$\Re(\lambda) \ge N \times \underline{a} \qquad \textit{and} \qquad E_\lambda \subset \mathbf{1}^\perp.$$

Proof. (a) is a straightforward application of the Lebesgue differentiation Theorem for expectation.
(b) is a consequence of (a).
(c) The transpose ${}^tA$ of A has a strict dominating diagonal structure *i.e.* ${}^tA_{ii} > 0$, ${}^tA_{ij} < 0$, $i \ne j$ and $\sum_j^t A_{ij} = 0$ for every i. Consequently, it follows from Gershgorin's Lemma (see [8]) that 0 is an eigenvalue of order 1 of ${}^tA$ (with $\mathbf{1}$ as an eigenvector and that all other eigenvalues have (strictly) positive real parts). Consequently $\mathrm{Ker}A$ is one dimensional. The fact that $A(\mathbb{R}^N) \subset \mathbf{1}^\perp$ is obvious so that this inclusion holds as an equality by the dimension formula. Hence all the eigenvectors not in $\mathrm{Ker}A$ are in $\mathbf{1}^\perp$. Set $\tilde{a}_i - a_i - \underline{a} \ge 0$, $i = 1, \ldots, N$. Then ${}^t(\tilde{A})$ has a dominating diagonal structure so that all its eigenvalues have non-negative real parts. Now if λ is an eigenvalue of A, it is obvious that $\lambda - N\underline{a}$ is an eigenvalue of $\tilde{A}$. Consequently $\Re(\lambda) \ge N\underline{a}$. □

Theorem 4.2 *Assume that Assumption (4.2) holds for every (V, D_i), $i \in \mathcal{I}_N$ and that $V \in L^{2+\delta}(\mathbb{P})$, $\delta > 0$. Set*

$$\gamma_n = \frac{c}{n}, \qquad n \ge 1, \qquad c > \frac{1}{2\Re(\lambda_{\min})}$$

where $\lambda_{\min}$ denotes the eigenvalue of $A^\infty := -Dh(r^\infty)_{|\mathbf{1}^\perp}$ with the lowest real part. Then

$$\frac{r^n - r^\infty}{\sqrt{\gamma_n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0; \Sigma_*)$$

where the asymptotic variance is given by

$$\Sigma^\infty = \int_0^\infty e^{u(A^\infty - \frac{Id}{2c})} C^\infty e^{u(A^\infty - \frac{Id}{2c})^*} du$$

where

$$C^\infty = \mathbb{E}\left(H(r^\infty, V, D_1, \ldots, D_N)H(r^\infty, V, D_1, \ldots, D_N)^t\right)_{|\mathbf{1}^\perp}$$

and $(A^\infty - \frac{Id}{2c})^*$ stands for the transpose operator *of $A^\infty - \frac{Id}{2c} \in \mathcal{L}(\mathbf{1}^\perp)$.*

Remark. The above claim is consistent since $u \mapsto H(r, v, \delta_1, \ldots, \delta_{N})^t\, u$ preserves $\mathbf{1}^{\perp}$.

Proof. First note that, since $r^{\infty} \in int(\mathcal{P}_{N})$, the above Lemma 4.2$(b)$ shows that (still making the confusion between the linear operator $Dh(r^{\infty})$ and its matrix representation in the canonical basis)

$$Dh(r^{\infty}) = -\frac{1}{N}\left[-a_j(r_j^{\infty}) + N a_i(r_i^{\infty})\,\delta_{ij}\right]_{1\le i,j\le N} \qquad \text{with } a_i(r) = \rho_i \mathbb{E}(V^2 f_{D_i}(rV, V)) > 0$$

Then, Lemma 4.2 (c) implies that $-Dh(r^{\infty})_{|\mathbf{1}^{\perp}}$ has eigenvalues with positive real parts, all lower bounded by $\min_i a_i(r_j^{\infty}) > 0$.

At this stage, one can apply the CLT for stochastic algorithms defined on $\mathbf{1}^{\perp}$ (see *e.g.* [3], p.341). □

5 The (ERG) setting: convergence

For the sake of simplicity, although it is not really necessary, we will assume in this section that

$$\operatorname{argmax} \Phi_{\mathcal{P}_{N}} = \{r^*\}.$$

In fact we will need to assume a stronger *ergodicity assumption* for each pair $(V^n, D_i^n)_{n\ge1}$, $i = 1, \ldots, N$ than $(ERG)_i$. In particular we will not assume *a priori* that the "input sequence" is stationary, but that, *a.s.*, its empirical measure converges fast enough toward a stationary regime (for the weak convergence of distributions).

So we assume that for every $i \in \{1, \ldots, N\}$, the sequence $(V^n, D_i^n)_{n\ge1}$ satisfies $(ERG)_i$ with a limiting distribution (V, D_i) satisfying the consistency assumption (3.1) and the continuity assumption (3.4).

Indeed, this first assumption implies by standard weak convergence arguments that for every $i \in \mathcal{I}_{N}$ and every $u \in \mathbb{R}_+$,

$$\frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{uV^k < D_i^k\}} - \mathbb{E}(V\mathbf{1}_{\{uV \le D_i\}}) \xrightarrow{a.s.} 0$$

since the (non-negative) function $f_u(v, y) := v\mathbf{1}_{\{uv\le y\}}$ is $\mathbb{P}_{(V,D_i)}$-*a.s.* continuous and $O(v)$ as $v \to \infty$ by (3.4).

In fact our technical assumption is to require a rate in the above *a.s.* ans L^2 convergence, namely we assume that there exists an exponent $\alpha_i \in (0, 1)$ such that

$$\forall\, u \in \mathbb{R}_+, \quad \frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{uV^k < D_i^k\}} - \mathbb{E}(V\mathbf{1}_{\{uV < D_i\}}) = O(n^{-\alpha_i}) \quad a.s. \text{ and in } L^2(\mathbb{P}) \tag{5.1}$$

We will occasionally refer to this property as *ergodicity at rate* α_i of the sequence $(V^n, D_i^n)_{n\ge1}$ toward (V, D_i).

Note that, when the sequence $(V^n, (D_i^n)_{i\in I_{N}})_{n\ge1}$ is i.i.d. and $V \in L^2(\mathbb{P})$, elementary martingale arguments show that it is ergodic at rate $\frac{1}{2} - \varepsilon$ for every $\varepsilon \in (0, 1/2)$. In particular this shows that the theorem below almost embodies the (IID) setting, except for some integrability assumption (on V).

Now we are in position to state the main convergence result of this section. We rely on the convergence theorem for stochastic algorithms established in [14]. For the reader's convenience its statement is reproduced in the Annex A. Note that in fact, stationarity of the couples $(V^n, D_i^n)_{n\geq 1}$ is not explicitly required in what follows.

Theorem 5.1 *Let $(V_n, D_n^1, \ldots, D_n^N)_{n\geq 1}$ be a [[stationary]] Feller homogeneous Markov chain and assume that $\sup_{n\geq 0} \mathbb{E}(V^n)^4 < +\infty$. Furthermore, assume that for every $i \in \mathcal{I}_N$, the sequence $(V^n, D_i^n)_{n\geq 1}$ is ergodic at rate $\alpha_i \in (0,1)$ toward (V, D_i). Assume that the distribution of (V, D_i) satisfies the consistency assumption (3.1) and the continuity assumption (3.4). If the step sequence $(\gamma_n)_{n\geq 1}$ satisfies*

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad \gamma_n = o(n^{\underline{\alpha}-1}) \quad \text{and} \quad \sum_{n\geq 1} n^{1-\underline{\alpha}} \max(\gamma_n^2, |\gamma_n - \gamma_{n+1}|) < \infty$$

where $\underline{\alpha} := \min_{i\in I_N} \alpha_i \in (0,1)$, then the algorithm defined by (3.8) with growth control parameter $\vartheta \in (0, 2/3)$ a.s. converges towards $r^\infty = \operatorname{argmax}_{\mathcal{P}_N} \Phi$.

TECHNICAL COMMENT.The above condition on the step sequence γ is satisfied as soon as $\gamma_n = \frac{c}{n^\beta}$ with $\beta \in (1-\underline{\alpha}, 1]$.

Proof. We aim at applying the general result for stochastic algorithms with markovian input dynamics in presence of a Lyapunov function established in [14] (see Annex A Theorem A.1 for its statement). The key is to notice that the state function of the algorithm is monotonic in r whatever the input y is, namely

$$\forall\, r,\, r' \in \mathcal{H}_N,\ r \neq r',\ \forall y \in \mathbb{R}_+^{N+1}, \quad \langle H(r,y) - H(r',y) | r - r' \rangle > 0.$$

Now let $(\mathcal{F}_n)_{n\geq 1}$ denote the natural filtration of the sequence Markov chain $Y^n := (V^n, D_1^n, \ldots, D_N^n)$, $n \geq 1$ and let $G^H(r,y)$ be the *pseudo-mean function* defined by

$$G^H(r,y) = \mathbb{E}(H(r, Y^{n+1}) \,|\, Y^n = y),\ n \geq 1.$$

For every $r \in \mathcal{H}_N$, $G^H(r,.)$ is continuous since the chain $(Y_n)_{n\geq 1}$ is Feller. It follows from the above monotonicity assumption satisfied by H that G^H also satisfies the Lyapunov assumption related to $L(r) = \frac{1}{2}|r - r^*|^2$ since

$$\forall\, r \in \mathcal{H}_N \setminus \{r^*\},\ \forall y \in \mathbb{R}_+^{N+1}, \quad \langle G^H(r,y) - G^H(r^*,y) | r - r^* \rangle > 0.$$

Moreover we have the following inequality for the function H

$$\forall r \in \mathcal{H}_n,\ \forall y \in \mathbb{R}^{N+1}, \quad |H(r,y)|^3 \leq C_H g(y)(1 + L(r))(1 + \log(1 + L(r))$$

where $C_H > 0$ and $g(Y) = V^3$. The fact that $\sup_{n\geq 0} \mathbb{E}(V^n)^4 < +\infty$ implies that the growth assumption on H is satisfied for $p = 2$. Consequently, we apply Theorem A.1 to prove the convergence of the algorithm. □

APPLICATION. If, for every $i \in \mathcal{I}_N$, $(V^n, D_i^n)_{n\geq 1}$ is an α-mixing stationary Markov chain with mixing coefficient $(\alpha_n)_{n\geq 1}$ satisfying Ibragimov's condition

$$\sum_{n\geq 1} \alpha_n^{\frac{2}{2+\delta}} < +\infty \qquad \text{for some } \ \delta > 0,$$

(which is satisfied in case of geometric α-mixing) then the sequence $(V^n, D_i^n)_{n\geq 1}$ is stable at rate β for every $\beta \in (0, 1/2)$ toward the stationary distribution (supposed to satisfy (3.1) and (3.4)).

EXAMPLE: An exponential discrete time Ornstein-Ulhenbeck model for $(V, D_1, \ldots, D_{_N})$.

$$V^n = v^0 e^{X_0^n}, \qquad D_i^n = d_i^0 e^{X_i^n},\ i = 1, \ldots, N$$

where v^0, d_1^0, $\ldots$, $d_{_N}^0$ are positive real numbers and the sequence $(X_n)_{n\geq 1}$ satisfies the linear auto-regressive dynamics

$$X^{n+1} = m + AX^n + B\,\Xi^{n+1}, n \geq 1,$$

with $m \in \mathbb{R}^{N+1}$, $A \in \mathcal{M}(N+1, N+1, \mathbb{R})$, $\|A\| < 1$, $B \in \mathcal{M}(N+1, M)$ with $\mathrm{rank}(B) = N+1$ $(\leq M)$ and $(\Xi^n)_{n\geq 1}$ an i.i.d. sequence of $\mathcal{N}(0; Id_M)$-distributed random variables. We assume that the sequence is stationary *i.e.* that the distribution of X_1 is the (Gaussian) invariant distribution (with covariance matrix C solution to the Lyapunov equation $C - ACA^* = BB^*$ where * stands for transpose). Then (see [5], , p. 99), the sequence $(X^n)_{n\geq 0}$ is geometrically α-mixing and subsequently so is $(V^n, D_1^n, \ldots, D_{_N}^n)_{n\geq 1}$.

Remark. In fact, it is straightforward that the transition $P(x, dy)$ of the above Gaussian auto-regressive Markov chain satisfies the following absolute continuity assumption

$$P(y, dy') = p(y, y')\lambda_{N+1}(dy') \quad \text{ with } \quad \forall\, y' \in \mathbb{R}_+^{N+1},\ p(y, y') > 0\ \lambda_{N+1}(dy)\text{-}a.e.$$

since its Gaussian density p is everywhere positive. Then, standard arguments based on ergodic theory (that can be adapted from those reviewed in the survey paper [17]) show that the ergodicity at rate β for every $\beta \in (0, 1/2)$ holds for the chain starting at any $y = (v, (d_i)_{i=1,\ldots,N}) \in \mathbb{R}_+^{N+1}$.

To establish a rate of convergence in this α-mixing framework, we need to prove that a solution to the Poisson equation related to the pseudo-transition of the process (θ_n, Y_n) as developed in [3]. In our modelling, the innovation process $(Y - n)_{n\geq 1}$ does not depend upon the state variable $\theta_n$parameter sequence which make us quite optimistic concerning a positive issue to this question. This guess is strongly supported by our numerical experiments (see Section 7. For some partial theoretical results in that directions we refer [13].

6 An alternative procedure based on a reinforcement principle.

Recently, inspired by the discussion developed by Almgren and Harts in [2] about liquidity estimation, Berenstein and Lehalle devised a "smart routing" recursive procedure of requests to be executed by a pool of N dark pools (see [4]). This procedure is not based on the opimization of a potential function but on a intuitive reinforcement mechanism. Let I_i^n be the profit induced by the execution of the order sent to dark pool i at time n. The proportion r_i^n of the global order V^{n+1} to be sent to dark pool i for execution at time $n+1$ is defined as proportional to this profit *i.e.* by

$$\forall\, i \in \mathcal{I}_{_N}, \qquad r_i^n := \frac{I_i^n}{\sum_j I_j^n}.$$

The updating of the random vector I^n is as follows

$$\forall\, n \geq 0,\ \forall\, i \in \mathcal{I}_{_N}, \quad I_i^{n+1} = I_i^n + \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right), \quad I_i^0 = 0.$$

The first equation models the idea of "reinforcement" since the proportion of orders sent for execution to dark pool i is proportional to the historical performances of this dark pool since the beginning of the procedure.

The second equation describes in a standard way – like in the optimization algorithm – the way dark pools execute orders.

Elementary computations show that the algorithm can be written directly in a recursive way in terms of a new vector valued variable

$$X^n = \frac{I^n}{n}, \qquad n \geq 1,$$

since

$$X_i^{n+1} = X_i^n - \frac{1}{n+1}\left(X_i^n - \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right)\right), \quad i \in \mathcal{I}_N.$$

This is a standard form a stochastic algorithm (with step $\gamma_n = \frac{1}{n}$).

Furthermore, note that setting $\underline{\rho} := \min_i \rho_i$,

$$\sum_{i\in\mathcal{I}_N} I_i^n \geq \underline{\rho} \min\left(\frac{1}{N}V^n, \min_{i\in\mathcal{I}_N} D_i^n\right)$$

since $r_i^n \geq \frac{1}{N}$ for at least one dark pool $i \in I_N$. Consequently, as soon as the sequence $(V^n, D_1^n, \ldots, D_N^n)$ is stationary and ergodic

$$\liminf_n \sum_{i\in\mathcal{I}_N} X_i^n \geq \underline{\rho} \lim_n \frac{1}{n}\sum_{k=1}^n \min\left(\frac{1}{N}V^k, \min_{i\in\mathcal{I}_N} D_i^k\right) = \underline{\rho}\,\mathbb{E}\min\left(\frac{1}{N}V, \min_{i\in\mathcal{I}_N} D_i\right) \ a.s.$$

So if we make the natural assumption that

$$\mathbb{E}\min\left(\frac{1}{N}V, \min_{i\in\mathcal{I}_N} D_i\right) > 0$$

then, $a.s.$, the algorithm X_n cannot converge to 0.

If we make the additional assumption that the sequence $(V^n, D_1^n, \ldots, D_N^n)$ is $i.i.d.$ then the algorithm is a discrete time (non homogenous) $\mathcal{F}_n$-Markov process with respect to the filtration $\mathcal{F}_n = \sigma((V^k, D_1^k, \ldots, D_N^k), k = 1, \ldots, n)$, $n \geq 0$, so that it admits the canonical representation

$$X_i^{n+1} = X_i^n - \gamma_{n+1}\left(X_i^n - \varphi_i(r_i^n)\right) + \gamma_{n+1}\Delta M_i^{n+1} \quad i \in \mathcal{I}_N, \quad n \geq 0,$$

where $\gamma_n = \frac{1}{n}$ and

$$\Delta M_i^n = \rho_i \min\left(r_i^{n-1}V^n, D_i^n\right) - \varphi_i(r^{n-1}), \quad i \in \mathcal{I}_N, \quad n \geq 1,$$

is an $\mathcal{F}_n$-martingale increment. Furthermore it is L^2-bounded as soon as $V \in L^2$.

In fact the specific difficulties induced by this algorithm are more in relation with its mean function

$$h : x \longmapsto \left(x_i - \varphi_i\left(\frac{x_i}{\sum_j x_j}\right)\right)_{1\leq i\leq N} \tag{6.1}$$

than with the martingale "disturbance term" $\gamma_{n+1}\Delta M_{n+1}$. Our first task will be to prove under natural assumptions the existence of a non degenerate equilibrium point. Then we will show why this induces the existence of many parasitic equilibrium points.

6.1 Existence of an equilibrium

In this section, we will need to introduce a new function associated to a generic order V and a generic dark pool with characteristics (ρ, D).

$$\psi(u) := \frac{\varphi(u)}{u}, \quad u > 0, \qquad \psi(0) = \varphi'(0) = \rho\,\mathbb{E}\,V\mathbf{1}_{\{D>0\}}. \tag{6.2}$$

If Assumption (3.1) holds then $\psi(0) < +\infty$ and ψ is continuous at 0. It follows from the concavity of φ and $\varphi(0) = 0$ that ψ is non-increasing. It is continuous as soon as φ is *e.g.* if Assumption (3.4) holds true.

Proposition 6.1 *Let $N \geq 1$. Assume that Assumption (3.1) holds for every couple (V, D_i), $i \in \mathcal{I}_N$.*
(a) There exists a $x^* \in \mathbb{R}_+^N$ such that

$$\sum_{i\in\mathcal{I}_N} x_i^* > 0 \quad \text{and} \quad \varphi_i\left(\frac{x_i^*}{\sum_{j\in\mathcal{I}_N} x_j^*}\right) = x_i^*, \quad i \in \mathcal{I}_N. \tag{6.3}$$

(b) Let ψ_i be the functions associated to dark pool $i \in \mathcal{I}_N$ by (6.2). Assume that for every $i \in \mathcal{I}_N$, ψ_i is (continuous and) decreasing on $[0, \infty)$ and that

$$\sum_{i\in\mathcal{I}_N} \psi_i^{-1}(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) < 1. \tag{6.4}$$

Then there exists $x^* \in int(\mathcal{P}_N)$ satisfying (6.3).

Proof. (a) We define for every $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$

$$\Psi(r) := \left(\frac{\varphi_i(r_i)}{\sum_{j\in\mathcal{I}_N} \varphi_j(r_j)}\right)_{i\in\mathcal{I}_N}.$$

This function maps the compact convex set $\mathcal{P}_N$ into itself. Furthermore it is continuous since, on the one hand, for every $i \in \mathcal{I}_N$, φ_i is continuous owing to the fact that (V, D_i) satisfies (3.1) and, on the other hand,

$$\sum_{j\in\mathcal{I}_N} \varphi_j(r_j) \geq \min_{j\in\mathcal{I}_N} \varphi_j\left(\frac{1}{N}\right) > 0.$$

Indeed, for every $i \in I_N$,

$$\varphi_j\left(\frac{1}{N}\right) \geq \frac{1}{N}\mathbb{E}\,\min(V, D_i) > 0$$

since $V > 0$ $\mathbb{P}$-*a.s.* and $\mathbb{P}(D_j = 0) < 1$. Then it follows from the Brouwer Theorem that the function Ψ has a fixed point r^*. Set for every $i \in \mathcal{I}_N$,

$$x_i^* = r_i^* \sum_{j\in\mathcal{I}_N} \varphi_j(r_j).$$

It follows immediately from this definition that

$$\forall\, i \in \mathcal{I}_N, \quad x_i^* = \varphi_i(r_i^*)$$

which in turn implies that $\sum_{j\in\mathcal{I}_N} \varphi_j(r_j^*) = \sum_{j\in\mathcal{I}_N} x_j^*$ so that $r_i^* = \frac{x_i^*}{\sum_{j\in\mathcal{I}_N} x_j^*}$, $i \in \mathcal{I}_N$.

(b) For every $i \in \mathcal{I}_N$ we consider the inverse of ψ_i defined on the interval $(0, \varphi_i'(0)]$. This function is decreasing continuous and $\lim_{v\to 0} \psi_i^{-1}(v) = +\infty$. Then, let Θ be the continuous function defined by

$$\forall\, \theta \in (0, \min_{i\in\mathcal{I}_N} \varphi_i'(0)], \quad \Theta(\theta) = \sum_{i\in\mathcal{I}_N} \psi_i^{-1}(\theta).$$

We know that $\lim_{\theta\to 0} \Theta(\theta) = +\infty$ and we derive from Assumption (6.4) that $\Theta(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) \leq 1$. So, owing to the (strict) monotonicity of θ, there exists $\theta^* \in (0, \min_{\in\mathcal{I}_N} \varphi_i'(0))$ such that $\Theta(\theta^*) = 1$. Set

$$r_i^* = \psi_i^{-1}(\theta^*), \quad i \in \mathcal{I}_N.$$

Then $r^* := (r_1^*, \ldots, r_N^*) \in int(\mathcal{P}_N)$ since $\sum_i r_i^* = 1$ by definition of θ^*. If $r_{i_0}^* = 0$, then $\theta^* = \psi_{i_0}(0) = \min_{i\in I_N} \varphi_i'(0)$ which is impossible. □

Corollary 6.1 *Assume that all the functions ψ_i are continuous and decreasing. If furthermore, the rebate coefficients ρ_i are equal (to 1) and if $\mathbb{P}(D_i = 0) = 0$ for every $i \in \mathcal{I}_N$ then there exists an equilibrium point lying in $int(\mathcal{P}_N)$.*

Proof. Under the above assumptions $\varphi_i'(0) = \mathbb{E}\, V > 0$. Consequently

$$\psi_i^{-1}(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) = \psi^{-1}(\psi_i(0)) = 0 < 1. \quad \square$$

COMMENTS. Unfortunately there is no hope to prove that all the equilibrium points lie in the interior of $\mathcal{P}_N$ since one may always adopt an execution strategy which boycotts a given dark pool or, more generally N_0 dark pools. So it seems hopeless to get the uniqueness of the equilibrium point. To be more precise, under the assumptions of claim (b) of the above Proposition 6.1, there exists at least one strategy involving a subset of $N - N_0$ dark pools $N_0 = 0, \ldots, N-1$ (one dark pool is needed at least). Elementary combinatorial arguments show that there are *at least* $2^N - 1$ equilibrium points.

So, from a theoretical point of view, we are facing a situation where there may be many parasitic equilibrium points, some of them being clearly parasitic. However it is quite difficult to decide *a priori*, even if we make the unrealistic assumption that we know all the involved distributions, which equilibrium are parasitic.

This is a typical situation encountered when dealing with procedures devised from a reinforcement principle.

However, one may reasonably hope that some of them are so-called "traps", that means equilibrium which repulsive at least in one noisy direction so that the algorithm escapes from it. Another feature described below suggests that a theoretical study of the convergence behaviour of this procedure would need a specific extra work.

The next natural question is to wonder whether an equilibrium x^* of the algorithm – namely a zero of h – is a target for the algorithm *i.e.* is attractive for the companion ODE, $\dot{x} = -h(x)$.

Proposition 6.2 *An equilibrium x^* satisfying (6.3) is locally uniformly attractive as soon as*

$$\sum_{j\in\mathcal{I}_N} \frac{x_j^*}{(\bar{x}^*)^2}\varphi_j'\left(\frac{x_j^*}{\bar{x}^*}\right) < 1 - \frac{1}{\bar{x}^*}\max_{i\in\mathcal{I}_N}\varphi'\left(\frac{x_i^*}{\bar{x}^*}\right)$$

where $\bar{x}^* = \sum_{i\in\mathcal{I}_N} x_i^*$.

Remark. In fact the following inequalities are satisfied by any equilibrium x^*:

$$1 - \frac{1}{\bar{x}^*}\varphi_i'\left(\frac{x_i^*}{\bar{x}^*}\right) > 0, \quad i\in\mathcal{I}_N.$$

This follows from the convexity of the function $\xi \mapsto \xi - \varphi_i\left(\frac{\xi}{\bar{x}^*}\right)$ which is zero at x_i^* with positive derivative. As a consequence, the right hand side in the above sufficient condition is always positive. However the above condition is just a criterion.

Proof. One computes the differential $Dh(x)$ of h at some $x\in\mathbb{R}_+^N$. Elementary computations show that

$$\forall\, i,\, j\in\mathcal{I}_N, \qquad \frac{\partial h_i}{\partial x_j}(x) = \delta_{ij}\left(1 - \frac{1}{\bar{x}}\varphi_i'\left(\frac{x_i}{\bar{x}}\right)\right) + \frac{x_i}{\bar{x}^2}\varphi_i'\left(\frac{x_i}{\bar{x}}\right).$$

As a consequence all the diagonal terms of $Dh(x^*)$ are positive. The above condition for all the eigenvalues of $Dh(x)$ to have positive real parts follows from a standard application of Gershgorin's Lemma to the transpose of $Dh(x)$. □

6.2 A competitive system

But once again, even if we could show that all equilibrium are noisy traps, the convergence would not follow for free since this algorithm is a associated to a so-called *competitive system*. A competitive differential system $\dot{x} = h(x)$ is a system in which the field $h : \mathbb{R}^N \to \mathbb{R}^N$ is differentiable and satisfies

$$\forall x\in\mathbb{R}^N,\ \forall\, i,\, j\in\mathcal{I}_N,\ i\neq j, \quad \frac{\partial h_i}{\partial x_j}(x) > 0.$$

As concerns Almgren and Harts's algorithm, the mean function h is given by (6.1), and under the standard differentiability assumption on the functions φ_i's,

$$\forall x\in\mathbb{R}^N, \quad \frac{\partial h_i}{\partial x_j}(x) = \varphi_i'\left(\frac{x_i}{x_1+\cdots+x_N}\right)\frac{x_i}{(x_1+\cdots+x_N)^2} > 0.$$

These systems are known to have possibly a non converging behaviour even in presence of a single (attracting) equilibrium. This is to be compared to their *cooperative* counterparts (with negative non diagonal partial derivatives) whose flow converge uniformly on compact sets toward the single equilibrium in that case. This property can be transferred to the stochastic procedure by the mean if the so-called ODE method which shows that the algorithm almost behaves like some trajectories of the Ordinary differential Equation associated to its mean field h (see *e.g.* [12, 3, 6] for an introduction). Cooperativeness and competitiveness are in fact some criterions which ensure some generalized monotonicity properties on the flow of the ODE viewed as a function of its space variable. For some background on cooperative and competitive systems we refer to [10, 11] and the references therein.

7 Numerical tests

The aim of this section is to compare the behaviour of both algorithms in different settings : the IID one, the ERG one and with pseudo-real data (whose construction is explained below).

We examine specifically two situations : abundance and shortage. What we call "abundance" is the fact that the mean of V is less that the sum of the means of the D_i, *i.e.* $\mathbb{E}V \leq \sum_{i=1}^{N} \mathbb{E}D_i$, and the "shortage" is the situation where we have the contrary, *i.e.* $\mathbb{E}V > \sum_{i=1}^{N} \mathbb{E}D_i$. We distinguish these two frameworks because the behaviour of both algorithms is different, but the most interesting setting to compare them is the shortage since it is the situation the most common in the market.

We present the performances of both algorithms and compare them to the strategy devised by an insider "oracle" who would know the true values of V and D_i. This "oracle" strategy is the best possible allocation. Indeed, this algorithm executes

$$\forall n \geq 1, \qquad \min\left(V^n, \sum_{i=1}^{N} D_i^n\right).$$

So we introduce in the following figures

- the allocation coefficients of the optimization algorithm and the reinforcement algorithm (just drawn in the IID setting as example because the best way to compare the two algorithms is to look at their performances),
- the ratios between the executed quantity and the sent quantity for the three algorithms (we name it *satisfaction*), *i.e.* for every $n \geq 1$,
  - $\dfrac{\min\left(V^n, \sum_{i=1}^{N} D_i^n\right)}{V^n}$ for the oracle,
  - $\dfrac{\sum_{i=1}^{N} \min\left(r_i^n V^n, D_i^n\right)}{V^n}$ for both algorithms.
- the ratios between the satisfaction index of both optimization and reinforcement algorithms and that of the oracle, *i.e.* for every $n \geq 1$

$$\frac{\sum_{i=1}^{N} \min\left(r_i^n V^n, D_i^n\right)}{\min\left(V^n, \sum_{i=1}^{N} D_i^n\right)}.$$

Remark 7.1 *We can prove that the allocation coefficients of the oracle algorithm read*

$$\forall n \geq 1,\ \widehat{r}_i^n = \frac{\min(V^n, D_i^n)}{\sum_{i=1}^{N} \min(V^n, D_i^n)},\ \forall 1 \leq i \leq N.$$

Moreover, when we simulate the data, we have chosen 10^4 simulations because it corresponds approximatively to the number of pseudo-real data observed in a day.

7.1 The IID setting

We consider here simulated data in the i.i.d. setting, where the quantity V and D_i, $i \in I_N$, are log-normal variables and $N = 3$. Then the variables V and D_i, $i \in I_N$, satisfy the assumptions of the CLT and we have the rate of convergence of the algorithm.

In the over-abundace framework, *i.e.* when $\mathbb{E}V \leq \min_i \mathbb{E}D_i$, the two algorithms conserve the initial allocation coefficients and if we change the initial value, we observe the same behaviour in non degenerate case (*i.e.* $r_i^0 < 1,\ 1 \leq i \leq N$). In the abundance setting, the two algorithms have a close behaviour with slightly different allocation coefficients, but the optimization satisfaction is better than the reinforcement one as expected.

The figure below represents the shortage situation. The initial value of the algorithms is $r_i^0 = \frac{1}{N}$, $1 \leq i \leq N$ and we set

$$\mathbb{E}V = \frac{3}{2}\sum_{i=1}^{N} \mathbb{E}D_i$$

with

$$\mathbb{E}D_i = i,\ 1 \leq i \leq N, \quad \mathrm{Var}(V) = 1, \mathrm{Var}(D_i) = 1, 1 \leq i \leq N \quad \text{and } \rho = \begin{pmatrix} 0.95 \\ 0.97 \\ 0.99 \end{pmatrix}.$$

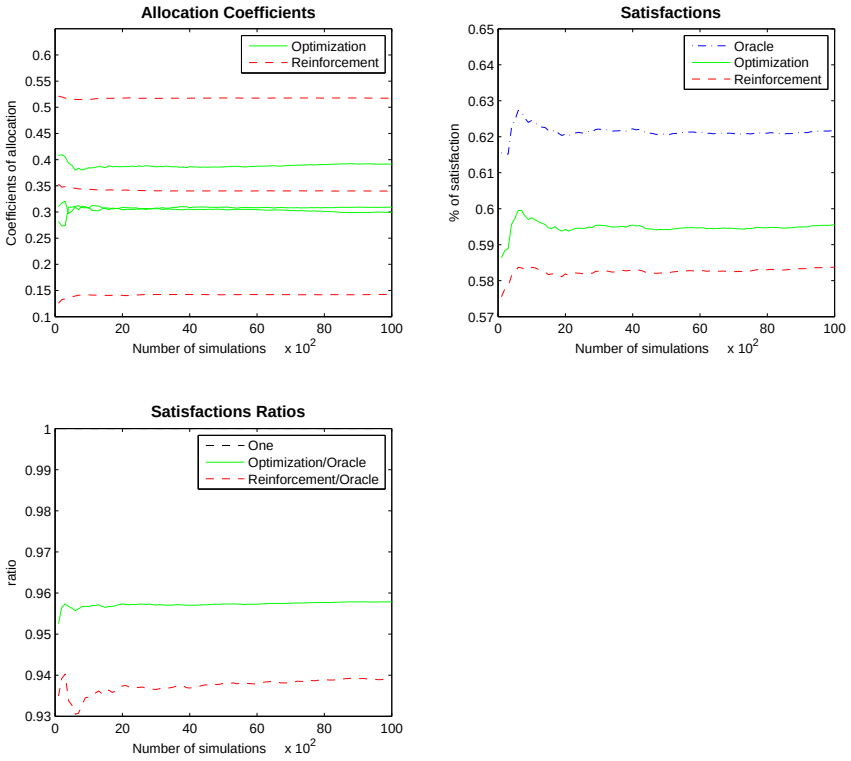

Figure 1: Case $N = 3$, $m_V = \frac{3}{2}\sum_{i=1}^{N} m_{D_i}$, $m_{D_i} = i$, $\sigma_V = 1$, $\sigma_{D_i} = 1$, $1 \leq i \leq N$.

The allocation coefficients are rather different and the rate of satisfaction of the optimization algorithm is better than the one of the reinforcement one. Further more the optimization procedure is closer to the oracle than to the reinforcement one. It seems that the optimization algorithm remains more robust when the variances fluctuate.

7.2 The ERG setting

We consider here simulated data in the ergodic setting, where the quantity V and D_i, $i \in I_N$, are exponentials of an Ornstein-Uhlenbeck process, *i.e.*

$$X^{n+1} = m + AX^n + B\Xi^{n+1},$$

where $\|A\| < 1$, $BB^* \in Gl(d, \mathbb{R})$ and

$$m = \begin{pmatrix} m_1 \\ \vdots \\ m_{N+1} \end{pmatrix} \in \mathbb{R}^{N+1}, \quad \Xi^{n+1} = \begin{pmatrix} \Xi_1^{n+1} \\ \vdots \\ \Xi_{N+1}^{n+1} \end{pmatrix} \sim \mathcal{N}(0, I_{N+1}) \ \ i.i.d., \quad e^{X^n} = \begin{pmatrix} V^n \\ D_1^n \\ \vdots \\ D_N^n \end{pmatrix}.$$

In the abundance situation, we observe that the two algorithms behave in the same way as in the IID setting. So we are interested in the shortage situation. The initial value of the algorithms is $r_i^0 = \frac{1}{N}$, $1 \le i \le N$ and we set

$$\rho = \begin{pmatrix} 0.95 \\ 0.97 \\ 0.99 \end{pmatrix}, \quad m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.7 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.3 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.2 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.02 & 0 & 0 & 0 \\ 0.01 & 0.9 & 0 & 0 \\ 0.01 & 0.01 & 0.6 & 0 \\ 0.01 & 0.01 & 0.01 & 0.3 \end{pmatrix}.$$

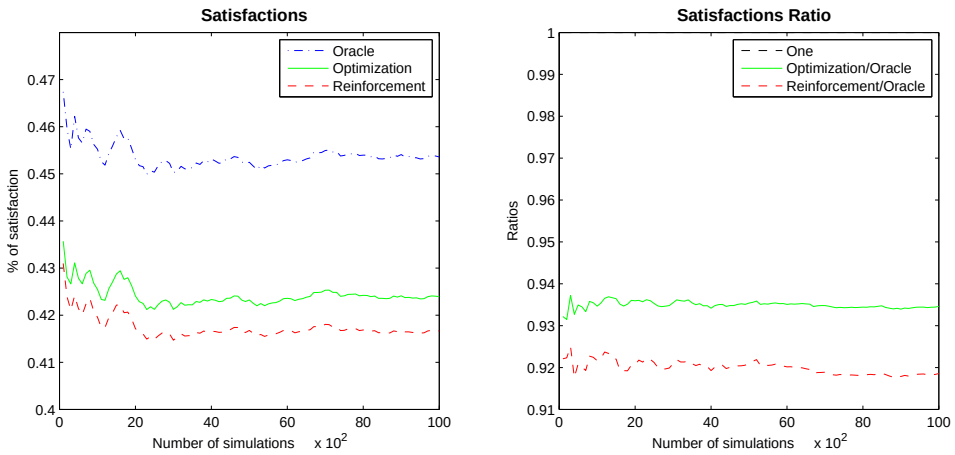

Figure 2: Case $N = 3$, $m_V \ge \sum_{i=1}^N m_{D_i}$, $\sigma_V \ge 1$, $\sigma_{D_i} \ge 1$, $1 \le i \le N$.

We remark that in the ergodic setting, when the variances are high, the optimization algorithm still over-performs the reinforcement algorithm but less significantly than in the i.i.d. setting. The difference between the satisfaction ratios of both algorithms equals to 1% and 2% w.r.t. the oracle one.

7.3 The pseudo-real data setting

Firstly we explain how the data are created. We consider an asset volume V which we want buy or sell, and the volumes which are offered by the N dark pools $(D_i)_{1\leq i\leq N}$. We have picked up data on the market for V and the D_i are building from the N assets which are the most correlated with V, denoted by S_i, $i = 1, \ldots, N$, and V by the mixing function

$$\forall 1 \leq i \leq N, \quad D_i := \beta_i \left((1 - \alpha_i)V + \alpha_i S_i \frac{\mathbb{E}V}{\mathbb{E}S_i} \right)$$

where $\alpha_i,\ i = 1, \ldots, N$ are the mixing coefficients and $\beta_i,\ i = 1, \ldots, N$ some weights.

Now, we can define the abundance and shortage situations. If all the β_i equal to 1, then all the dark pools D_i have the same mean as V : this is an abundance framework and we are sure to receive the desired volume V. If $\sum_{i=1}^{N} \beta_i \leq 1$, then $\mathbb{E}\left[\sum_{i=1}^{N} D_i\right] \leq \mathbb{E}V$: this is a shortage framework and we use the algorithm to find the optimal allocation.

To best study both algorithms, we add to the optimization, reinforcement and oracle algorithms an other called "naive" because it allocates at each step

$$\tilde{r}_i^n = \frac{\mathbb{E}D_i}{\sum_{i=1}^{N} \mathbb{E}D_i}, \forall i,\ \forall n.$$

This gives us another comparison element.

The simulations presented here have been made with the asset BNP and the four most correlated assets with BNP, so $N = 4$. As we work we real volumes, the data used here extend on many days, 11 exactly. So the behaviour of the quantity of securities to execute changes according to the day, but especially to the needs. To explains this changes in the response of the algorithms, we have underlined the days by drawing vertical lines to separate the days of execution. We place in the shortage situation : we set

$$\rho = \begin{pmatrix} 0.94 \\ 0.96 \\ 0.98 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.2 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 0.4 \\ 0.6 \\ 0.8 \\ 0.2 \end{pmatrix}.$$

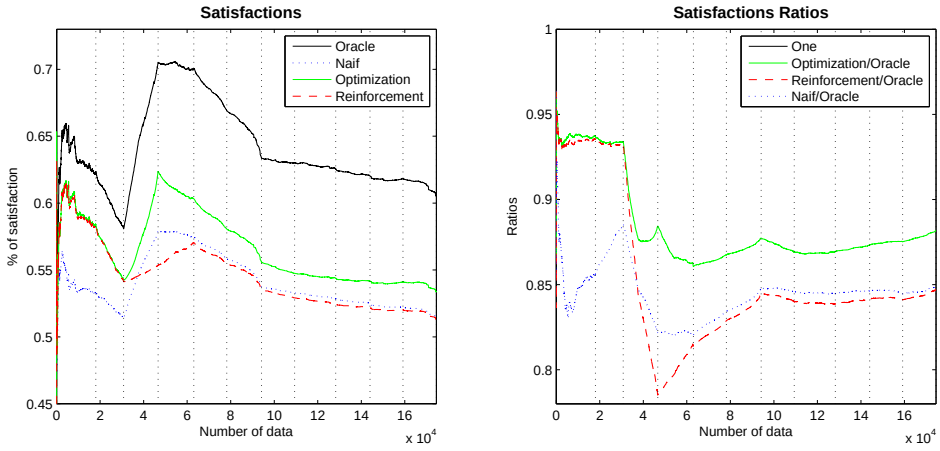

Figure 3: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0 < \alpha_i \leq 0.2$ and $r_i^0 = 1/N$ $1 \leq i \leq N$

We observe that the optimization is more reactive than the reinforcement one and its rate of satisfaction is 2% higher. What it is surprising is the fact that the naive algorithm is slightly better than the reinforcement one.

8 Provisional remarks

8.1 Toward more realistic mean execution functions

One natural idea is to take into account that the rebate may depend on the quantity rV sent to be executed by the dark pool. The mean execution function of the dark pool can be modeled by

$$\forall\, r \in [0,1], \qquad \varphi(r) = \mathbb{E}(\rho(rV)\min(rV, D)) \tag{8.1}$$

where the rebate function ρ is a non-negative, bounded, non-decreasing right differentiable function.

For the sake of simplicity, we assume that (V, D) satisfies (3.4). The right derivative of φ reads

$$\varphi_r'(r) = \mathbb{E}\left(\rho_r'(rV)V\min\left(rV, D\right)\right) + \mathbb{E}\left(\rho(rV)V\mathbf{1}_{\{rV<D\}}\right), \tag{8.2}$$

with in particular $\varphi'(0) = \rho(0)\,\mathbb{E}(V\mathbf{1}_{\{D>0\}}) > 0$ as above. The main gap is to specify the function ρ so that φ remains concave which is the key assumption to apply the convergence theorem. Unfortunately the choice for ρ turns out to strongly depends on the (unknown) distribution of the random variable D. Let us briefly consider the case where V and D are independent and D has an exponential distribution $\mathcal{E}(\lambda)$.

First note that the function g defined by $g(u) := \mathbb{E}(u \wedge D)$, $u \geq 0$ is given by

$$\forall\, u \geq 0, \quad g(u) = \frac{1 - e^{-u\lambda}}{\lambda}$$

so that, owing to the independence of V and D,

$$\forall\, r \geq 0, \quad \varphi(r) = \mathbb{E}(\rho(rV)g(rV)).$$

At this stage, φ will be concave as soon as the function $\rho\, g$ is so. Among all possible choices, elementary computations show that a class of possible choices is to consider $\rho = g^\theta$ with $\theta \in (0, \lambda]$. Of course this may appear as not very realistic since the rebate function is structural feature of the different dark pools.

However several numerical experiments not reproduced here testify that both algorithms are robust to a realistic choice for the function ρ *e.g.* a non-decreasing and stepwise constant.

Another natural extension is to model the fact that the dark pool may take into account the volume rV to decide which quantity will really executed rather than simply the *a priori* deliverable quantity D. One reason for such a behaviour is that the dark pool may wish to preserve the possibility of future transactions with other clients.

One way to model this phenomenon is to introduce a *delivery function* $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$, non-decreasing and concave w.r.t. its first variable and satisfying $0 \leq \psi(x, y) \leq y$, so that the new mean execution function is as follows:

$$\varphi(r) = \rho\, \mathbb{E}\left(\min(rV, \psi(rV, D))\right). \tag{8.3}$$

It is clear that the function φ is concave (as the minimum of two concave functions) and bounded. In this case, the first (right) derivative of φ reads

$$\varphi'_r(r) = \rho\, \mathbb{E}\left(V\left(\mathbf{1}_{\{rV<\psi(rV,D)\}} + \psi'_x(rV, D)\mathbf{1}_{\{rV\geq\psi(rV,D)\}}\right)\right) \tag{8.4}$$

where ψ'_x denotes the right derivative with respect to x. In particular $\varphi'_r(0) = \rho\, \mathbb{E}(V\mathbf{1}_{\{D>0\}}) > 0$.

As concerns the implementations resulting from these new execution functions, the adaptation is straightforward. Note for the optimization procedure under constraints that, firstly, the "edge" functions R_i functions are not impacted by the type of the execution function. On the other hand the definition of the functions H_i or the updating of the variables I^n for the reinforcement procedure should be adapted in accordance with the representations (8.2) and (8.4) of the new mean execution function φ'.

Example: We consider for modelling the quantity delivered by the dark pool i a function where we can define a minimal quantity required to begin to consum D_i, namely

$$\psi_i(rV, D_i) = D_i \mathbf{1}_{\{rV > s_i D_i\}}$$

where s_i is a parameter of the dark pool i assumed to be deterministic.

Pseudo-real data setting

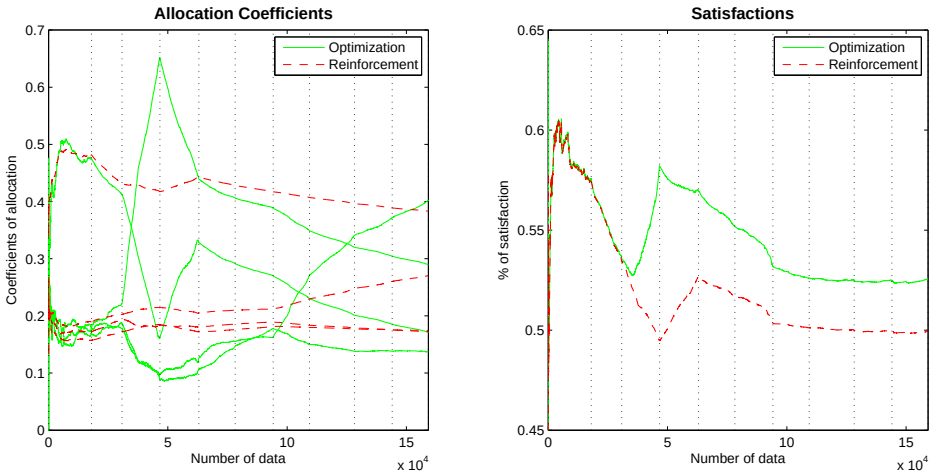

Figure 4: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0 < \alpha_i \leq 0.2$ and $r_i^0 = 1/N$, $1 \leq i \leq N$, $s = (0.3, 0.2, 0.2, 0.3)^t$.

8.2 Optimization *vs* reinforcement ?

For practical implementation what conclusions can be drawn form our investigations on both procedures. Both reach quickly a stabilization/convergence phase close to optimality. The reinforcement algorithm leaves the simplex structurally stable which means the proposed dispatching at each time step is realistic whereas the stochastic Lagrangian algorithm may sometimes need to be corrected. However, in a high volatility context, the stochastic Lagrangian algorithm clearly prevails with performances that may be significantly better performance. This optimization procedure also relies on established convergence results in a rather general framework (stationary α-mixing input data).

Given the computational cost of these procedures which is close to zero, a good strategy is probably to implement both in parallel.

Appendix

A A general result for convergence of stochastic approximation with ergodic innovation

We consider an algorithm of the following form

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1}), \quad n \geq 0 \tag{A.1}$$

where $\theta_0 \in \mathbb{R}^d$, $(Y_n)_{n\geq 0}$ is an $\mathbb{R}^q$-valued homogeneous Markov chain evolving on $\mathbb{R}^q$ with transition $P = (P(x, dy))_{x\in\mathbb{R}^q}$ defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\gamma_n)_{n\geq 1}$ is a nonincreasing sequence of positive real numbers and H is a Borel function from $\mathbb{R}^d \times \mathbb{R}^q$ to $\mathbb{R}^d$.

We will say that the Markov chain $(Y_n)_{n\geq 0}$ is ν-ergodic (under $\mathbb{P}$) if, for every $p\in [1, +\infty)$,

$$\forall f \in L^p(\nu), \quad \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) \longrightarrow \int_{\mathbb{R}^q} f d\nu \quad \mathbb{P}\text{-}a.s. \text{ and in } \ L^p(\mathbb{P}). \tag{A.2}$$

We need some additional notations to define the mean function of the algorithm, so we set as a *pseudo mean function*

$$G^H(\theta, y) := \mathbb{E}\left[H(\theta, Y_{n+1}) \mid Y_n = y\right] \tag{A.3}$$

and the stochastic approximation procedure defined by (A.1) gets a recursive zero search of the function below

$$h(\theta) := \int G^H(\theta, y)\nu(dy) \tag{A.4}$$

Let $\beta\in (0, 1)$, $p \in [1, \infty)$. We denote by $\mathcal{V}_{\beta,p}$ the class of functions whose convergence rate in (A.2) $\mathbb{P}$-$a.s.$ and in $L^p(\mathbb{P})$ is n^β, namely

$$\mathcal{V}_{\beta,p} = \left\{ f : \mathbb{R}^q \to \mathbb{R}^q \mid \frac{1}{n}\sum_{k=1}^{n} f(Y_k) - \int f d\nu \overset{\mathbb{P}\text{-}a.s. \ \&\ L^p(\mathbb{P})}{=} O(n^{-\beta}) \right\}. \tag{A.5}$$

Now we are in a position to state the convergence theorem.

Theorem A.1 *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (A.4), $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ a Borel function and $(Y_n)_{n\geq 0}$ be a Feller homogeneous Markov chain satisfying (A.2). Assume there exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}^+$ satisfying*

$$\nabla L \text{ is Lipschitz continuous and } |\nabla L|^2 \leq C\,(1 + L) \tag{A.6}$$

such that the function $G^H(\theta, y)$ defined by (A.3) satisfies the local mean-reverting *assumption*

$$\forall \theta \in \mathbb{R}^d\backslash\{\theta^*\}, \ \forall y \in \mathbb{R}^q, \ \left\langle \nabla L(\theta) \mid G^H(\theta, y) - G^H(\theta^*, y)\right\rangle > 0. \tag{A.7}$$

and that there exists $\beta\in (0, 1/2)$ such that

$$\forall \theta \in \mathbb{R}^d, \ G^H(\theta, \cdot) \in \mathcal{V}_{\beta,p}. \tag{A.8}$$

Moreover, assume that H satisfies the following (pseudo-)linear growth assumption

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad |H(\theta, y)| \leq cg(y)(1 + L(\theta))^{\frac{1}{3}}(1 + \ln(1 + L(\theta)))^{\frac{1}{3}} \tag{A.9}$$

where $g \in L^{3\vee\frac{2p}{p-1}}(\nu)$ and $\sup_{n\geq 1} \|g(Y_n)\|_{3\vee\frac{2p}{p-1}} < +\infty$ (or if $(Y_n)_{n\geq 0}$ is stationary, $g(Y_0) \in L^{3\vee\frac{2p}{p-1}}(\mathbb{P})$).

Let $\gamma = (\gamma_n)_{n\geq 1}$ be a non negative non increasing sequence of gain parameters satisfying

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad n^{1-\beta}\gamma_n \longrightarrow 0, \quad \text{and} \quad \sum_{k\geq 1} k^{1-\beta} \max\left(\gamma_k^2, |\Delta\gamma_{k+1}|\right) < +\infty. \tag{A.10}$$

Then, the recursive procedure defined by (A.1) satisfies

$$\theta_n \xrightarrow{a.s.} \theta^*.$$

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