Understanding Production in the Performing Arts: A Production Function for German Public Theatres

Marta Zieba and Carol Newman*

Abstract
The production structure for the performing arts is complicated by a number of factors making it difficult to estimate production technologies using a theoretical framework built for standard applications. However, understanding the nature of production and the way in which decisions are made by performing arts firms is particularly important given that many performing arts organisations are funded by government. Public funding of performing arts organisations is justified where socially desirable objectives are fulfilled. The public good component of output makes an important dimension of firms' production decisions unobservable while the principal-agent problem reduces the incentive for firms to behave as cost minimisers. Both may result in an observed production structure which is uneconomic. In this paper we re-visit these issues using a new and extensive dataset for German public theatre. We aim to explore the extent to which the standard laws of production that apply in other sectors of the economy hold for performing arts institutions.

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* Marta Zieba (ziebam@tcd.ie) and Carol Newman (cnewman@tcd.ie) are at the Department of Economics, Trinity College Dublin, Ireland.

1. Introduction

The production structure for the performing arts is complicated by a number of factors making it difficult to estimate production technologies using a theoretical framework built for standard applications. However, understanding the nature of production and the way in which decisions are made by performing arts firms (and other non-profit enterprises) is particularly important given the sometimes substantial subsidies that they receive from the government. Baumol and Bowen (1965) were the first to identify the economic problem facing producers in the live performing arts sector (theatre, opera, music and dance). Technological progress in the production of artistic output is not possible since live performing arts belong to the stable productivity sector of the economy. This implies that while labour costs associated with the production of an artistic performance will increase over time in line with other sectors of the economy, productivity will remain unchanged.[1] As a result, performing arts organisations will fall into financial difficulties since costs will inevitably increase over time relative to revenues. Thus, most performing arts organisations are subsidized by the state or private donors, or are run by governments.[2] Public funding of performing arts organisations is justified where socially desirable (but not necessarily profitable) objectives are fulfilled, such as ensuring high quality, diverse performances at affordable prices. The public good component of output makes an important dimension of firms' production decisions unobservable while the principal-agent problem reduces the incentive for firms to behave as cost minimisers. Both may result in an observed production structure which is uneconomic. While the former may be justifiable from a social perspective the latter represents inefficiency yielding welfare losses, particularly to taxpayers.

The literature on production in the performing arts is sparse. Since the early efforts of Throsby (1977) who estimated a production function for Australian performing Arts companies, and Gapinski (1980; 1984; 1988) who used US and British data to explore the production structure of performing arts firms, the authors are unaware of any attempts to empirically estimate production functions for the sector, an exercise so common to other sectors of the economy.[3] The main hindrance to this process surrounds the definition and measurement of output which is complicated by its public good component and the role of demand factors. The availability of data which allows for adequate controls of these factors has prevented the literature from moving forward. In this paper we re-visit these issues using a new and extensive dataset for German public theatre. We aim to explore the extent to which the standard laws of production that apply in other sectors of the economy hold for performing arts institutions. What kind of technological characteristics do the performing arts have which can be recognized as "different"? We draw from Gapinksi's (1980; 1984) work with the aim of testing the extent to which his results for the US and the UK hold for a more recent sample.

[1] For example, the output per man-hour of an actor playing Romeo is fixed. In addition, is relatively difficult to reduce the number of actors in the "Romeo and Juliet" performance.

[2] There are few exceptions where high performing arts are produced by a profit maximising firm. One example is Broadway theatre in the USA.

[3] A number of studies have been approached from a cost function perspective. See for example Globerman and Book (1974), Lange *et al.* (1985); Taalas (1997).

Germany presents an interesting case study for exploring these issues. There are approximately 150 public theatres in existence in Germany today.[4] German public theatres are state-owned and are financed by either the federal region or the municipalities, depending on the licence holders.[5] Rather than being part of the public administration sector, as are public schools, police and the health service, for example, they are considered public service providers.[6]. In contrast to the public administration sector which are entirely financed by taxes or fees set by law, German theatres are public enterprises which earn revenues on the market through the services that they provide. However, as already discussed, there is also a strong public good component to the provision of artistic output and there are a number of public goals which German public theatres must also aim to fulfil. As such, while public theatres will aim to maximise revenue they will do so under the constraint that certain non-private benefits to society are also achieved. For example, they may be required to produce performances of a high quality and variety, or set ticket prices at a level that is accessible to the public. They should also care about other non-private benefits to society such as cultural heritage or national cohesion. As a result, a gap will inevitably exist between the cost of providing services and the revenues received from the provision of the service at the social optimum. Public subsidies are justified on the grounds that they fill this gap. The state and tax payers can thus be viewed as shareholders of these public enterprises.[7]

Over the last 13 years, German theatres have suffered average losses of 70 per cent. This immense burden on tax payers has led to the closure or privatization of many public theatres. Understanding why such losses have occurred is important in justifying such closures. It may be the case that the public good considerations that German public theatres must consider in their objective function are such that it results in uneconomic production from a profit maximizing perspective but the attainment of other social goals. It may also be, of course that due to a soft budget constraint, state-owned theatres are simply inefficiently run. In this paper we explore the nature of production in German public theatres in an attempt to explore these issues.

The paper is structured as follows. In Section 2 the peculiarities specific to production in the performing arts sector are explored. Artistic output and the inputs required for its production are defined and the objectives of performing arts firms and the production horizon are discussed. Section 3 presents the theoretical model used to explore the production technology of German public theatres and presents a range of testable hypotheses. The data are presented in Section 4 while in Section 5 the

[4] There has been little variation over time in the number of operating theatres, with the exception of the addition of approximately 70 East German theatres post-reunification in 1991.

[5] German public theatres have different legal forms such as Regiebetrieb (organisation form of public administration) or Eigenbetrieb (typical public legal form for public enterprise). It is also quite common that they are in the form of a Limited Liability Company where the only shareholder is the state.

[6] This definition is also compatible with the statistical register of economic sectors in Germany, prepared by Federal Statistics Office where German public theatres belong with other public and private enterprises to other services sector such as culture, sport and leisure.

[7] The market for artistic output is partly competitive as there are no regulations that exclude other enterprises from the sector. In addition to the public theatres in operation in Germany there are 230 large private theatres, some of which are moderately funded through donations, and an additional 2000-3000 small private theatres ('free theatres') which are exclusively profit-oriented.

empirical model and results are discussed with the emphasis placed on analysing the production structure in German public theatres. Section 6 concludes the paper.

2. Technological considerations in the production of artistic output

2.1 Defining Output

The definition and quantification of output is complicated by a number of factors. The output of an artistic production process is not observed and cannot be measured directly. In the case of the performing arts, we observe such quantities as the number of performances given, the number of separate productions, the number of seats/tickets available for a single production or the number of visitors or tickets sold, much like output in the service or tertiary sector (Throsby and Withers, 1979). Artistic output is often thought about in terms of the "cultural experience" associated with its consumption. For example, a measure of the long-run output of a firm might be the number of separate plays produced in a given season, with each play considered an individual cultural experience. More commonly, since each individual attending a given performance consumes the cultural experience, output is often measured as the number of paid attendances (or tickets sold) during a period of time (Gapinski, 1980; 1984; Throsby, 1994; Globermann and Book, 1974).

The cultural experience, however, can be different for each individual attending a given performance (Heilbrun and Grey, 1993). The exact nature of the cultural experience each individual receives from a given performance is not discernable and will depend on the tastes and artistic interpretation of each individual. This dimension of output is not observed by the firm or the econometrician and as such cannot be considered in the analysis. The only influence the firm can have over this process is through inputs that they use.

An additional consideration is the public good component of artistic output. In addition to the benefits gained by individuals who pay to attend a performance, the performing arts generate non-private benefits to the rest of society. For example, a performing arts institution is producing not only an individual cultural experience but also spiritual well-being for its people, such as national identity, social cohesion and national prestige (O'Hagan, 1998). The artistic product promotes national culture and also international recognition. Furthermore ideas created in the arts are inspiration for television, cinema and industrial design. The arts may be used to celebrate the values of society on the one hand and to confront, question and change these values on the other hand. As such, even those who do not consume artistic output in the physical sense may still derive utility from its production.

Within a production function framework, only individuals who purchase a ticket for a performance will be included in the quantification of output, but many more may enjoy the non-exclusive non-rival components of its production. The existence of these positive externalities means that actual artistic output may be much greater than that which is observed through ticket sales. However, since the firm does not receive revenue for the production of positive externalities, they will not be considered in the firms' production decision. If profit maximising, the firm will produce at a lower level of output than is socially optimal. Since the positive externality is unobserved and unquantifiable, we can merely examine the production technology associated with observed "physical" cultural experiences. However, the purpose of this paper is to

analyse the firm's production technology which is unaffected by the immeasurable part of output. We interpret the public component of the artistic output as a by-product of the artistic output, thus not affecting the technological characterisation.

The production function is further complicated by the fact that production and consumption are strongly interlinked in the performing arts since output is consumed at the same time as it is produced, and it can not be accumulated for later consumption. This has important implications for the estimation of the supply curve for performing arts organisations (Throsby, 1994). Expanding output will not only depend on the technology at hand (the quantity of inputs employed and technical efficiency), but it will also depend on whether the consumer is willing to come to the theatre or opera on that particular occasion. This will depend on demand factors such as the quality of the particular performance, the price of the ticket, income of the inhabitants in the region where the performing arts company is located, the weather, the number of tourists, marketing and transportation etc. Ignoring demand effects will lead to a false interpretation of the supply curve. For example, Throsby and Withers (1979) and Throsby (1994) found that the number of tickets sold decreased at the end of the season and interpreted this as decreasing returns to scale in production. However, if the fall in ticket sales is as a result of a change in demand side factors leading to a fall in consumption, the technology may actually have constant returns to scale.[8] As a result, in supply side analyses of the kind conducted in this paper, factors that normally only appear in demand side studies may also be of relevance.

2.2 Classification of Inputs

Inputs are divided into two distinct categories, primary and secondary factors of production (Gapinski, 1980).[9] This categorisation is based on the idea that there are certain inputs which are essential to the production of an artistic performance and others which are not. For example, a performance could not take place without artists. Hence the labour of artists is the most important factor in the production of artistic output. Artists can therefore be classified as primary labour inputs. The performing labour such as actors or choir members, are not only an instrument in producing a final good, they are also the end product in the performing arts production (Withers, 1977). Apart from artists, there are also secondary labour inputs. In contrast with primary inputs they may have a small impact or even no impact on the production of the cultural experience. These labour inputs include maintenance and administrative staff, further technical and additional house staff (ancillaries), such as ushers, box office and help staff, front-of-house staff, cleaning personnel and stage hands.

Capital resources are also important for the artistic production process although they are not as fundamental as labour. Capital can also be differentiated by primary and secondary resources. The borderline between the two types of capital inputs is not well-defined and must be set from case to case. In general, we can define capital resources as primary, when they have a direct influence on the production of the

[8] Heilbrun and Grey (1993) used the number of seats available for sale as a measure of output avoiding the overlap between supply and demand factors. However, this definition of output is not consistent with our previous interpretation of artistic output as a cultural experience which occurs as a result of direct contact with the audience.

[9] Throsby (1977) and Throsby and Withers (1979) differentiated between operating and investment capital (fixed and variable inputs).

artistic output. For example, props, sound equipment, musical instruments, décor and costumes, stage machinery, lighting rigs, electrical gear, energy, property rights, scripts and royalties. Even if the primary capital inputs are not as important as labour, the artistic production can also not take place without them (or at least the quality of the performance will be questioned). Secondary capital inputs may include other capital assets and capital stock (e.g. building facilities provided for the audience, stage, fitting rooms and other venues and associated facilities) and investment capital (e.g. book collections and libraries, publications and marketing services).

2.3 The Firms Objective Function

Standard production theory may not apply in the case of the performing arts due to the fact that the objectives of the firm are unclear. If profit-maximising, a performing arts firm will attempt to maximise the number of paid attendances subject to cost restrictions. However, since most performing arts organisations are non proprietary, such an objective may not apply. Non-profit performing arts firms are very often regulated by state, heavily subsidized or are run solely by government (as is the case in Germany). As such, the non-private benefits associated with production may also be of importance in the firm's production decisions. However, as discussed above, the public good aspect of the performing arts is not observed and so a firm that incorporates these aspects into its objective function may appear uneconomic on the basis of the quantifiable production function. The process is further complicated by the fact that the cost constraints facing the firm may not be given by the market. For example, if the objective of the firm is to maximise attendance, price may be set at an unprofitable level in an attempt to achieve this.[10] Despite these irregularities, it is still reasonable to assume that even non-profit performing arts firms will aim to maximise the number of attendances. However, since other unobserved goals may coexist, we would expect the production technology of performing arts organisations to differ from the usual case. In this paper we attempt to identify these differences.

3. The Model

The purpose of this paper is to review the characteristics of the technology used in the production of artistic output, specifically for performing arts organisations. This will be achieved through the estimation of a production function, appropriately specified to take into account the non-standard features of production associated with the performing arts. In this section, we present these non-standard features and pose a number of hypotheses about the underlying technology that we wish to test. Following this we present a flexible production function that will allow for each of these irregularities.

3.1 Testable features of the technology

Sign on the Marginal product

A common feature of most production technologies is the law of positive and diminishing marginal product. For performing arts organisations there may be different constellations of marginal product for different types of artistic output. In the short run, during a single production period, we could postulate that the marginal

[10] One could certainly envisage this occurring in cases where performing arts organizations face no cost/budget restrictions such as cases where the firm is subsidized by the state in the form of covered deficits.

product will be positive, since the number of performances for a single production can be increased by employing more variable inputs such as actors, singers, energy *etc*. which are used in the course of each performance. We could also assume that the marginal product will diminish at the end of the season due to demand effects since demand tends to fall toward the end of a season (Throsby, 1994). It may also be possible for the production of artistic output to contain regions of negative marginal products due to demand effects. For example, if a performing arts firm decides during one production to give more performances and thus employs more variable inputs but nobody attends these performances, the marginal product will be negative. In fact this will be the case wherever the venue is not operating to full capacity since inputs cannot be adjusted.

In the long run all inputs are variable and as such we can assume that the capacity of the venue, labour and other capital used to set up single productions can be adjusted. Thus we might expect that firms will operate in the economic region of the isoquant. As with the short run, however, demand effects will determine the sign on the marginal product. On the one hand, since artistic output depends on demand effects and demand depends on the quality of the artistic output produced, increasing certain inputs such as rehearsal time, usage of materials, costumes, labour of technicians, directors and other primary labour inputs will increase the quality of output and thus the amount of artistic output produced. The extent to which output will actually expand will depend on demand factors.

Value of the Marginal Product

In perfectly competitive markets, a profit maximising firm will produce output such that marginal revenue is equal to marginal cost which is equal to the price. Thus, the price of each input should equal the value of the marginal revenue product of that input. If this does not hold, we can conclude that inputs are being used excessively. In the context of performing arts organisations, since profit maximisation is unlikely to be the firm's objective, it is possible that inputs will not be employed efficiently.[11]

Returns to scale

Increasing, decreasing and constant returns to scale are all feasible in the context of performing arts organizations. For example, by increasing the number of artists employed, specialization in different performances can occur. These performances can be re-run allowing artists to perform with better quality, potentially increasing more than proportionally the number of attendances. This type of specialisation, however, lessens the repertoire choice, by reproducing simple and famous productions which attract large audiences, for example. If a theatre company is non-profit and one of its objectives is repertoire diversification, then such economies of scale could not be exploited. Thus, while technically possible, it will depend on the nature of the performing arts organization and its objectives. In addition, scale economies of this kind will only be possible up to a certain point since output for each performance is constrained to the capacity of the venue. For example, if all inputs used for a

[11] Some caution should be exercised in interpreting such a result since it is also likely that the ticket price of a non-profit performing arts firm will not be the market price that would be chosen by a profit maximising firm. For example, if the objective of the firm is to maximise attendance, price may be set at an unprofitable level in an attempt to achieve this. One could certainly envisage this occurring in cases where performing arts organizations face no cost/budget restrictions such as cases where the firm is subsidized by the state in the form of covered deficits.

performance are doubled, output can only be increased to its maximum, given by the auditorium size. Output will also be constrained by demand and thus to the population in the particular region where the performing arts firm is located. The size of the market may be insufficient to fully exploit the available scale economies. If these environmental constraints are exogenous and fixed then as the number of performances increases, decreasing returns to scale will eventually set in.

Substitutability between factors of production
The longer the period of production the more substitutability between different inputs is possible as more and more inputs will be variable. For the production of artistic output in the short run the elasticity of substitution will be low because primary inputs cannot be substituted during a run of a single performance or a single production. The assumption of limited substitutability between factors of production can be relaxed in the long run when the production plan can be adjusted and different productions can be chosen. Different productions will require different inputs so that a performing arts institution could substitute, for example, more labour intensive productions for more capital intensive productions (Gapinski, 1979). Furthermore, there is also the possibility of substitution not only within the productions but also within different art forms. A performing arts firm could substitute more labour intensive productions such as operas or symphonies for theatre, musical theatre or chamber orchestra performances which require more capital inputs. The substitution possibilities in the performing arts will vary over inputs and the time horizon of production.

3.3 The Production Function
The unique features of production in the performing arts suggests that standard well-behaved homogenous production functions (such as the Cobb-Douglas) which impose unnecessary restrictions on the production function parameters will not be appropriate. The transcendental production function, originally introduced by Halter et al. (1957), was adapted by Gapinski (1980; 1984) to accommodate the required level of flexibility (see Equation (1)).

$$Y_{it} = c \cdot \prod_{k}^{K} X_{ikt}^{\alpha_k} e^{\beta_k X_{ikt}} \cdot e^{\sum_{l}^{L} (\gamma_l Z_{ilt} + \delta_l Z_{ilt}^2)} \qquad k = 1,.....,K \qquad l = 1,.....,L \qquad (1)$$

where Y_{it} is the output of firm i in time period t, c is a constant term, X_k are K primary inputs, Z_l are L secondary inputs, and α_k, β_k, γ_l and δ_l are coefficients to be estimated. We can see from equation (1) that artistic output can not be produced if there are no primary inputs employed ($X_i=0$). For example, if there are no artists, an artistic performance cannot take place. In contrast, artistic output can be produced in the absence of secondary inputs, such as maintenance of administrative staff, and so if $Z_i=0$, artistic output can still have positive values.

This specification allows for both positive and negative marginal products that can be either decreasing or increasing. Similarly, there are no restrictions placed on the marginal rate of the technical substitution or the output elasticity. Variable returns to scale are also accommodated. For formulae on each of the components of technology of interest in this paper see the Appendix.[12] The transcendental production function can display technology which is not monotonic and not well-behaved. Because

[12] For a detailed derivation of each of these components contact the authors.

movements in the marginal products are accommodated, the same output level can be associated with two values of the input (Gapinski, 1980). For such technology the isoquants may be circles or semicircles. This implies that production can occur in an uneconomic region and that production can change depending on different input values.

Taking logs of the production function and including fixed firm and time effect terms (to control for unobservable characteristics of the theatre in the case of the former, and over time in the case of the latter, that may impact on output) and a statistical noise term, the full empirical model is given in equation (2).[13]

$$\ln Y_{it} = \ln c_0 + c_i + \lambda_t + \alpha_k \sum_{k=1}^{2} \ln X_{ikt} + \beta_k \sum_{k=1}^{2} X_{ikt} + \sum_{l=1}^{2} \left(\gamma_l Z_{ilt} + \delta_l Z_{ilt}^2 \right) + u_{it} \qquad (2)$$

where Y_{it} is the artistic output of theatre *i* in time period *t*, X_{ikt} are the primary inputs (*k=1,2*), Z_{ilt} are the secondary inputs (*l=1,2*), c_i are the firm specific fixed effects, λ_t are the fixed time effects and u_{it} is the statistical noise term with zero mean and constant variance which we assume is uncorrelated with the parameters of the model.

Applying the fixed effects model for German public theatres is appropriate given the fact that there may be specific individual unobservable characteristics which do not change over time but may influence the attendance number. These can be for example the geographical location of the theatre, population size, infrastructure of the region where the theatre company acts, size of the theatre company, also quality and prestige and other regional, environmental factors which do not change in time. In addition, the managerial style or the quality of the inputs which was not explicitly specified in the model may be theatre specific. For example, the derived inputs quantities from financial data may include some unobservable quality differences in theatres or it may be the case that more talented actors earn a higher salary thus increasing the number of man hours for some theatres. Excluding fixed effects which control for these unobservables will mean that the error term u_{jt} will be correlated with the independent variables leading to inconsistent estimates. The fixed effects approach controls for these factors. It is also the case that there may be factors exogenous to the model that cause the production function to shift. For example, technological change, changes in government policy for German public theatres and other external effects.

4. The Data

4.1 The structure of the German public theatre sector

German public theatre can be described as "Dreispartentheater" (three branch theatres) meaning that many have drama, music theatre (opera/operetta/musical) and ballet/dance at their disposal. This implies that a variety of performing arts forms are generally offered by single theatre enterprises.[14] In major cities, however, for

[13] The inclusion of theatre and time specific effects reduces the bias associated with omitting variables from the model.

[14] Occasionally puppet and figure theatre and children's and youth theatre are also provided. As such, German public theatre can also be termed Mehrspartentheater (Multiple-Branch-Theatre).

example Berlin, Munich, Dortmund, Hamburg or Magdeburg, the branches of theatre tend to be separate.[15] In addition, about 82 orchestras are integrated with public theatres. The orchestra's main task is to play in music theatre but they also stage additional concerts. Theatres also employ independent cultural orchestras to play in musical theatre (opera, operetta, ballet, dance, musical etc.).

German public theatres are also described as "repertory" theatres. This means that the performances of each production included in the repertoire are spread over the theatre season which lasts 12 months. The production program is prepared and published at the beginning of the season. During a particular season numerous different productions are presented. For large theatres up to 20-25 new productions are performed in a season, with few evenings where the same production is shown.[16]

The rich and varied repertoire of German public theatres has implications for the inputs used in the production process. To offer such wide variety and high quantity of performances, German public theatres must have their own artistic ensemble consisting of solo artists, choir, ballet and theatre orchestra members. Artists are employed on permanent or a temporary basis (contracts of one, two or occasionally three years' duration) to perform for the entire theatre season. The theatre also employs artistic management (e.g. directors and dramaturges) and some artistic-technical staff such as stage designers. The contracts for artists are regulated by a special contractual agreement called "Normalvertrag Bühne" which provides a framework for issues such as working hours, minimum salary etc. This agreement is valid for all artists in Germany, not only for those employed at German public theatres. Support staff consisting of technicians, administrators and house staff are also employed on a full (permanent or temporary contracts) or part time basis. Finally, they have their own venues, which often consist of one large and several small auditoriums granted to them by the state.

While the licence holder is the principal of the theatre enterprise, as with other public enterprises, German public theatres also have their own theatre management. The most important managing body in theatre is the artistic director (called "Intendant") which is chosen by the theatre's licence holder. As such it can be considered the 'agent'. For all theatres, the artistic director decides the artistic production programme, repertoire and ensemble in association with other artistic management such as dramaturges or stage managers. Representative bodies of the licence holder are also involved in the management of the theatre with the responsibility of assuring that the artistic director fulfils his artistic and other public obligations. As such the artistic direction of the theatre is not completely autonomous. For example, the artistic director must inform the representative authorities in advance on the content of the artistic programme. Another member of theatre management is the administrative director responsible for the budget and administration. The detailed organisation of theatre management and the control mechanisms of the managing bodies differ depending on the legal form of the entity. For all legal forms however it holds that the

[15] In Hamburg, for instance, there are two municipal drama theatres and one municipal opera house.

[16] For example, in a public theatre in Schwerin (Mecklenburgisches Staatstheater) there are about 35 performances offered during one month with the repertoire ranging from Brecht's "Puntila", through Lessing's "Emilia Galotti", Ibsen's "Ghosts" Chekov's "The Seagull", Puccini's "La Boheme", Mozart's "The Magic Flute", to several ballet performances, puppet theatres evenings and a few farces.

licence holder (the municipality or federal region) may influence the production of artistic output through their representatives in the theatre management.

Most of the labour inputs and all capital inputs apart from buildings and land are obtained on the competitive factor market. Artist turnover is high and many are invited as guest artists for one production season as part-time employees. Haunschild (2003) found that the labour market for artists in Germany is flexible and competitive. This also applies to artistic directors and other artistic management such as dramaturges or stage designers. In most cases support staff are employed as private sector workers with the exception of theatres organised in public legal form (e.g. Regiebetrieb) where administrators (including the administrative director) and technicians are employed as civil servants. These employees may be protected by special labour law regulations which exclude labour market competition.

4.2 The data set

A panel data set of all public theatres that operated between 1991/92 and 2003/04 in Germany are included in the analysis. The data are taken from the yearly Theatre Reports prepared each theatre season by the German Stage Association (available since 1965). Artistic output is measured as the total number of visitors to the theatre including aggregate ticket sales and complementary tickets issued. Attendances at guest performances by theatres at other locations (sight touring) are also included.[17]

The data for the factor inputs are based on yearly expense data transformed into real values using data on wage rates, other income statistics and price deflators taken from the Federal and Regional Statistics Offices. Information on the capacity of the venue is also utilised. A description of each of the data sources used is provided in Table 1. Following Tobias (2003), expenses reported for the fiscal year are transformed into yearly theatre season equivalents.[18]

Personnel expenses are used to construct the labour inputs. Two separate labour inputs are included in the production function: artists (X_1) which includes artistic directors, stage managers, solo artists for operetta and opera, solo artists for drama, ballet members, choir members and members of theatre orchestras; and ancillaries (Z_1) which includes technicians (technical and artistic-technical staff) and administration and house staff.[19] The latter is included as a secondary labour input. Man hours are obtained by aggregating personnel expenses for each theatre season and dividing by relevant average wage rates for that season.[20] Consideration should be given to the quality of the artistic labour input in particular. Andersson and Andersson (2006) identify one component of artist quality as the objective technical ability of artists which can be judged by skilled professionals such as artistic managers and directors.[21] Tobias (2003) finds that the marginal returns of artistic

[17] Almost all theatres produce these types of guest performances and they are important in the total output of the theatre amounting to 14 per cent for the 13 years of the sample period.
[18] See Krebs (1996) and Widmayer (2000) for alternative approaches.
[19] Originally technicians and administration/house staff were included as separate labour inputs, however, early empirical analysis revealed that they were structurally identical in terms of their impact on output and as such the two were merged.
[20] For data sources on wage rates see Table 1. Specific details on how wage rates are computed are available on request.
[21] The second dimension can be judged only by individual visitors of the performance and as such in unobservable.

expenses are positive in terms of quality, it is reasonable to assume that the quality of artists (as judged by such professionals) will be reflected in their salaries. Dividing by the average wage rate for all artists provides us with a measure of 'quality adjusted' man hours.[22]

In the absence of direct information capital input flows for performing arts institutions, proxy capital input variables are constructed using data on expenses, costs and commodity usage. The primary capital input (X_2) are defined as operative expenses which include administration costs, cost for renting and leasing of facilities, cost for décor and costumes, publications, copy right costs and materials, expenses for guest performances/sight touring, guest performances by foreign ensembles and other operating expenditures. Prior to aggregation, non-personnel expenses are converted into real values using appropriate capital price deflators. The secondary capital input (Z_2) is measured using expenses on other financial projects which include the allocation of reserves and expenses for further investment granted to the theatres by their licence holders. We also include a proxy variable for the value of capital stock which is taken to be the number of seats for each season times the number of venues (measuring theatre capacity) valued at the property value per square meter of building land available from the Federal Statistics Office in Germany. Prior to aggregation of the secondary capital input, the non-personnel expenses included in this measure and value of the capital stock are deflated using an appropriate capital price index.

In total, 174 theatres appear in the sample over the course of the period.[23] Table 2 provides the basic descriptive statistics for the full sample used. Considerable variation exists in artistic output which ranges from 2,758 to 616,234 visitors per theatre season. The input variables also vary considerably across theatres. The variation "within" theatres over time is also quite large.

5. The Empirical Results

The results for the empirical model (given by equation (2)) of German public theatres are presented in Table 3.[24] The first column displays the result including individual fixed effects while the second column extends the model to incorporate time effects also. The models have reasonable explanatory power with an overall R-squared of 77 per. Most of the parameters are significant at the 1 or 5 per cent level. The estimated coefficients (for the two-way error components model – column 2) are used to explore the technological properties of the sector as outlined in Section 3.[25] The relevant statistics are presented in Table 4. A capital-labour ratio of 70 per cent is found. This

[22] In the empirical model, theatre fixed effects control for quality differences across theatres. The 'quality adjusted' man hours measure only needs to capture differences in the quality of artists within a specific theatre across seasons.

[23] Missing data linked to the adjustment process after the German Unification in 1991, results in nine theatres being completely excluded from the sample and an additional 14 theatres excluded for a season. The data were also fitted for the balanced panel of observations as a check for possible sample selection bias. The results do not change.

[24] The model is estimated using STATA Version 8.2. White's robust standard errors are reported. A Hausman test validates the use of the within estimator while an F-test indicates that a pooled model estimated using OLS would produce inconsistent estimates.

[25] See the Appendix for the formulae used to compute each component of the production technology discussed and to draw the isoquants.

is consistent with Gapinski's (1980) finding that the capital-labour ratio for performing arts in the US varied from 35 per cent to 126 per cent.

Sign on the Marginal product

The mean marginal product is positive and diminishing for all factors of production. As expected, the largest marginal product occurs for artists followed by primary capital, highlighting the importance of primary inputs in artistic production. Ancillaries, the secondary labour inputs follow next while the marginal product for secondary capital is the lowest.

Figure 1 presents the estimated long run transcendental production function for each of the four factors of production. The shape of the function is depicted for every input by setting the other inputs to their mean levels. The values are scale adjusted for clarity. The minimum value of the marginal product for artists (see Table 4) is always positive, thus increasing the working hours of artists will always increase output. The shape of the production function reveals that the marginal product for artists is at first positive and then diminishes to a threshold level (815,455 man hours in a year) after which it increases. Since the maximum value for the rate of change in the marginal product for artists is positive, some theatres do in fact reach this level of production. For all other inputs the marginal product is at first positive and diminishing until it reaches a critical point after which it is negative. Below this critical point production becomes uneconomic. As expected, for secondary inputs, for a certain level of attendance, zero quantity of these inputs is required. German public theatres could produce on average 130,776 visitors without secondary capital and 107,422 visitors without ancillaries with the assumption that other inputs are employed at their mean values.

Gapinski (1980) also finds that the marginal product of labour is greater than the marginal product of capital, consistent with the fact that the most important factor in the production of the arts is artistic labour. He also finds the same result for primary inputs: that is, in the long run the marginal product is positive and declining for primary inputs. For secondary inputs he finds a positive but increasing marginal product.

The sign on the mean of the output elasticities and their variation display a similar pattern to that of marginal product. On average they are always positive. As before, the output elasticity is greatest for artists indicating that a 1 per cent increase in artistic man hours results in 0.21 per cent increase in the attendance number. Ancillaries also have a relatively high elasticity as might be expected given, for example, the role of technicians and house staff in the day to day running of the performances and administrators in the coordination of ticket sales. In comparison, the elasticities on the capital inputs are small.

Value of the Marginal Product

As proposed by Gapinski (1980), a useful way of testing whether inputs are efficiently used is to compare the estimated marginal products to those of a profit maximiser in a competitive industry. For profit maximisation, the factor price of an input should equal the value of the marginal revenue product of that input. Thus, if the marginal product is smaller than the ratio of the factor and output price, the input is used excessively. Here, the marginal cost per unit of labour is calculated using the average

wage rate while the marginal cost per unit of capital is calculated using convex combinations of the singular capital indices used to deflate expenses calculated separately for primary and secondary capital. The marginal revenue product for each input is computed as the marginal product times the average real ticket price deflated by the Consumer Price Index. The average values of the input prices and their marginal revenue products are presented in Table 5. In all cases the marginal product is substantially below the ratio of the input prices indicating that all inputs are used excessively. Gapinski (1980) also found that both capital and labour inputs were used excessively in the long run production of non-profit theatres in the UK and the US.

Returns to Scale

The returns to scale coefficients, presented in Table 4, are always smaller than the constant scalar (λ) for input increases (when λ>1) and greater than the constant scalar for input decreases (when λ<1). Thus, there is evidence of decreasing returns to scale over the whole range of inputs. For example, even if we increase all inputs of production at German public theatres three-fold, artistic output will only increase by 50 per cent. One interpretation of this finding is that German public theatres belong to the stable sector of the economy, where technological progress is not possible. However, it may also reflect diseconomies of scale such as capacity, geographical or population constraints. Gapinski (1980) also found decreasing returns to scale for theatre, opera and symphony.

Substitutability between Factors of Production

The mean marginal rate of technical substitution, presented in Table 4, is positive for every input pair indicating that artistic production takes place in the region on the isoquant which is convex and downward sloping. In Figure 2, isoquants are drawn for every input pair holding output and the other inputs at their mean levels. The isoquants resemble semi-circles for artists-primary capital, artists-ancillaries and artists-secondary capital. For the rest of the inputs pairs the isoquants resemble complete circles. The arrows on the isoquants indicate the mean level of the inputs actually used at German public theatres. This confirms that German public theatres produce in the economic region on the isoquant with positive marginal products for all input pairs and where the marginal rate of technical substitution is also positive. Gapinski (1980) finds that artists and capital are not generally substitutable, with other inputs exhibiting some degree of substitutability.

To measure the substitution possibilities, the partial substitution elasticises are calculated. The values for all input pairs are above zero indicating that the technology at German public theatres is not represented by a fixed-proportion production function. The substitution possibilities are smallest between artists and capital and are below unity. For all other inputs pairs the substitution elasticises are greater than one. For example the substitution elasticity between artists and ancillaries is quite high at 4.73 implying that labour is mostly substitutable for labour. The substitution elasticity between primary capital and ancillaries is smaller at 3.86.[26] Also of note is the

[26] This implies that although primary capital can be substituted for secondary labour, there may be some substitution constraints, especially for technicians and some help staff. For example, technicians may be needed at the repertoire theatre where one performance will generally be rehearsed during the day and another one performed in the evening. German public theatres consist very often only of one building with a large stage and several small venues with smaller auditoriums. Thus, the sets and stage

extraordinarily high substitution elasticity associated with secondary capital and the other inputs. The secondary capital represents the capital stock and investment capital that is not directly needed to produce an attendance. The high substitution elasticity suggests that in terms of generating attendance, all inputs can be substituted with capital investments such as performance broadcasting, media and marketing activities. This can be explained through the impact that such investments may have, not only on attendances but also on the other inputs. For example, the artists themselves could become more motivated as a result of such investments and their increased popularity could attract more visitors to the theatres. Thus, media marketing could lead to labour-saving technological progress and even greater levels of artistic output.

6. Conclusions

The findings of this study suggest that the production technology in use by German public theatres occurs in the economic region with positive and diminishing marginal product for all factors of production. This is also confirmed by the estimated marginal rates of technical substitution between input pairs which reveal that production occurs in the region on the isoquant which is convex and downward sloping. As expected, artists are found to be the most important factors of production with the largest output elasticity. These results confirm Gapinski's (1980) findings.

Although the marginal product on average is positive and diminishing, it is of a small magnitude suggesting that the possibilities for increasing output in the performing arts are limited. In addition, there is evidence to suggest that German public theatres do not minimise costs and use all inputs to excess with the value of marginal product covering on average only 5 per cent of marginal costs. Similar to Gapinski (1980) we find that all inputs are used excessively in German public theatres.

The high degree of substitutability between inputs has important implications for the production choices of German public theatres. In particular, the high substitution possibilities between secondary capital and other factors of production (including artists), suggests that there may be opportunities for labour-saving technological progress. This suggests that contrary to Baumol and Bowen's (1965) observation, performing arts in Germany may have the ability to achieve output enhancing technological advances. If this is the case, then capacity constraints may present the biggest challenge. The evidence of decreasing returns to scale for all input levels suggests that capacity and geographical constraints may be a significant barrier to output expansion. As such, theatres may look to new ways of overcoming such constraints. In this context guest performances and sight touring become more important. Another way to solve this problem may be to outsource theatre venues.

German theatres are required to gain their revenue on the market while at the same time implementing other public goals such as fair production of artistic output (e.g. high quality and low ticket price) and the provision of non-private benefits. As such it is not expected that they should behave as profit maximisers since they must fulfil the latter objectives. Public subsidies should fill the gap between revenues and cost, caused by these non-profit goals. The results presented here suggest that while

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must be set up and down every day. Technicians have to work in shifts. Under these conditions the stage technicians and stage hands become particularly important and can not be substituted for capital.

German public theatres are economic, they do not behave optimally from a market perspective given their obligation to fulfil these other non-private benefits. In the absence of such obligations (such as a case where they were all privatised) the nature of the product provided by these theatres would have to change to reflect profit maximising objectives. This would no doubt lead to a change in the nature of performing arts organisations in Germany which currently produce a very unique, complex and diverse cultural good of very high quality.

One possibility for overcoming the economic problem facing publicly funded performing arts organisations is to change the corporate governance structure to reflect both artistic and financial goals. Management should implement the goals and the licence holders should control the fulfilment of these goals. The municipalities and regions as licence holders of their theatres have extensive control possibilities and can also set the goals for German public theatres more precisely. For the exact definition of what these goals should be further research will be needed. Firstly, productivity in the sector should be analysed with a view to determining the types of theatres which work well from a cost efficiency perspective and where the problem areas lie. Secondly, an analysis of demand side factors may help in understanding how output can be expanded. For example, knowledge of price elasticities, how artistic quality affects demand as well as the characteristics of those who visit the theatre may all contribute to this debate.

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Table 1: Data sources and variables

Variable name	Description	Data source
Artistic output (Y): theatre attendance	**Total number of visitors** during yearly theatre season, aggregated together with guest performances and free attendances	Theatre Report, Deutscher Bühnenverein 1991/92-2003/04, Table 3
Artists (X_1): artistic directors/ management, solo performing personnel in operetta, opera, drama, ballet members, choir members, members of the theatre orchestras	**Man-Hours** Calculated by dividing yearly personnel expenses by regional and yearly wage rates available for the public and private service sector	Theatre Report , Deutscher Bühnenverein 1991/92-2003/04, Table 6 Regional Statistics Office, National Accounts of Federal Regions (Volkswirtschaftliche Gesamtrechnungnen der Länder, Reihe 1, Länderergebnisse Band 2.)
Ancillaries (Z_1): Technical and artistic-technical staff Administration and House Staff		Federal Statistics Office, National Accounts „Volkswirtschaftliche Gesamtrechnungen, Inlandsproduktbere-chnung Lange Reihen ab 1970“
Primary Capital (X_2): Operative Non-Personnel Expenses including: Administration, Rents & Leases, Décor and Costumes, Publications, Copy right and materials, guest performances, other operating expenditures.	**Non-personnel Expenses** deflated by numerous PPI and CPI indexes according to the capital input structure for German public theatres and adjusted for regional differences between West and East Germany using Purchasing Power Parity Index;	Theatre Report , Deutscher Bühnenverein 1991/92-2003/04, Table 6, Table 1 Federal Statistical Office: www.genesis.destatis.de, www-ec.destatis.de, Index der Erzeugerpreise Lange Reihen 1995-2006, Statistisches Jahrbuch 1996
Secondary Capital (Z_2): Expenses on different financial projects and value of capital stock	**Value of capital stock** calculated when multiplying capacity of venue (total number of seats in theatres) with the property value, deflated with index for capital stock	Institute of Economic Research, Halle, Report 2003 Regional Statistics Office: www.statistik-portal.de

Table 2: Summary statistics

Variable		Mean	Std. Dev.	Minimum	Maximum
Artistic output Y	overall	151224.9	108983	2758	616234
	between		104363	4877.6	558425.1
	within		25331	-198991.6	473255.7
Artists X_1	overall	352094.4	347699	3327.6	1890081
	between		333356	4013.8	1779708
	within		47103	36365.6	700029
Ancillaries Z_1	overall	245136.4	223557	0	1348629
	between		218413.2	0	1259910
	within		26714.91	36396.97	462786.7
Primary Capital X_2	overall	267723.7	248036.8	286.4	1492720
	between		235287	2052.2	1238160
	within		64012.9	-84332	774531.8
Secondary Capital Z_2	overall	77182.09	240938.2	244.6	3643305
	between		184484.8	244.6	2115080
	within		142204	-1977324	1605408
Natural log Y	overall	11.6391	0.8449	7.9223	13.3314
	between		0.8370	8.4437	13.2307
	within		0.1723	10.3302	12.9717
Natural log X_1	overall	12.2123	1.2273	8.1100	14.4521
	between		1.2361	8.2937	14.3917
	within		0.1561	10.8693	13.6592
Natural log X_2	overall	12.0491	1.0770	5.6574	14.2161
	between		1.1048	7.41640	14.0211
	within		0.2054	10.2901	13.1122
Ancillaries squared Z_1^2	overall	1.10e+11	2.08e+11	0	1.82e+12
	between		1.99e+11	0	1.59e+12
	within		2.85e+10	-8.80e+10	4.63e+11
Secondary capital squared Z_2^2	overall	6.40e+10	7.31e+11	59809.59	1.33e+13
	between		5.23E+11	59809.59	6.63E+12
	within		4.76E+11	-6.56E+12	6.71E+12

Table 3: Production function estimates for German Public Theatres (1991/92-2003/04)

$\ln Y$	Two-way error component model	One-way error component model
$\ln c_0$	8.2077*** (0.6155)	8.2441*** (0.6309)
α_1	0.1183** (0.0427)	0.1111** (0.0427)
β_1	2.65e-07* (1.21e-07)	2.99e-07* (1.20e-07)
α_2	0.1452*** (0.0346)	0.1495*** (0.0340)
β_2	-2.20e-07* (1.01e-07)	-2.93e-07** (1.04e-07)
γ_1	9.95e-07** (3.81e-07)	1.08e-06** (3.75e-07)
δ_1	-4.96e-13* (2.77e-13)	-6.15e-13* (2.74e-13)
γ_2	2.30e-07*** (5.69e-08)	2.17e-07*** (5.89e-08)
δ_2	-6.21e-14*** (1.66e-14)	-5.81e-14** (1.70e-14)
Time effects	yes	no
R-squared overall	0.77	0.77
F	7.42*** (20,1698)	12.39*** (8,1710)
Hausman Test Chi-Squared	21.24*	25.25***
Number of observations	1882	1882

Table 4: Properties of production structure

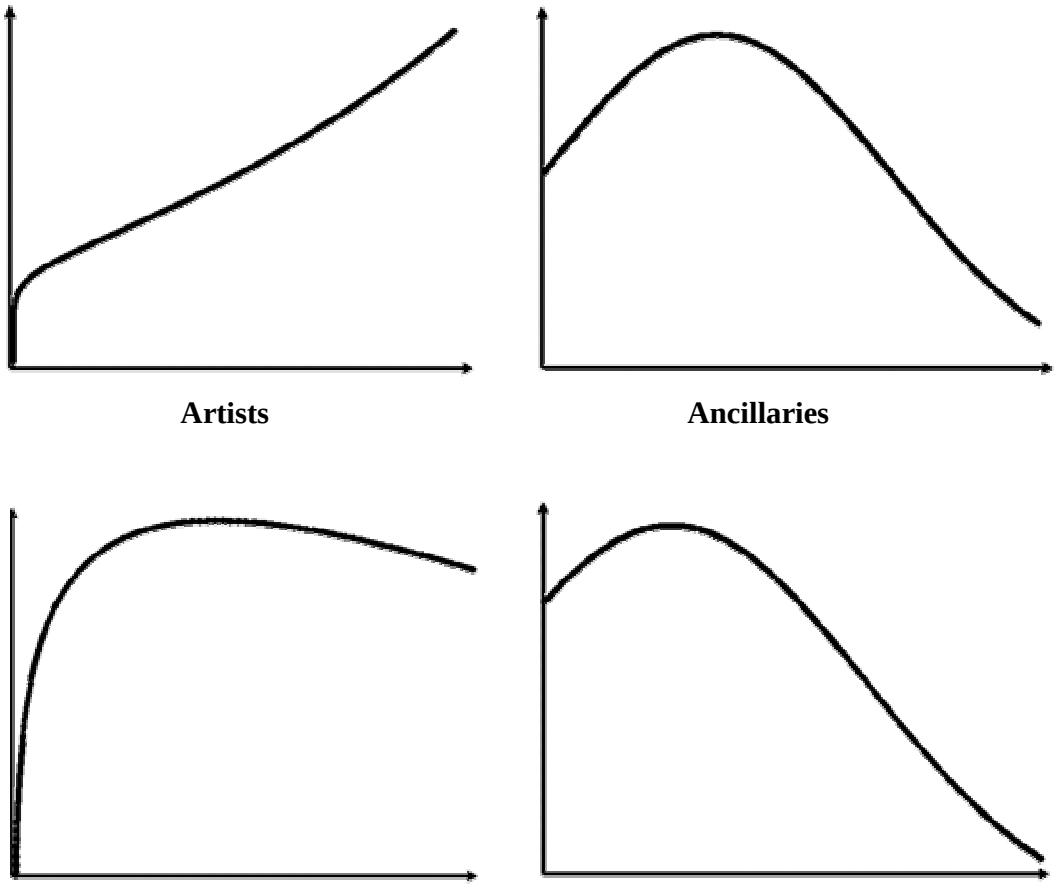
	Mean	Standard Deviation	Minimum	Maximum
Capital	597231	556694	4199	3006655
Labour	344906	391644	1601	4727361
Estimated Output (Attendance)	125167	57711	21168	330473
Capital-Labour Ratio	0.70	0.47	0.16	5.05
Marginal product				
Artists	0.1332	0.1227	0.0611	1.3377
Ancillaries	0.0820	0.0246	-0.0984	0.1388
Primary Capital	0.1119	0.3921	-0.0369	10.888
Secondary Capital	0.0270	0.0128	-0.0564	0.0731
Δ *Marginal Product*				
Artists	-4.39E-06	2.17E-05	-0.000351	2.4E-08
Ancillaries	-6.17E-08	6.80E-08	-3.28E-07	-7.16E-11
Primary Capital	-4.10E-05	1.05E-03	-0.032247	-1.43E-08
Secondary Capital	-9.49E-09	4.90E-09	-3.98E-08	-1.51E-09
Output Elasticity				
Artists	0.2116	0.0921	0.1192	0.6191
Ancillaries	0.1347	0.0861	-0.4624	0.2495
Primary Capital	0.0863	0.0546	-0.1832	0.1451
Secondary Capital	0.0098	0.0451	-0.8106	0.1064
MRTS				
Artists, Primary Capital	6.6735	190.15	-585.99	8115.5
Artists, Ancillaries	2.2553	4.8975	-1.6272	141.31
Primary Capital, Ancillaries	2.2631	17.374	-41.4205	509.67
Artists, Secondary Capital	7.9703	15.384	-2.2672	155.72
Ancillaries, Secondary Capital	3.3717	1.0540	-2.8212	9.3526
Primary Capital, Secondary Capital	9.4041	75.026	-5.0003	2204.6
Partial Substitution Elasticity				
Artists, Primary Capital	0.7081	0.3917	-0.8093	1.1068
Artists, Ancillaries	4.7270	13.638	-0.0903	164.83
Primary Capital, Ancillaries	3.8555	15.952	-0.5408	190.09
Primary Capital, Secondary Capital	40.025	124.64	-0.8063	1908.8
Artists, Secondary Capital	57.596	114.90	0.0705	1722.8
Ancillaries, Secondary Capital	130.56	349.58	0.0027	3899.4
Returns to Scale Coefficient Λ				
$\lambda = 0.25$	0.5913	0.0679	0.4012	0.6933
$\lambda = 0.75$	0.8837	0.0280	0.8000	0.9749
$\lambda = 1.25$	1.1055	0.0323	0.9041	1.2124
$\lambda = 1.75$	1.2884	0.1138	0.4947	1.6621
$\lambda = 2.00$	1.3669	0.1617	0.3001	1.8896
$\lambda = 3.00$	1.6087	0.3719	0.0109	2.6647
$\lambda = 4.00$	1.7584	0.5787	5.E-05	3.1756

Table 5: Value of Marginal Product

	Mean	Standard Deviation	Minimum	Maximum
Marginal costs as factor prices				
Wage rate per hour	20.32	3.03	11.36	24.97
Primary capital index price	9.82	0.50	4.65	10.80
Secondary capital index price	8.48	1.14	5.20	10.37
Real ticket price	11.90	7.15	0.83	59.77
Value of marginal product as marginal revenue				
Artists	1.32	0.92	0.14	11.42
Ancillaries	0.98	0.68	-2.46	4.62
Primary Capital	0.73	1.24	-1.50	33.14
Secondary Capital	0.37	0.42	-1.19	3.16
Value of marginal product in % of factor prices				
Artists	6.39	4.24	1.05	75.50
Ancillaries	4.76	3.13	-10.28	22.52
Primary Capital	7.44	12.11	-14.60	316.41
Secondary Capital	4.40	4.87	-12.49	43.60

Figure 1: Estimated Production Functions

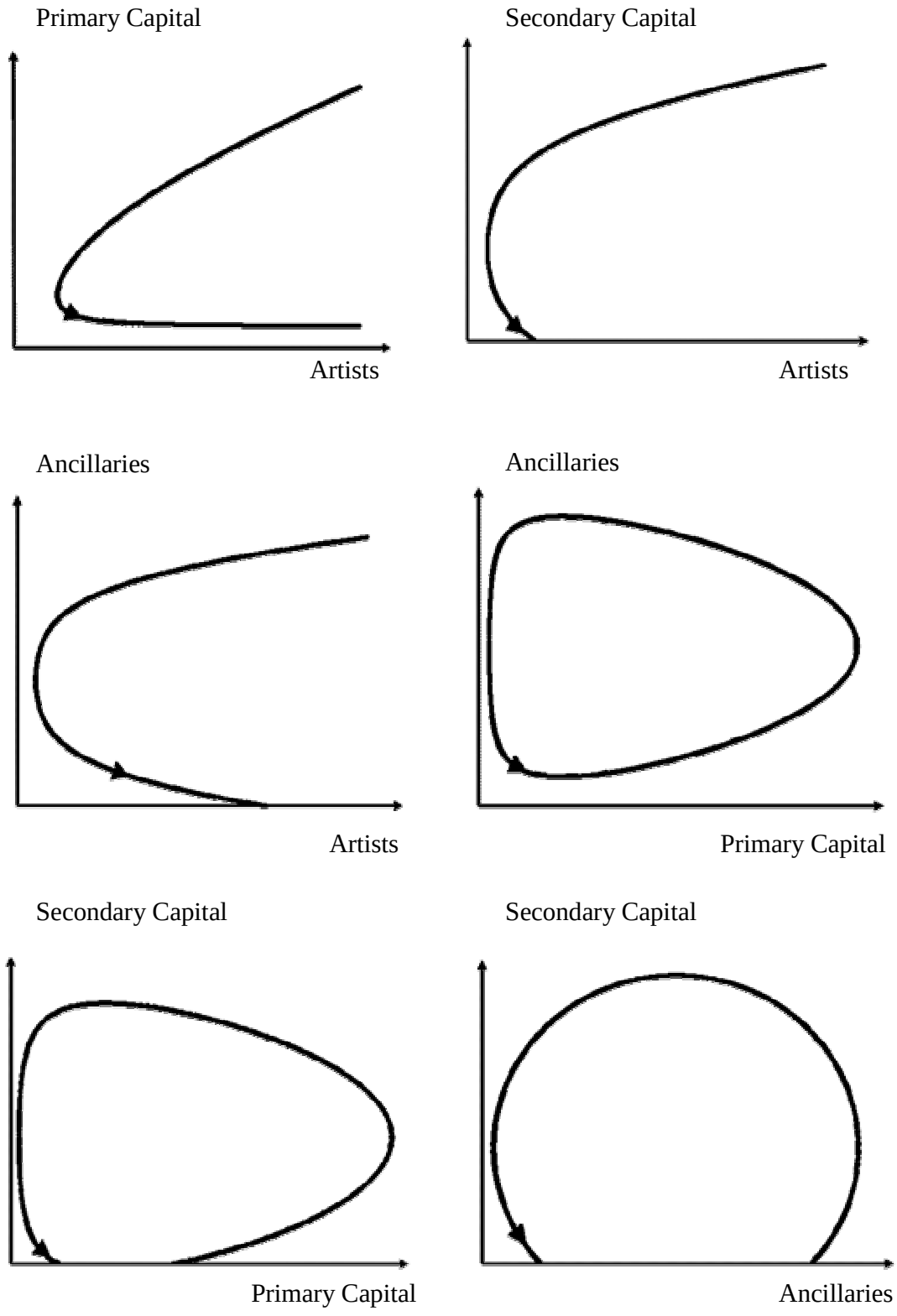
Artistic Output measured as Attendance



Artists **Ancillaries**

Primary Capital **Secondary Capital**

Figure 2: Isoquants for input pairs



Appendix[27]

Marginal product

The estimated production function given in equation (2) can be used to compute the marginal product for each of the inputs and their rate of change. For the primary inputs these are:

$$MP_{X_k} = \left(\frac{\alpha_k}{X_k} + \beta_k \right) \cdot Y \qquad \frac{\partial MP_{X_k}}{\partial X_k} = \left(\left(\frac{\alpha_k}{X_k} + \beta_k \right)^2 - \frac{\alpha_k}{X_k^2} \right) \cdot Y$$

For the secondary inputs these are:

$$MP_{Z_l} = (\gamma_l + 2\delta_l Z_l) \cdot Y \qquad \frac{\partial MP}{\partial Z_i^2} = \left[(\gamma_i + 2\delta_i Z_i)^2 + 2\delta_i \right] \cdot Y$$

The sign on the marginal product and its rate of change will depend on the estimated values for α_k and β_k in the case of primary inputs and γ_l and δ_l in the case of secondary inputs.

To determine whether the marginal product is positive or negative, critical values are computed by setting the first derivative of output with respect to each of the inputs to zero and solving for X_k and Z_l, respectively. The threshold values are given by:

$$X_k^0 = \frac{-\alpha_i}{\beta_i}$$

$$Z_l^0 = -\frac{\gamma_l}{2\delta_l}$$

Examining the rate of change in marginal product will determine the shape of the production function. Setting the second derivative of the production function to zero and substituting X_k^0 and Z_l^0 for X_k and Z_l respectively provides the critical points. These are given by:

$$Y(X_0)'' = -\frac{\beta_k^2}{\alpha_k}$$

$$Y(Z_0)'' = 2\delta_l$$

Output Elasticites

For primary input X_k:

[27] For ease of illustration the firm and time subscripts are suppressed.

$$\varepsilon_k = \frac{\partial Y}{\partial X_k} \cdot \frac{X_k}{Y} = MP_{X_k} \cdot \frac{X_k}{Y} = (\frac{\alpha_k}{X_k} + \beta_k) \cdot Y \cdot \frac{X_k}{Y} = \alpha_k + \beta_k X_k$$

For secondary input Z_l:

$$\eta_l = \frac{\partial Y}{\partial Z_l} \cdot \frac{Z_l}{Y} = MP_{Z_l} \cdot \frac{Z_l}{Y} = (\gamma_l + 2Z_l\delta_l) \cdot Y \cdot \frac{Z_l}{Y} = (\gamma_l + 2Z_l\delta_l)Z_l$$

Marginal rate of technical substitution (MRTS)
Between two primary inputs X_k, X_j is:

$$MRTS_{X_iX_j} = \frac{dX_j}{dX_i} = \frac{MP_{Xi}}{MP_{Xj}} = Y \cdot \left(\frac{\alpha_i}{X_i} + \beta_i \right) \div Y \cdot \left(\frac{\alpha_j}{X_j} + \beta_j \right) = \frac{X_j(\alpha_i + \beta_i X_i)}{X_i(\alpha_j + \beta_j X_j)}$$

Between two secondary inputs Z_l, Z_j is:

$$MRTS_{Z_iZ_j} = \frac{dZ_j}{dZ_i} = \frac{MP_{Zi}}{MP_{Zj}} = \frac{(\gamma_i + 2Z_i\delta_i) \cdot Y}{(\gamma_j + 2Z_j\delta_j) \cdot Y} = \frac{(\gamma_i + 2Z_i\delta_i)}{(\gamma_j + 2Z_j\delta_j)}$$

Between one primary and one secondary input, X_k,Z_l:

$$MRTS_{X_iZ_i} = \frac{dZ_i}{dX_i} = \frac{MPx_i}{MP_{Z_i}} = \frac{(\alpha_i / X_i + \beta_i) \cdot Y}{(\gamma_j + 2Z_j\delta_j) \cdot Y} = \frac{(\alpha_i / X_i + \beta_i)}{(\gamma_j + 2Z_j\delta_j)}$$

MRTS can be either positive or negative. Negative MRTS indicates that the firm is operating in an uneconomic region. If MRTS is positive but increasing the assumption of a convex production function is violated.

Partial substitution elasticities

According to Meyer (1970, p.231), the general formula for partial substitution elasticity for both homothetic and homogenous production functions for two primary inputs X_k, X_j:
Is given by: [28]

$$\sigma(X_k, X_j) = \frac{d\ln(X_k / X_j)}{d\ln\left|dX_k / dX_j\right|}$$

$$= \frac{MP_{X_k} \cdot MP_{X_j}(X_j MP_{X_j} + X_k MP_{X_k})}{-X_j X_k \left(\frac{dMP_{X_j}}{dX_j} \cdot \left(MP_{X_k}\right)^2 - 2\frac{dMP_{X_j}}{dX_k} \cdot MP_{X_j} \cdot MP_{X_k} + \frac{dMP_{X_k}}{dX_k}\left(MP_{X_j}\right)^2 \right)}$$

Substituting for MP_{X_k} and MP_{X_j} yields:

$$\sigma(X_k, X_j) = \frac{\varepsilon_k + \varepsilon_j}{\frac{\varepsilon_j}{\varepsilon_k}\left(\alpha_k + \alpha_j\left(\frac{\varepsilon_k}{\varepsilon_j}\right)^2\right)}$$

where $\varepsilon_k, \varepsilon_j$ are output elasticises of the inputs X_k,X_j respectively.

Using the same approach we find:

$$\sigma(X_k, Z_l) = (\varepsilon_k + \eta_l) \Bigg/ \left(\frac{\eta_l}{\varepsilon_k}\left(\alpha_k - 2\delta_l Z_l^{\,2}\left(\frac{\varepsilon_k}{\eta_l}\right)^2\right)\right)$$

$$\sigma(Z_l, Z_j) = -(\eta_l + \eta_j) \Bigg/ \left(\frac{2\eta_j}{\eta_l}\left(\delta_l Z_l^{\,2} + \delta_j Z_j^{\,2}\left(\frac{\eta_l}{\eta_j}\right)^2\right)\right)$$

Variable partial substitution elasticities indicate that substitution possibilities may differ along an isoquant.

[28] See also Gapinski (1980), p. 580.

Returns to scale

According to a general formula, returns to scale can be expressed by a coefficient Λ:[29]

$$\Lambda = \frac{Y'}{Y}$$

Y is the original output level, Y' is the new output level which is obtained by multiplying all inputs by a constant λ. The general formula for k primary inputs and l secondary inputs is given by:

$$\Lambda = \lambda^{\sum_k \alpha_k} \exp\left((\lambda - 1) \cdot \left[\sum_k (\beta_k X_k) + \sum_l \left(\gamma_l Z_l + (\lambda + 1) \delta_l Z_l^2 \right) \right] \right)$$

When the inputs are increased by the same constant so that $\lambda > 1$ then:

If $\Lambda > \lambda$ there is existence of increasing returns to scale

If $\Lambda = \lambda$ there is existence of constant returns to scale

If $\Lambda < \lambda$ there is existence of decreasing returns to scale

When the inputs are decreased by the same constant scalar so that $\lambda < 1$, then

If $\Lambda < \lambda$ there is existence of increasing returns to scale

If $\Lambda = \lambda$ there is existence of constant returns to scale

If $\Lambda > \lambda$ there is existence of decreasing returns to scale

[29] Gapinski (1980), Meyer (1970), Cowell (1986)