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Minimum cost spanning tree problems with indifferent agents

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Abstract

We consider an extension of minimum cost spanning tree (mcst) problems where some agents do not need to be connected to the source, but might reduce the cost of others to do so. Even if the cost usually cannot be computed in polynomial time, we extend the characterization of the Kar solution (Kar (2002, GEB)) for classic mcst problems. It is obtained by adapting the Equal treatment property: if the cost of the edge between two agents changes, their cost shares are affected in the same manner if they have the same demand. If not, their changes are proportional to each other. We obtain three variations on the Kar solution, that are differentiated and characterized using stability, fairness and manipulation-proofness properties.

JEL Classification: C71, D63

Keywords: Minimum cost spanning tree; Steiner tree; cost sharing; Shapley value

1 Introduction

Minimum cost spanning tree (mcst) problems study situations where a group of agents need to connect to a source in order to obtain a good or service. If an edge in the network is used, there is a cost, which is independent of the number of agents that use that edge to connect to the source. There is a large literature on the cost-sharing problem, with the first use of game-theoretic tools in Bird (1976). That article showed that the core is always non-empty and proposes a method, now known as the Bird solution, that is always in the core.

We propose an extension to the mcst model: instead of assuming that all agents desire to be connected to the source, we allow for a subset of agents to be indifferent to such a connection. While they have no demand, we keep the agents as part of the problem, as their cooperation means that agents that want to be connected can potentially use their locations to connect to the source at a cheaper cost. We then have the non-trivial problem of potentially compensating these agents for their contributions to a lower cost.

This extension allows us to cover a wide range of problems. Obviously, if all agents want to be connected, we are back in the classic mcst problem. If only one agent wants to be connected, this turns out to be equivalent to a shortest path problem. The problem considered is also close to Steiner tree problems (see Hwang and Richards (1992) for a review), where some freely available nodes can be used by anyone. Steiner tree problems and the general problem considered here share the unpleasant characteristics that the minimum cost of the project usually cannot be computed in polynomial time and that the core can be empty. An attempt to provide cost-sharing solutions to Steiner tree problems is Bergantinos et al. (2011). The difference with our setup is that we don't consider these nodes as unoccupied, and we allow for compensation to their owners. If such compensations were disallowed, the problem would become a Steiner tree problem.

The relevance of the extension is not only theoretical: the minimum cost spanning tree problem applies well to physical distribution networks of goods like water, gas or electricity. However, most of these projects involve states or cities that do not want the good (or are not its main destination) but

that host part of the network. Conflicts with these agents are frequent, as the use of their territory usually requires compensation.[1]

The cooperation between agents that need to be connected to the source and agents that can provide them with cheaper connections is similar to the general technological cooperation models (Trudeau (2009a), Bahel and Trudeau (2012)) where agents provide an unspecified service (sharing patents, know-how, providing access to suppliers, etc.) that reduces the overall cost. In the mcst problems, these issues have been raised, notably when discussing if it is acceptable to subsidize some agents, as the Kar solution does (Bergantinos and Vidal-Puga (2007), Trudeau (2012)).

In the minimum cost spanning three literature, three of the main cost sharing solutions (the Bird solution, the folk solution (Feltkamp et al. (1994), Bergantinos and Vidal-Puga (2007)) and the cycle-complete solution (Trudeau (2012))) are designed to be in the core. Given that in our context the core can empty, there is less appeal in extending these methods to our context. We therefore focus on the Kar solution (Kar (2002)), which is itself an adaptation of the Shapley-Shubik method.

This characterization of Kar uses Group Independence, a property that says that if there is never any gains for two groups to cooperate on the construction of the network, then we can compute the cost shares independently on each group. It also uses the property of Equal Treatment, that says that if the cost of edge (i, j) goes down, the cost shares of agents i and j should change by the same amount, a notion this is very natural as the edge that generates the saving is jointly owned by the two agents. If Group Independence can be extended easily to our context, it is not trivial to adapt Equal Treatment. It seems natural to keep the requirement if i and j are identical with respect to their desire to connect to the source, as that edge can be used to connect i or j to the source. If i wants to be connected but not j, it does not seem as natural to require them to have equal changes to their cost shares, as the agents play different roles. In particular, the edge can only be used to connect i. Therefore, we only require that the changes be proportional to each other. It is a consistency requirement, as this guarantees that the way we treat demanders versus non-demanders stays constant throughout, regardless of the magnitude of the change in cost, the way it affects the optimal configuration or the identity of the agents involved.

For a given factor of proportionality, it turns out to be sufficient to characterize a unique cost sharing solution. In particular, we obtain three variations on the Shapley value when we require that agents that do not want to be connected be unaffected, that agents that want to be connected be unaffected, or that both types have equal changes to their cost shares. It is a rare example of a characterization of a (non-trivial) method for a cost-sharing problem that might not be solvable in polynomial time.

We then introduce a series of properties to differentiate the three solutions. The properties are related to stability, the rent given to non-demanders and the non-manipulability of demands. The properties are used to characterize the three solutions.

The paper is divided as follows: Section 2 describes the model and provides an example on how the optimal configuration is computed. Section 3 shows that the core can be empty. Section 4 describes and characterizes the family of methods obtained by extending Kar's characterization for mcst problems. Section 5 differentiates and characterizes three of those methods. As our model covers many different cases explored in the literature, links with these models are clarified in Section 6.

2 The model

In order to define mcst problems with indifferent agents, we first define classic mcst problems.

2.1 Minimum cost spanning tree problems

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of potential participants and $N \subseteq \mathcal{N}$ be the set of actual participants that need to be connected to the source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set $Z \subseteq \mathcal{N} \cup \{0\}$,

[1] Negative externalities coming with the network are another reason for conflict. It is not modelled here.

define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with c_e representing the cost of edge e. Let $\Gamma(N)$ be the set of all cost vectors when the set of agents is N, with $N \subseteq \mathcal{N}$. Let Γ be the set of all cost vectors, for all possible N. Since c assigns a cost to all edges e, we often abuse language and call c a cost matrix. A minimum cost spanning tree problem is a triple $(0, N, c)$. Since 0 does not change, we omit it in the following and simply identify a mcst problem as (N, c), with $N \subseteq \mathcal{N}$ and $c \in \Gamma(N)$.

A cycle p_{ll} is a set of $K \geq 3$ edges (i_k, i_{k+1}), with $k \in [0, K-1]$ and such that $i_0 = i_K = l$ and $i_1, ..., i_{K-1}$ distinct and different than l. A path p_{lm} between l and m is a set of K edges (i_k, i_{k+1}), with $k \in [0, K-1]$, containing no cycle and such that $i_0 = l$, $i_K = m$ and $i_1, ..., i_{K-1}$ distinct and different than l and m. A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges. Its associated cost is $\sum_{e \in t} c_e$.

In a classic mcst problem, the minimum cost of connecting N to the source and the associated minimum cost spanning tree is obtained using Prim's algorithm, which has $|N|$ steps. First, pick an edge $(0, i)$ such that $c_{0i} \leq c_{0j}$ for all $j \in N$. We then say that i is connected. In the second step, we choose an edge with the smallest cost connecting an agent in $N \setminus \{i\}$ either directly to the source or to i, which is connected. We continue until all agents are connected, at each step connecting an agent not already connected to an agent already connected or to the source. Let $\hat{C}(N, c)$ be the associated cost. Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $\hat{C}(S, c)$ be the cost of the mcst problem (S, c^S).

2.2 Minimum cost spanning tree problems with indifferent agents

To define minimum cost spanning tree problems with indifferent agents, we need to distinguish agents that need a connection to the source from those that do not. Let $N^d \subseteq \mathcal{N}$ be the set of players that desire a connection to the source. Let $N^t \subseteq \mathcal{N} \setminus N^d$ be the set of players that are not interested in being connected to the source, but are ready to let others use their locations to connect to the source. Again, since 0 does not change, we omit it in the following and simply identify a mcst problem with indifferent agents as $((N^d, N^t), c)$, with $N^d \cup N^t = N \subseteq \mathcal{N}$ and $c \in \Gamma(N)$.

To obtain the cost for a mcst problem with indifferent agents $((N^d, N^t), c)$, we proceed as follow. First, we compute the cost function $\hat{C}$ of the mcst problem, as if all agents wanted to be connected to the source. Then, for a coalition $S \subseteq N$, the cost $C(S, ((N^d, N^t), c)$ is the cost to connect agents in $S \cap N^d$, with the help of agents in $S \cap N^t$. Therefore, $C(S, ((N^d, N^t), c) = \min_{T \subseteq (N^t \cap S)} \hat{C}(S \cap N^d \cup T, c)$ with the convention that $\hat{C}(\emptyset, c) = 0$. Given these definitions, we say that C is the stand-alone cost function associated with the problem $((N^d, N^t), c)$.

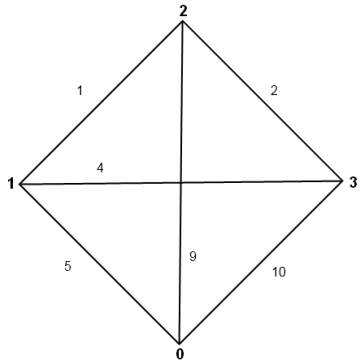


Figure 1: Example with 3 players

Figure 1 shows an example with 3 players, with identity of the players appearing in bold. Other numbers are the cost of the corresponding edges. Here are the corresponding values for $\hat{C}$ and C, first when both agents 2 and 3 want to be connected, then when only agent 3 has such a demand.

S	$\hat{C}(S)$	$C(S,(\{\{2,3\},\{1\}),c)$	$C(S,(\{\{3\},\{1,2\}),c)$
$\{1\}$	5	0	0
$\{2\}$	9	9	0
$\{3\}$	10	10	10
$\{1,2\}$	6	6	0
$\{1,3\}$	9	9	9
$\{2,3\}$	11	11	10
$\{1,2,3\}$	8	8	8

For example, to compute $C(\{1,3\},(\{\{2,3\},\{1\}),c)$, either agent 3 connects without help at a cost of $\hat{C}(\{3)\} = 10$ or with the help of agent 1 at a cost of $\hat{C}(\{1,3)\} = 9$.

2.3 Cost sharing solutions

For a problem $((N^d,N^t),c)$, a cost allocation $y \in \mathbb{R}^N$ assigns a cost share to each agent, and the budget balance condition is $\sum_{i\in N} y_i = C(N,(N^d,N^t),c)$.

A cost sharing solution (or rule) assigns a cost allocation $y((N^d,N^t),c)$ to any admissible mcst problem with indifferent agents $((N^d,N^t),c)$.

3 Possible emptiness of the core

As in the Steiner tree problem, and unlike in minimum cost spanning tree problems, the core can be empty for mcst problems with indifferent agents. This means that we might not be able to find an allocation that removes for all coalitions the incentives to secede from the large group. Formally, the core is the set of allocations $y \in \mathbb{R}^N$ such that $\sum_{i\in S} y_i \leq C(S,(N^d,N^t),c)$ for all $S \subset N$ and $\sum_{i\in N} y_i = C(N,(N^d,N^t),c)$. The following example shows that it can be empty.[2]

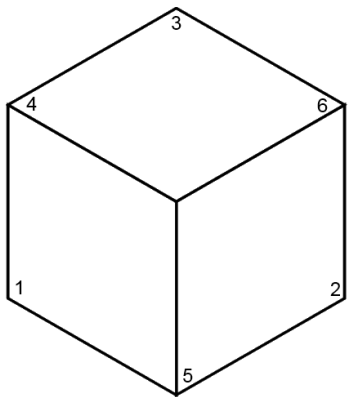


Figure 2: Example with empty core

We have an illustration of cost matrix c in Figure 2, with represented edges having a cost of one while unrepresented edges have a cost of ten. Suppose that $N^d = \{1,2,3\}$ and $N^t = \{4,5,6\}$. It is

[2] This example was provided by Gustavo Bergantinos, Juan Vidal-Puga, Maria Gomez-Rua and Leticia Lorenzo during a visit at the University of Vigo.

easy to see that the cost for the grand coalition is 5, while the costs for coalitions $\{1,2,5\}$, $\{1,3,4\}$ and $\{2,3,6\}$ are all of 3. This generates the following core constraints:

$$\begin{aligned} y_1 + y_2 + y_5 &\leq 3 \\ y_1 + y_3 + y_4 &\leq 3 \\ y_2 + y_3 + y_6 &\leq 3. \end{aligned}$$

Summing these up, we obtain

$$\begin{aligned} 2(y_1 + y_2 + y_3) + y_4 + y_5 + y_6 &\leq 9 \\ y_1 + y_2 + y_3 &\leq 4 \text{ (using } \sum_{i=1}^{6} y_i = 5) \end{aligned}$$

This means that $y_4 + y_5 + y_6 \geq 1$. However, since $N^t = \{4,5,6\}$, $C(\{4,5,6\},(N^d,N^t),c) = 0$. Thus, the core is empty.

4 Extending Kar's characterization

Kar's characterization uses two properties: Group Independence and Equal Treatment.[3] Group Independence conveys the idea that if two groups of agents always connect to the source in an independent manner, then computing the cost shares on the whole group or independently on the two groups makes no difference. This is a natural property, and without it, we allow for one group to subsidize the other, for costs it is clearly not responsible for. The concept applies as well to our problem than to mcst problems. To formally define the property we define *irrelevant edges* as follows: for $i,j \in N$, we say that (i,j) is an irrelevant edge if $c_{ij} \geq c_{0i}, c_{0j}$. Otherwise, we say it is a *relevant edge*. Relevant edges will be used by coalition $\{i,j\}$ if both agents want to connect to the source and nobody cooperates with them. Irrelevant edges will not be used in that situation, or in any other.

Group Independence: if there exists $S \subseteq N^d$ and $T \subseteq N^t$ such that (i,j) is an irrelevant edge for all $i \in S \cup T$, $j \in N \backslash (S \cup T)$, then $y_i((N^d,N^t),c) = y_i((S,T),c^{S \cup T})$ if $i \in S \cup T$ and $y_i((N^d,N^t),c) = y_i((N^d \backslash S, N^t \backslash T), c^{N \backslash (S \cup T)})$ if $i \in N \backslash (S \cup T)$.

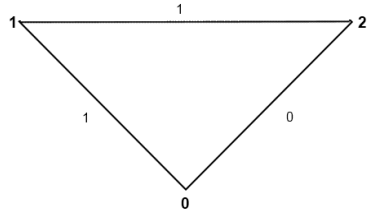


Figure 3: Two-player example

Consider a problem where $\{1\} \subseteq N^d$ and c is as illustrated in Figure 3. By Group Independence, we must have that agent 1 pays 1 and agent 2 pays nothing, whether he wants to be connected or not. Suppose now that the cost of the edge $(1,2)$ drops to zero. This generates a saving of 1, as agent 1 can now connect to the source through the node of agent 2. How should we allocate this saving between the two agents? For mcst problems, Kar (2002) uses the Equal Treatment principle: since the agents have an equal property right to an edge between them, the savings generated should be split equally.

[3] To be exact, Group Independence is divided in two distinct properties in Kar (2002). However, a solution satisfies these two properties if and only if it satisfies Group Independence.

When we have more than two players, the concept is naturally extended: if the cost of the edge (i, j) decreases, with all else constant, the changes in the cost shares of i and j will be equal. Notice that this principle does not mean that each agent pockets half of the savings: the shares of other agents could also be affected, with the combined changes to the shares of i and j possibly larger or smaller than the savings. Formally, we adapt the Equal Treatment principle to our context as follows:

Equal Treatment: Let c, c' be such that $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else for $i, j \in N$. Then,

$$y_i((N^d, N^t), c') - y_i((N^d, N^t), c) = y_j((N^d, N^t), c') - y_j((N^d, N^t), c).$$

What makes this property particularly attractive for mcst problems is the fact that in addition to a pair of agents having equal property rights to the edge that runs between them, they also have an equal desire to be connected to the source. Alone, each agent i would build the edge $(0, i)$ to connect to the source. Cooperation between agents 1 and 2 gives them the possibility to connect one of the agents through the edge $(1, 2)$. If one of the agents does not want to be connected, this symmetry is gone, as edge $(1, 2)$ can only be used to connect the agent that wants a link to the source.

Still, it is natural to keep the property if both agents have an identical desire to connect to the source, a property that we formalize as follows:

Equal Treatment for Equal Demands (ETED): Let c, c' be such that for $i, j \in N$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else If $i, j \in N^d$ or $i, j \in N^t$, then

$$y_i((N^d, N^t), c') - y_i((N^d, N^t), c) = y_j((N^d, N^t), c') - y_j((N^d, N^t), c).$$

Consider now the case of agents that are not identical with respect to their demand. In the example above, we might now believe that because the two agents play different roles in the creation of the saving, we should allow them different shares of it. It is not clear what should be the shares of the demander and the non-demander. However, it seems very natural to assume that once we have decided on how the saving should be split, this decision should be applied consistently in any case where we need to arbitrage between the share of a demander and a non-demander. With more than two agents, this means that the changes in the cost share of the demander should be proportional to the changes in the cost share of the non-demander. The coefficient of proportionality reflects our views on how to share savings between the agents. This idea is formalized as follows:

Proportional Treatment for Unequal Demands (PTUD): Let $\alpha \in \mathbb{R}$, c, c' be such that for $i \in N^d$ and $j \in N^t$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. Then,

$$\alpha \left[y_i((N^d, N^t), c') - y_i((N^d, N^t), c) \right] = (1 - \alpha) \left[y_j((N^d, N^t), c') - y_j((N^d, N^t), c) \right].$$

It could be argued that it would be more natural to write the above equality as

$$\left[y_i((N^d, N^t), c') - y_i((N^d, N^t), c) \right] = \beta \left[y_j((N^d, N^t), c') - y_j((N^d, N^t), c) \right].$$

However, our choice of notation allows us to define three interesting cases. If $\alpha = 0$, then $j \in N^t$ is unaffected by the change. If $\alpha = 1$, then the cost share of $i \in N^d$ stays constant. Finally, if $\alpha = \frac{1}{2}$, it is just as if we applied Equal Treatment to agents that do not have the same demand.

We are now ready for the main result.

Theorem 1 *Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands characterize a unique cost sharing solution for each value of $\alpha \in \mathbb{R}$, the parameter in Proportional Treatment for Unequal Demands*

Proof. The proof bears similarities to the one in Kar (2002), although the use of ETED and PTUD instead of Equal Treatment requires a few more steps.

First, suppose that c has no relevant edges. By multiple applications of Group Independence and budget balance, we have $y_i(\left(N^d, N^t\right), c) = y_i((\{i\}, \emptyset), c^{\{i\}}) = c_{0i}$ if $i \in N^d$ and $y_i(\left(N^d, N^t\right), c) = y_i((\emptyset, \{i\}), c^{\{i\}}) = 0$ if $i \in N^t$.

In the following, $\left(N^d, N^t\right)$ is fixed. To simplify the notation, we write $y(c)$ for $y(\left(N^d, N^t\right), c)$ where there is no confusion. Fix $\alpha \in \mathbb{R}$, the parameter in PTUD and suppose that for all c with $m-1$ relevant edges, all cost sharing methods satisfying the properties coincide with $y(c)$. We show that it implies that they coincide with $y(c)$ if c has m relevant edges.

We proceed by contradiction. Suppose that there exists $z(c)$ that satisfies Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands but such that $z(c) \neq y(c)$.

By Group Independence, we can suppose that for any S in N, there exists a relevant edge (i, j), with $i \in S$ and $j \in N \backslash S$. (If it wasn't the case, we could apply Group Independence and $y_i(\left(N^d, N^t\right), c) = y_i(\left(N^d \cap S, N^t \cap S\right), c^S)$ for all $i \in S$.).

Suppose that for $k, l \in N$, (k, l) is a relevant edge. Let c^{kl} be such that $c^{kl}_{kl} = \max(c_{0k}, c_{0l})$ and $c^{kl}_e = c_e$ for all $e \neq (k, l)$. Thus, c^{kl} has $m-1$ relevant edges.

By our induction assumption, we have $y(c^{kl}) = z(c^{kl})$. If $k, l \in N^d$ or $k, l \in N^t$, we have

$$\begin{aligned} y_k(c) - y_l(c) &= y_k(c^{kl}) - y_l(c^{kl}) \text{ by ETED,} \\ &= z_k(c^{kl}) - z_l(c^{kl}) \text{ by induction,} \\ &= z_k(c) - z_l(, c) \text{ by ETED.} \end{aligned}$$

Therefore, we have

$$y_k(c) - z_k(c) = y_l(c) - z_l(c). \tag{1}$$

If $k \in N^d$ and $l \in N^t$, we have

$$\begin{aligned} \alpha\left[y_k(c) - y_k(c^{kl})\right] &= (1-\alpha)\left[y_l(c) - y_l(c^{kl})\right] \text{ by PTUD,} \\ \alpha\left[y_k(c) - z_k(c^{kl})\right] &= (1-\alpha)\left[y_l(c) - z_l(c^{kl})\right] \text{ by induction,} \\ \alpha y_k(c) - (1-\alpha) y_l(c) &= \alpha z_k(c^{kl}) - (1-\alpha) z_l(c^{kl}). \end{aligned}$$

We also have that

$$\begin{aligned} \alpha\left[z_k(c) - z_k(c^{kl})\right] &= (1-\alpha)\left[z_l(c) - z_l(c^{kl})\right] \text{ by PTUD,} \\ \alpha z_k(c) - (1-\alpha) z_l(c) &= \alpha z_k(c^{kl}) - (1-\alpha) z_l(c^{kl}). \end{aligned}$$

Combining the two results, we obtain

$$\alpha y_k(c) - (1-\alpha) y_l(c) = \alpha z_k(c) - (1-\alpha) z_l(c).$$

Rearranging, we have

$$\alpha\left[y_k(c) - z_k(c)\right] = (1-\alpha)\left[y_l(c) - z_l(c)\right]. \tag{2}$$

For any $i, j \in N$, by our assumption that for any S in N there exists a relevant edge (m_1, m_2) with $m_1 \in S$ and $m_2 \in N \backslash S$, there exists $i_1, i_2, ..., i_L$ such that (i_l, i_{l+1}) is relevant for all $l = 1, ..., L-1$ and $i_1 = i$, $i_L = j$. Thus, by multiple applications of (1) and (2), we obtain that equation (1) holds for any pair of agents k, l such that $k, l \in N^d$ or $k, l \in N^t$ and that equation (2) holds for any $k \in N^d$ and $l \in N^t$, regardless of whether the edge (j, k) is relevant or not.

We consider two cases depending on the value of α.

Case i) $\alpha = 1$

Equation (2) simplifies to $y_k(c) - z_k(c) = 0$ for all $k \in N^d$. By budget balance, we must have that $\sum_{i \in N}\left[y_i(c) - z_i(c)\right] = 0$, which here simplifies to $\sum_{i \in N^t}\left[y_i(c) - z_i(c)\right] = 0$. By equation (1), we have that $y_k(c) - z_k(c) = y_l(c) - z_l(c)$ for all $k, l \in N^t$. Directly, this implies that $y_i(c) = z_i(c)$ for all $i \in N$, yielding the desired contradiction.

Case ii) $\alpha \neq 1$

By equation (1), we have that $y_k(c) - z_k(c) = y_l(c) - z_l(c) \equiv \beta \in \mathbb{R}$ for all $k, l \in N^d$. By equation (2), $y_l(c) - z_l(c) = \frac{\alpha}{1-\alpha}\beta$ for all $l \in N^t$. By budget balance, we must have that

$$\begin{aligned} \sum_{i \in N} [y_i(c) - z_i(c)] &= 0 \\ \beta \left(\left|N^d\right| + \left|N^t\right| \frac{\alpha}{1-\alpha} \right) &= 0. \end{aligned}$$

This gives us two possibilities: either $\beta = 0$ or $\alpha = \frac{|N^d|}{|N^d|-|N^t|}$. Suppose we set $\alpha = \frac{|N^d|}{|N^d|-|N^t|}$. This possibility is clearly incompatible with Group Independence as separating S from $N \backslash S$ would change the number of demanders and non-demanders in each sub-problems. Therefore, we must have that $\beta = 0$, which means that $y_i(c) = z_i(c)$ for all $i \in N$, yielding the desired contradiction. ■

Lemma A.1 in Appendix shows that the properties are independent.

The theorem informs us that for each α, there is a unique solution satisfying the properties. Unfortunately, given the complexity of the underlying optimization problem, we are not able to provide an expression for all the members of this family. However, we identify three members of the family, corresponding to the three special cases of α identified above.

First, we define the extension of the Kar solution, which is nothing more than the Shapley-Shubik solution, to our setting. Let $y^K((N^d, N^t), c) = Sh(C(\cdot, (N^d, N^t), c))$.

Next, we define a variant of the Shapley-Shubik method that disregards the contributions of agents in N^t. Let $\bar{C}$ be defined for all $S \subseteq N^d$ and such that $\bar{C}(S, (N^d, N^t), c) = C(S \cup N^t, (N^d, N^t), c)$. Then, $y_i^0((N^d, N^t), c) = Sh_i(\bar{C}(\cdot, (N^d, N^t), c))$ for all $i \in N^d$ and $y_i^0((N^d, N^t), c) = 0$ for all $i \in N^t$. Notice that to compute $\bar{C}$, we assume that demanders can always use the locations of non-demanders as they wish, without providing compensations.

Finally, the third solution takes the opposite stand and assigns all of the gains coming from technological cooperation to the agents in N^t. Let C^{N^d} be defined for all $S \subseteq N^d$ and such that $C^{N^d}(S, (N^d, N^t), c) = C(S, (N^d, N^t), c)$. Let $\tilde{C}$ be defined for all $S \subseteq N^t$ and such that $\tilde{C}(S, (N^d, N^t), c) = C(S \cup N^d, (N^d, N^t), c) - C(N^d, (N^d, N^t), c)$. Then, $y_i^1((N^d, N^t), c) = Sh_i(C^{N^d}(\cdot, (N^d, N^t), c))$ for all $i \in N^d$ and $y_i^1((N^d, N^t), c) = Sh_i(\tilde{C}(\cdot, (N^d, N^t), c))$ for all $i \in N^t$. Here, C^{N^d} is as if demanders never had access to the locations of non-demanders. $\tilde{C}$ is the function assigning the cost savings generated by the locations of any coalition of non-demanders.

We have the following result:

Lemma 1 *Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands characterize*

i) y^0 *if* $\alpha = 0$.

ii) y^K *if* $\alpha = \frac{1}{2}$.

iii) y^1 *if* $\alpha = 1$.

Proof. In Appendix. ■

As α summarizes how we value the contribution of non-demanders versus demanders, if we don't value the contribution of non-demanders ($\alpha = 0$), we obtain a cost sharing method where non-demanders receive nothing and demanders can use their location without compensation. If we don't value the contribution of demanders ($\alpha = 1$), non-demanders share the benefits their locations create, while demanders only get credit for the economies of scale. If we assume that the demand and advantageous locations are as important in creating savings ($\alpha = \frac{1}{2}$) we obtain the Kar solution that applies the Shapley value to the stand-alone game.

Example 1 *We provide an example to illustrate the solutions proposed above. We find in Figure 4 the structure, with agents appearing in bold. Other numbers are the cost of the corresponding edges.*

The following table provides the cost shares prescribed by y^0, y^K *and* y^1, *as well as for the case with a generic* α. *Shares are computed for three cases, first when only agent 3 wants to be connected,*

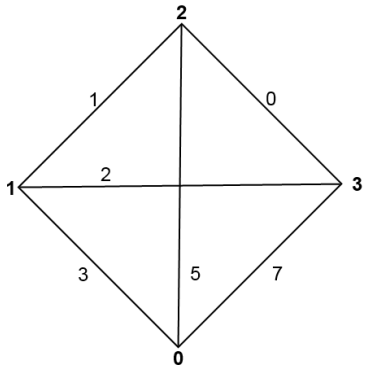


Figure 4: Illustration of cost sharing solutions

then when agents 2 and 3 need to be connected and finally when the connection is desired by all three agents. For the case with a generic α, cost shares were computed in the following manner: we started by modifying the cost matrix so there were no relevant edges (so that $c_{12} = 5$, $c_{13} = c_{23} = 7$), obtaining cost shares by Group Independence. One by one, relevant edges were added. Moving from k relevant edges to $k+1$, we obtain a system of equations by ETED and/or PTUD, as well as by budget balance. We then solved the system to obtain the cost shares.

	$N^d = \{3\}$	$N^d = \{2,3\}$	$N^d = \{1,2,3\}$
y^0	$(0,0,4)$	$\left(0, \frac{3}{2}, \frac{5}{2}\right)$	$\left(\frac{1}{6}, \frac{7}{6}, \frac{8}{3}\right)$
y^K	$\left(-\frac{11}{18}, -\frac{11}{18}, \frac{47}{9}\right)$	$\left(-\frac{5}{6}, \frac{5}{3}, \frac{19}{6}\right)$	$\left(\frac{1}{6}, \frac{7}{6}, \frac{8}{3}\right)$
y^1	$\left(-\frac{3}{2}, -\frac{3}{2}, 7\right)$	$\left(-1, \frac{3}{2}, \frac{7}{2}\right)$	$\left(\frac{1}{6}, \frac{7}{6}, \frac{8}{3}\right)$
y^α	$\left(-\frac{\alpha[3\alpha^2+2\alpha+1]}{(1+\alpha)^2}, -\frac{\alpha[3\alpha^2+2\alpha+1]}{(1+\alpha)^2}, \frac{2[3\alpha^3+4\alpha^2+5\alpha+2]}{(1+\alpha)^2}\right)$	$\left(\frac{3\alpha^2-4\alpha}{2-\alpha}, \frac{6-\alpha-2\alpha^2}{2(2-\alpha)}, \frac{10+\alpha-4\alpha^2}{2(2-\alpha)}\right)$	$\left(\frac{1}{6}, \frac{7}{6}, \frac{8}{3}\right)$

A first observation is that α enters in a non-linear fashion in the cost shares, and thus that y^K is in no way a weighted average of y^0 and y^1. The last case, where $N^d = \{1,2,3\}$, is a classic mcst problem. In that case, PTUD is not necessary, as all agents have an equal desire to be connected to the source. Thus, all methods prescribe the same cost shares. It is also interesting to see that even though we have the intuition that a higher α is more favorable to agents in N^t, cost shares for these agents are not necessarily decreasing in α. For example, $y_1((\{2,3\},\{1\}),c)$ reaches a minimum at $\alpha = \frac{6-2\sqrt{3}}{3} \approx 0.845$. When $\alpha = 1$, non-demanders get all of the benefits coming from their location, but only their contributions when joining the whole group of demanders are considered. For a lower α, the contributions to smaller groups of demanders also get partially remunerated.

We conclude this section by noting once again that the mcst problem with indifferent agents is not solvable in polynomial time. There are very few examples of characterizations of solutions for such problems. While ETED and PTUD provide structure to the cost sharing methods, it is somewhat inevitable, as we know very little about the problem and its associated cost function.

5 Characterizing the three solutions

In order to distinguish the three methods defined in the previous section, we define a set of simple properties. We start by requiring that no agent pay more than its stand alone cost, a weak stability property.

Individual Rationality: Let $c \in \Gamma(N)$. We have that $y_i((N^d,N^t),c) \leq C(\{i\},(N^d,N^t),c)$. More precisely, $y_i((N^d,N^t),c) \leq c_{0i}$ if $i \in N^d$ and $y_i((N^d,N^t),c) \leq 0$ if $i \in N^t$.

It should be clear that even though Theorem 1 holds for any $\alpha \in \mathbb{R}$, Individual Rationality limits us to $\alpha \in [0,1]$. To see this, start with a c with no relevant edge. Then, $y_i = c_{0i}$ for all $i \in N^d$ and $y_i = 0$ for all $i \in N^t$, by Group Independence. Let c' be identical except that we now have a relevant edge between $i \in N^d$ and $j \in N^t$. If $\alpha \notin [0,1]$, one agent will see its cost share decrease, while it will increase for the other. Any increase violates Individual Rationality.

The next properties take advantage of the differences between demanders and non-demanders. In general, when an agent wants to be connected to the source, we take into account two different factors to compute her cost share: how costly it is to connect her to the source, of which she should pay a positive share, and how much savings is obtained by letting others connect through her location, for which she could receive a subsidy. For an agent that does not want to be connected to the source, only the second consideration matters. Following these observations, the next two properties are very natural.

Demand Monotonicity: Let $c \in \Gamma(N)$ and $i \in N^t$. We have that $y_i((N^d \cup \{i\}, N^t \setminus \{i\}), c) \geq y_i((N^d, N^t), c)$.

Demand Ranking: Let $c \in \Gamma(N)$ and $i \in N^t$, $j \in N^d$. If $c_{ik} = c_{jk}$ for all $k \in N_0 \setminus \{i,j\}$, then $y_i((N^d, N^t), c) \leq y_j((N^d, N^t), c)$.

Demand Monotonicity imposes that if an agent indifferent to a connection to the source suddenly desires such a connection, his change of mind cannot decrease his cost share. In standard cost sharing models, it was introduced by Moulin (1995) and used, among many others, by Friedman and Moulin (1999) and Moulin and Sprumont (2006). Demand Ranking requires that if two agents are located in symmetric locations but only one of them wants to be connected to the source, that agent cannot pay less than the other one. The property is similar in nature to the Fair Ranking property of Moulin and Shenker (1992). In addition to making agents responsible for their demand, Demand Monotonicity and Demand Ranking are also required if the agents can manipulate their (reported) demand to pay a smaller share.

The final two properties deal with the rent that demanders pay to non-demanders for the use of their locations. Since we have seen that the core can be empty, we propose a weaker stability condition that requires that the rent paid by non-demanders be no larger than the savings they generate as a group. Stated otherwise, the demanders, as a group, should not be worse off when they cooperate with non-demanders. The property was introduced by Bahel and Trudeau (2012).

Fair Rent: Let $c \in \Gamma(N)$. We have that $\sum_{i \in N^d} y_i((N^d, N^t), c) \leq C(N^d, (N^d, N^t), c)$.

One trivial way of satisfying Fair Rent is to never subsidized any of the non-demanders. Our final property requires that solutions sometimes subsidize non-demanders. It thus eliminates solutions that completely ignore the presence of non-demanders.

Positive Rent : There exists N^d, N^t, $c \in \Gamma(N)$ and $i \in N^t$ such that $y_i((N^d, N^t), c) < 0$.

The following Lemma describes which properties are satisfied by our three solutions.

Lemma 2 *i) y^0 satisfies Individual Rationality, Demand Ranking and Fair Rent but fails Demand Monotonicity and Positive Rent.*

ii) y^K satisfies Individual Rationality, Demand Monotonicity, Demand Ranking and Positive Rent but fails Fair Rent.

iii) y^1 satisfies Individual Rationality, Fair Rent and Positive Rent but fails Demand Monotonicity and Demand Ranking.

Proof. In Appendix. ■

The properties can also be used to characterize the solutions.

Theorem 2 *A cost sharing solution y that satisfies Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands also satisfies*

i) Demand Ranking and Fair Rent if and only if $y = y^0$.

ii) Individual Rationality, Demand Monotonicity, and Demand Ranking if and only if $y = y^K$.

iii) Individual Rationality, Fair Rent and Positive Rent if and only if $y = y^1$.

Proof. In the following, we use the fact that by Theorem 1, a solution that satisfies Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands is uniquely defined by α, the parameter in PTUD.

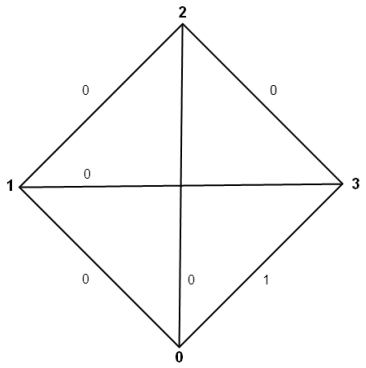


Figure 5: Example, Theorem 2

Consider the cost matrix c depicted in Figure 5. Let c^i be the matrix where we change the cost of $(i,3)$ to zero and $c^{1,2}$ the cost matrix where edges $(1,3)$ and $(2,3)$ both have a cost of zero. The shares for these cost matrices are given in Appendix, for a generic α (the parameter in PTUD) and for $N^t = \{1,2\}$ and $\{1\}$.

It is clear that any $\alpha \notin [0,1]$ violates Individual Rationality. Consider c^2. We have that $y_2((\{2,3\},\{1\}),c^2) = -\frac{1}{2}$ and that $y_2((\{3\},\{1,2\}),c^2) = -\alpha$. Demand Monotonicity imposes that $y_2((\{3\},\{1,2\}),c^2) \leq y_2((\{2,3\},\{1\}),c^2)$, or $\alpha \geq \frac{1}{2}$.

Consider $c^{1,2}$. We have that $y_1((\{2,3\},\{1\}),c^{1,2}) = -\frac{\alpha(1-\alpha)}{2-\alpha}$ and $y_2((\{2,3\},\{1\}),c^{1,2}) = -\frac{\alpha}{2(2-\alpha)}$. Notice that 1 and 2 have the same cost to connect to the source and to agent 3. We can thus apply Demand Ranking, which imposes that $y_1((\{2,3\},\{1,2\}),c^{1,2}) \leq y_2((\{2,3\},\{1\}),c^{1,2})$, Demand Ranking is thus satisfied if $\alpha \in \left[0,\frac{1}{2}\right]$ or $\alpha > 2$.

By Fair Rent, we must have that $y_1((\{2,3\},\{1\},c^{1,2}) \geq 0$, (as agents 2 and 3 can connect independently at a cost of zero). Fair Rent is thus satisfied if $\alpha \leq 0$ or if $\alpha \in [1,2)$.

i) The restrictions imposed by Demand Ranking and Fair Rent leave us with $\alpha = 0$. By Lemma 1 , our only candidate is y^0. By Lemma 2, y^0 satisfies all properties.

ii) The restrictions imposed by Individual Rationality, Demand Monotonicity and Demand Ranking leave us with $\alpha = \frac{1}{2}$. By Lemma 1, our only candidate is y^K. By Lemma 2, y^K satisfies all properties.

iii) The restrictions imposed by Individual Rationality and Fair Rent leaves us with $\alpha = 0$ or $\alpha = 1$. By Lemma 1 , it leaves us with y^0 and y^1. By Lemma 2, y^0 fails Positive Rent while y^1 satisfies it. y^1 also satisfies the other properties. ■

The previous theorem allows for interesting contrasts between the solutions. If we fear manipulations and require both Demand Monotonicity and Demand Ranking, y^K is the only individually rational solutions left. This should come as no surprise, as y^K, because it satisfies Equal Treatment, makes no distinction between demanders and non-demanders and simply looks at the cost they generate. If we put the accent on a stability property like Fair Rent but still want to remunerate non-demanders, y^1 is the only individually rational choice. y^0 turns out be characterized by combining an incentive property (Demand Ranking) and a stability property (Fair Rent). This result can be seen as the impossibility of combining Fair Rent, Demand Ranking and Positive Rent.

It should be noted that given that we do not have closed forms for the solutions of Theorem 1 for which $\alpha \notin \left\{0,\frac{1}{2},1\right\}$, it is difficult to fully appreciate some of these properties. In particular, among these solutions, only y^K is known to satisfy Demand Monotonicity. It is possible that the characterization of y^K could be done by only adding Demand Monotonicity to the properties of Theorem 1. Given

that y^1 satisfies Demand Monotonicity when there are three players but not always for four players, it seems unlikely that we can fully appreciate its impact on α.

In a similar fashion, the lack of closed-form solutions with $\alpha \notin \left\{0, \frac{1}{2}, 1\right\}$ prevents us from completely verifying the independence of the properties we added. For example, we know of no methods that satisfy Fair Rent and Positive Rent but not Individual Rationality, although the results above tell us that the candidate would be such that $\alpha \in (1, 2)$. Given that we do not know what these methods look like, it is impossible to verify if they satisfy Fair Rent.

6 Linear network, shortest path and Steiner problems

The case where $N^d = N$ is the well-known minimum cost spanning tree model. It turns out that at the opposite end of the spectrum, when $\left|N^d\right| = 1$, we have another well-known model. In that case, we only need to connect one agent to the source, with the (potential) help of other members of N. In our model, it means that we need to find the cheapest path from the source to i. The problem thus becomes a shortest path problem, solvable in polynomial time (Dijkstra (1959)). As for the mcst problem, the corresponding stand-alone cost game always has a non-empty core.

We can also see that it is equivalent to a much more general problem: suppose that agents in N have a demand $q \in \mathbb{R}_+^N$. They need to be connected to the source to obtain these goods, which have to be transported from the source to the nodes of these agents. We still have the cost matrix c, but now the cost to send x units of goods on edge (i, j) is $c_{ij}x$. A linear network problem is (N, c, q), the set of agents, the cost matrix and the demand vector. It is easy to see that it is also a shortest path problem. For each agent i, we find the shortest path connecting him to the source. The optimal network configuration for N is obtained by combining the individual shortest paths. See Trudeau (2009b) and Rosenthal (2013) for a discussion of this case. Since the network can be built in a decentralized manner, it seems very natural to impose that cost shares could also be decentralized and that $y(N, c, q) = \sum_{j \in N} y(N, c, e^j)q_j$, where $e^j \in \mathbb{R}_+^N$ is such that $e_k^j = 1$ if $j = k$ and $e_k^j = 0$ otherwise. Therefore, assigning cost shares to the whole linear network problem can be done by assigning cost shares for individual units of demand and summing up. But then, the linear network problem (N, c, e^j), which consists in finding the shortest path between the source and j to send him one unit of demand, is identical to the mcst problem with indifferent agents $((\{j\}, N \backslash \{j\}), c)$, where we need to build a mcst from the source to j.

It is worth noting that in that case, y^K corresponds to the Shapley value studied in Rosenthal (2013). The linear network model is also very similar to the model in Trudeau (2009a), which is a general technological cooperation model with constant returns to scale. The only difference is that there is no underlying network structure. For cases where we do not want to reward demanders (for instance if it generates pollution), a method is proposed that starts by assigning each agent its stand-alone cost before giving rebates related to the savings generated by its technology. It corresponds to y^1 for linear network problems. For cases where $\left|N^d\right| > 1$, y^1 is a very natural extension of the solution of Trudeau (2009a) that deals with economies of scale.

As mentioned in the introduction, we can view the minimum cost Steiner problem of Bergantinos et al. (2011) as our model with the added constraint that $y_i = 0$ for all $i \in N^t$. If we add to this constraint Group Independence and ETED, we characterize y^0, which is nothing but the Shapley-Shubik solution for such problems.

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A Appendix

A.1 Proof of Lemma 1

Proof. Theorem 1 shows that Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands characterize a unique solution for any $\alpha \in \mathbb{R}$, the parameter in PTUD. We thus need to show that the solutions satisfy Group Independence and Equal Treatment for Equal Demands, as well as Proportional Treatment for Unequal Demands for the corresponding α.

i) We obtain y^0 if $\alpha = 0$.

We show that y^0 satisfies Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands for $\alpha = 0$, proving that y^0 is the unique solution when $\alpha = 0$.

Suppose that for all $i \in S, j \in N\backslash S$, there are no relevant edges. Then, $\bar{C}((R \cup T) \cap N^d, (N^d, N^t), c) = \bar{C}(R \cap N^d, (N^d, N^t), c) + \bar{C}(T \cap N^d, (N^d, N^t), c)$ for all $R \subseteq S$ and $T \subseteq N\backslash S$. By the properties of the Shapley Value, $Sh_i(\bar{C}(\cdot, (N^d, N^t), c)) = Sh_i(\bar{C}(\cdot, (N^d \cap S, N^t \cap S), c^S))$ for all $i \in S \cap N^d$ and $Sh_j(\bar{C}(\cdot, (N^d, N^t), c)) = Sh_j(\bar{C}(\cdot, (N^d\backslash S, N^t\backslash S), c^{N\backslash S}))$ for all $j \in N^d\backslash S$. Since $y_i^0((N^d, N^t), c) = 0$ for all $i \in N^t$ and all c, Group Independence is satisfied.

Let c, c' be such that for $i, j \in N^d$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. The edge (i, j) can only be used by a coalition S if $i, j \in S$. Therefore, $\bar{C}(T, (N^d, N^t), c') = \bar{C}(T, (N^d, N^t), c)$ for all $T \subseteq N$ such that $\{i, j\} \subsetneq T$. By the properties of the Shapley value, $y_i^0((N^d, N^t), c') - y_i^0((N^d, N^t), c) = y_j^0((N^d, N^t), c') - y_j^0((N^d, N^t), c)$. Since $y_i^0((N^d, N^t), c) = 0$ for all $i \in N^t$ and all c, the part of ETED that applies to agents in N^t is satisfied trivially. Therefore, ETED is satisfied.

Let c, c' be such that for $i \in N^d$ and $j \in N^t$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. Since $y_j^0((N^d, N^t), c) = y_j^0((N^d, N^t), c') = 0$, PTUD is satisfied for $\alpha = 0$.

ii) We obtain y^K if $\alpha = \frac{1}{2}$.

We show that y^K satisfies Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands for $\alpha = \frac{1}{2}$, proving that y^K is the unique solution when $\alpha = \frac{1}{2}$.

Suppose that for all $i \in S$, $j \in N \backslash S$, there are no relevant edges. Then, $C(R \cup T, (N^d, N^t), c) = C(R, (N^d, N^t), c) + C(T, (N^d, N^t), c)$ for all $R \subseteq S$ and $T \subseteq N \backslash S$. By the properties of the Shapley Value, $Sh_i(C(\cdot, (N^d, N^t), c)) = Sh_i(C(\cdot, (N^d \cap S, N^t \cap S), c^S))$ for all $i \in S$ and $Sh_j(C(\cdot, (N^d, N^t), c)) = Sh_j(C(\cdot, (N^d \backslash S, N^t \backslash S), c^{N \backslash S}))$ for all $j \in N \backslash S$. Group Independence is satisfied.

Let c, c' be such that for $i, j \in N^d$ or $i, j \in N^t$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. The edge (i, j) can only be used by a coalition S if $i, j \in S$. Therefore, $C(T, (N^d, N^t), c') = C(T, (N^d, N^t), c)$ for all $T \subseteq N$ such that $\{i, j\} \subsetneq T$. By the properties of the Shapley value, $y_i^K((N^d, N^t), c') - y_i^K((N^d, N^t), c) = y_j^K((N^d, N^t), c') - y_j^K((N^d, N^t), c)$. Therefore, ETED is satisfied.

Let c, c' be such that for $i \in N^d$ and $j \in N^t$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. The edge (i, j) can only be used by a coalition S if $i, j \in S$. Therefore, $C(T, (N^d, N^t), c') = C(T, (N^d, N^t), c)$ for all $T \subseteq N$ such that $\{i, j\} \subsetneq T$. By the properties of the Shapley value, $y_i^K((N^d, N^t), c') - y_i^K((N^d, N^t), c) = y_j^K((N^d, N^t), c') - y_j^K((N^d, N^t), c)$. Thus, PTUD is satisfied for $\alpha = \frac{1}{2}$.

iii) We obtain y^1 if $\alpha = 1$.

We show that y^1 satisfies Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands for $\alpha = 1$, proving that y^1 is the unique solution when $\alpha = 1$.

Suppose that for all $i \in S$, $j \in N \backslash S$, there are no relevant edges. Then, $C^{N^d}((R \cup T) \cap N^d, (N^d, N^t), c) = C^{N^d}(R \cap N^d, (N^d, N^t), c) + C^{N^d}(T \cap N^d, (N^d, N^t), c)$ for all $R \subseteq S$ and $T \subseteq N \backslash S$. By the properties of the Shapley Value, $Sh_i(C^{N^d}(\cdot, (N^d, N^t), c)) = Sh_i(C^{N^d}(\cdot, (N^d \cap S, N^t \cap S), c^S))$ for all $i \in S \cap N^d$ and $Sh_j(C^{N^d}(\cdot, (N^d, N^t), c)) = Sh_j(C^{N^d}(\cdot, (N^d \backslash S, N^t \backslash S), c^{N \backslash S}))$ for all $j \in N^d \backslash S$. We also have that $\tilde{C}((R \cup T) \cap N^t, (N^d, N^t), c) = \tilde{C}(R \cap N^t, (N^d, N^t), c) + \tilde{C}(T \cap N^t, (N^d, N^t), c)$ for all $R \subseteq S$ and $T \subseteq N \backslash S$. By the properties of the Shapley value $Sh_i(\tilde{C}(\cdot, (N^d, N^t), c)) = Sh_i(\tilde{C}(\cdot, (N^d \cap S, N^t \cap S), c^S))$ for all $i \in S \cap N^t$ and $Sh_j(\tilde{C}(\cdot, (N^d, N^t), c)) = Sh_j(\tilde{C}(\cdot, (N^d \cap T, N^t \cap T), c^{N \backslash S}))$ for all $j \in N^t \backslash S$. Group Independence is satisfied.

Let c, c' be such that for $i, j \in N^d$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. The edge (i, j) can only be used by a coalition S if $i, j \in S$. Therefore, $C^{N^d}(T, (N^d, N^t), c') = C^{N^d}(T, (N^d, N^t), c)$ for all $T \subseteq N$ such that $\{i, j\} \subsetneq T$. By the properties of the Shapley value, $y_i^1((N^d, N^t), c') - y_i^1((N^d, N^t), c) = y_j^1((N^d, N^t), c') - y_j^1((N^d, N^t), c)$. Let c, c' be such that for $i, j \in N^t$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. The edge (i, j) can only be used by a coalition S if $i, j \in S$. Therefore, $\tilde{C}(T, (N^d, N^t), c') = \tilde{C}(T, (N^d, N^t), c)$ for all $T \subseteq N$ such that $\{i, j\} \subsetneq T$. By the properties of the Shapley value, $y_i^1((N^d, N^t), c') - y_i^1((N^d, N^t), c) = y_j^1((N^d, N^t), c') - y_j^1((N^d, N^t), c)$. Therefore, ETED is satisfied.

Let c, c' be such that for $i \in N^d$ and $j \in N^t$, $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else. Since $C^{N^d}(S, (N^d, N^t), c) = C^{N^d}(S, (N^d, N^t), c')$ for all $S \subseteq N^d$, we have that $y_i^1((N^d, N^t), c) = y_i^1((N^d, N^t), c')$. Therefore, PTUD is satisfied for $\alpha = 1$. ■

A.2 Independence of the properties of Theorem 1

Lemma A.1 *The properties of Theorem 1 are independent.*

Proof. We need to show that as soon as we remove one of the three properties in Theorem 1 (Group Independence, Equal Treatment for Equal Demands and Proportional Treatment for Unequal Demands), another solution appears.

Group Independence: Let $y_i^a = \frac{C(N,(N^d,N^t),c)}{|N|}$ for all $i \in N$. Trivially, y^a satisfies ETED and PTUD. Let $N^d = N = \{1,2\}$, $c_{01} = 0$, $c_{02} = 1 = c_{12}$. By Group Independence we should have that $y_1 = 0$, $y_2 = 1$ but $y_1^a = y_2^a = \frac{1}{2}$.

Equal Treatment for Equal Demands: Proceed as in the proof of Theorem 1, with the following modification. If we remove a relevant edge between i and j, with $i,j \in N^d$ or $i,j \in N^t$, apply the following equality:

$$(1 + c_{0i}c_{0j}^2)\left[y_i^b((N^d,N^t),c') - y_i^b((N^d,N^t),c)\right] = (1 + c_{0i}^2c_{0j})\left[y_j^b((N^d,N^t),c') - y_j^b((N^d,N^t),c)\right]$$

while if $i \in N^d$ and $j \in N^t$ apply

$$y_i^b((N^d,N^t),c') - y_i^b((N^d,N^t),c) = y_j^b((N^d,N^t),c') - y_j^b((N^d,N^t),c).$$

It is easy to see that if $c_{0i}, c_{0j} > 0$ and $c_{0i} \neq c_{0j}$, ETED is not satisfied. The second equality assures us that PTUD is satisfied and corresponds to the case where $\alpha = \frac{1}{2}$. Group Independence is satisfied by definition.

Proportional Treatment for Unequal Demands: Proceed as in the proof of Theorem 1, with the following modification. If we remove a relevant edge between i and j, with $i \in N^d$ and $j \in N^t$ apply

$$\begin{aligned} y_i^c((N^d,N^t),c') - y_i^c((N^d,N^t),c) &= y_j^c((N^d,N^t),c') - y_j^c((N^d,N^t),c) \text{ if } min(c_{0i},c_{0j}) = 0 \\ y_j^c((N^d,N^t),c') - y_j^c((N^d,N^t),c) &= 0 \text{ if } min(c_{0i},c_{0j}) > 0 \end{aligned}$$

By definition, PTUD is not satisfied while Group Independence and ETED are satisfied by definition. ■

A.3 Proof of Lemma 2

Proof. i) y^0 satisfies Individual Rationality, Demand Ranking and Fair Rent but fails Demand Monotonicity and Positive Rent.

By definition, Individual Rationality is satisfied if $i \in N^t$. By the properties of the Shapley value,

$$y_i^0((N^d,N^t),c) \leq \bar{C}(\{i\},(N^d,N^t),c) \leq C(\{i\},(N^d,N^t),c)$$

for $i \in N^d$. Individual Rationality is satisfied.

Suppose that for $i \in N^d$, $j \in N^t$, $c_{ik} = c_{jk}$ for all $k \in N_0 \backslash \{i,j\}$. By definition, we will thus have that $\bar{C}(S \cup \{i\},(N^d,N^t),c) \geq \bar{C}(S,(N^d,N^t),c)$ for all $S \subseteq N^d \backslash \{i\}$, as the benefits that agents in S can obtain by connecting through agent i's location can already be obtained by connecting through agent j's location. By definition, $y_i^0((N^d,N^t),c) \geq 0 = y_j^0((N^d,N^t),c)$. Therefore, Demand Ranking is satisfied.

Fair Rent is trivially satisfied as

$$\sum_{i \in N^d} y_i^0((N^d,N^t),c) = C(N,(N^d,N^t),c) \leq C(N^d,(N^d,N^t),c).$$

It also implies that Positive Rent is not satisfied.

Consider c^2 described in Section A.4. We have that $y_2^0((\{3\},\{1,2\}),c^2) = 0$ and $y_2^0((\{2,3\},\{1\}),c^2) = -\frac{1}{2}$, violating Demand Monotonicity.

ii) y^K satisfies Individual Rationality, Demand Monotonicity, Demand Ranking and Positive Rent but fails Fair Rent.

By the properties of the Shapley value, $y_i^0((N^d,N^t),c) \leq C(\{i\},(N^d,N^t),c)$.

For $i \in N^t$, we have that $C(S \cup \{i\}, (N^d, N^t), c) \leq C(S \cup \{i\}, (N^d \cup \{i\}, N^t \setminus \{i\}), c)$ and $C(S, (N^d, N^t), c) \leq C(S, (N^d \cup \{i\}, N^t \setminus \{i\}), c)$ for all $S \subseteq N \setminus \{i\}$. By the properties of the Shapley value, $y_i((N^d, N^t), c) \leq y_i((N^d \cup \{i\}, N^t \setminus \{i\}), c)$. Demand Monotonicity is satisfied.

Suppose that for $i \in N^d$, $j \in N^t$, $c_{ik} = c_{jk}$ for all $k \in N_0 \setminus \{i, j\}$. We have that $C(S \cup \{i\}, (N^d, N^t), c) \geq C(S \cup \{j\}, (N^d, N^t), c)$ for all $S \subseteq N \setminus \{i, j\}$. By the properties of the Shapley value, $y_i((N^d, N^t), c) \geq y_j((N^d, N^t), c)$. Demand Ranking is satisfied.

Consider $c^{1,2}$ described in Section A.4. We have that $y_1^K((\{2,3\}, \{1\}), c^{1,2}) = -\frac{1}{6}$. However, by Fair Rent, we should have that $y_1^K((\{2,3\}, \{1\}), c^{1,2}) = 0$ as agents 2 and 3 can connect independently at a cost of zero. Fair Rent is not satisfied. It also shows that Positive Rent is satisfied.

iii) y^1 satisfies Individual Rationality, Fair Rent and Positive Rent but fails Demand Monotonicity and Demand Ranking.

By the properties of the Shapley value,

$$y_i^1((N^d, N^t), c) \leq C^{N^d}(\{i\}, (N^d, N^t), c) = C(\{i\}, (N^d, N^t), c)$$

for $i \in N^d$. Also, since all marginal contributions are non-positive in $\tilde{C}$, $y_i^1((N^d, N^t), c) \leq 0$ for all $i \in N^t$. Individual Rationality is satisfied.

Fair Rent is trivially satisfied as

$$\sum_{i \in N^d} y_i^1((N^d, N^t), c) = C(N^d, (N^d, N^t), c).$$

Consider $c^{1,2}$ described in Section A.4. We have that $y_1^1((\{2,3\}, \{1\}), c^{1,2}) = 0$, $y_2^1((\{2,3\}, \{1\}), c^{1,2}) = -\frac{1}{2}$. Demand Ranking is not satisfied

Consider a problem with $N = \{1,2,3,4\}$ and c' such that $c'_e = 0$ for all edges except $(0,3)$ and $(0,4)$. We have that $y_2^1((\{3,4\}, \{1,2\}), c') = -\frac{1}{2}$ and $y_2^1((\{2,3,4\}, \{1\}), c') = -\frac{2}{3}$. The example shows that Demand Monotonicity is not satisfied but that Positive Rent is. ■

A.4 Shares for c, c^1, c^2 and $c^{1,2}$ (Figure 5)

We are interested in the shares for the cases where $N^t = \{1,2\}$ and $N^t = \{1\}$. The following table gives the shares for a generic α, the parameter in PTUD. A discussion follows on how they were obtained.

d	$y((\{3\}, \{1,2\}), d)$	$y((\{2,3\}, \{1\}), d)$
c	$(0,0,1)$	$(0,0,1)$
c^1	$(-\alpha, 0, \alpha)$	$(-\alpha, 0, \alpha)$
c^2	$(0, -\alpha, \alpha)$	$(0, -\frac{1}{2}, \frac{1}{2})$
$c^{1,2}$	$\left(-\frac{\alpha^2}{\alpha+1}, -\frac{\alpha^2}{\alpha+1}, \frac{2\alpha^2}{\alpha+1}\right)$	$\left(-\frac{\alpha(1-\alpha)}{2-\alpha}, -\frac{\alpha}{2(2-\alpha)}, \frac{\alpha(3-2\alpha)}{2(2-\alpha)}\right)$

The cost shares for c are trivially obtained by Group Independence. Then, we can find the shares of c^1 and c^2 by applying ETED or PTUD. For example, by Group Independence, we have that $y_1((\{3\}, \{1,2\}), c^2) = 0$. Letting $y_3((\{3\}, \{1,2\}), c^2) = w$ and $y_2((\{3\}, \{1,2\}), c^2) = -w$ (by budget balance), we obtain, by PTUD, that

$$\begin{aligned}(1-\alpha)\left[y_2((\{3\}, \{1,2\}), c) - y_2((\{3\}, \{1,2\}), c^2)\right] &= \alpha\left[y_3((\{3\}, \{1,2\}), c) - y_3((\{3\}, \{1,2\}), c^2)\right] \\ (1-\alpha)\, w &= \alpha(1-w)\end{aligned}$$

Solving, we find that $\alpha = w$. To find $y((\{2,3\}, \{1\}), c^2)$ we proceed in the same manner, using ETED instead of PTUD.

Notice that c^1 differs from $c^{1,2}$ only by edge $(2,3)$ while c^2 differs from $c^{1,2}$ only by edge $(1,3)$. Therefore, to find $y((\{3\}, \{1,2\}), c^{1,2})$ we need to solve the following system of equations:

$$\begin{aligned}(1-\alpha)\left[y_1((\{3\}, \{1,2\}), c^1) - y_1((\{3\}, \{1,2\}), c^{1,2})\right] &= \alpha\left[y_3((\{3\}, \{1,2\}), c^1) - y_3((\{3\}, \{1,2\}), c^{1,2})\right] \\ (1-\alpha)\left[y_2((\{3\}, \{1,2\}), c^2) - y_1((\{3\}, \{1,2\}), c^{1,2})\right] &= \alpha\left[y_3((\{3\}, \{1,2\}), c^2) - y_3((\{3\}, \{1,2\}), c^{1,2})\right] \\ \sum_i y_i((\{3\}, \{1,2\}), c^{1,2}) &= 0.\end{aligned}$$

The first two equations are obtained using PTUD and the third by budget balance. To find $y((\{2,3\},\{1\}),c^{1,2})$, we proceed as above, but use ETED to generate the second equation.