Point-Rationalizability in Large Games*

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Abstract: In this paper, I characterize *point-rationalizability* in large non-anonymous games with three different formulations of societal responses, and also propose an implicit dynamic process that is informed by Guesnerie's *eductive* notions. Given the introspection and "mentalizing" that the point-rationalizability notions presuppose, a motivation behind the work is to examine their viability in situations where the terms *rationality* and *full information* can be given a more parsimonious, and thereby more analytically viable, expression.

Keywords: Large games, Nash equilibria, point-rationalizability, closed under rational behavior (curb), societal response, distribution, integration, transformed statistics.

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1 Introduction

Games with a finite number of players are not able to adequately represent many economic environments. In order to capture the numerical negligibility of individuals in the economy, many papers employ a continuum of players. A large number of such papers have recently used the idea of "rationalizability" which amounts to common knowledge of the rationality of the agents. For example, Guesnerie (1992) introduces the concept of strong rationality in a specific market with a continuum of producers; and Evans and Guesnerie (2003) study the strong rationality for dynamics in macroeconomics. These works use an intuitive or model-specific definition of the idea of rationalizability. The goal of my paper is to provide a formal characterization of rationalizability in general game environments with a continuum of agents.

Unlike in general equilibrium structures, as fashioned in the mid-1960s and 1970s by Aumann (1964), Schmeidler (1972), and others following them, a continuum of players is now studied in a context where interdependence is made explicit and rendered analytically tractable. In the "theory of large games," a player's payoff depends, in addition to his or her own actions, on those of "each and everyone else." In contrast to games with finite players, each agent in a large game is strategically-negligible, and payoffs depend only on a statistical summary[1] of their actions. The *other* is no longer a player or a fully delineated group of players, but rather the society or the collective that is the formalized subject of the game. A player's actions, then, are influenced by how he or she conceives and conceptualizes the society of which he or she is a part, rather than how he or she conceptualizes specific individuals. The theory of large games has been well established for certain results such as the existence results of Nash equilibrium.[2] In this paper, my objective is to examine one notion of rationalizability in the context of large games.[3]

Formally, the concept of rationalizability has been introduced into game theory independently by Bernheim (1984) and Pearce (1984). They assert that agents only use strategies that are best responses to their forecasts and therefore, as in Nash equilibrium, some strategies in the action set will never be played. Unlike Nash equilibrium, however, rationalizability does not assume that players correctly predict the actions of the other players. Instead, rationalizability only assumes common knowledge of the rationality of the players. In this context,

[1] In the sequel, and especially in the informal discussion, I may use "state" or "societal response" interchangeably with "statistical summary".

[2] See the survey and the references in Khan and Sun (2002).

[3] In a related work, Jara-Moroni (2008) focuses on a specific type of large games; I shall discuss his work in the sequel.

rationality means that all players only use strategies that are best responses to some possible strategy profiles that actually are played by others, and common knowledge[4] of rationality means that every player knows all other players are rational, and every player knows all other players know that everyone else is rational, and so on. Bernheim (1984) also analyzes another concept *point-rationalizability,* which assumes rationality of point-belief which is to say, any player will not choose a strategy which is not a best response to any strategy profile of his or her opponents, while rationalizability assumes that a player will not choose a strategy which is not expected utility maximizing given any subjective beliefs about opponents' strategies. This paper will only focus on point-rationalizability. The point-rationalizable strategy profiles can be defined as those which survive the *iterated elimination*[5] of strategies which are never best response. Or, equivalently, the set of point-rationalizable strategy profiles should be the maximal subset of strategy profiles such that any strategy of an individual player is a best response to some combination of opponents' strategies within the subset.[6] Bernheim (1984) shows that these characterizations are well-defined and equivalent.

However, Bernheim (1984), Pearce (1984) and the literature[7] that follows restrict attention to games with *a finite number of players*. In this paper, I combine the literature on large games with the literature on rationalizability and characterize point-rationalizability for large games. As mentioned, this is motivated by the fact that the idea of rationalizability appears in the applied work with a continuum of players, despite the fact that a formal characterization of rationalizability in general game models is lacking. Specifically, I adapt the two characterizations in Bernheim (1984) to large games. The set of point-rationalizable societal responses is demonstrated to be well-defined as a result of the iterative elimination of strategies that are not best responses to forecasts of the society. This captures the idea of the characterization in Bernheim (1984) as well as that of Pearce (1984). Moreover, similar to Bernheim (1984), I show the set is also related to some curb set. Thus, by focusing on point-rationalizability within a large game framework, this solution set can be justified by the epistemic assumption that it is common knowledge that only best responses are ever chosen, while at the same time such a characterization will not deviate significantly from the Nash equilibrium. Moreover, the epistemic interpretation of point-rationalizability may also provide a plausible justification for equilibrium play. For example, if the point-rationalizable

[4]One can refer to Aumann (1976b) for a formal definition of common knowledge.

[5]This process is exactly the one used by Pearce (1984) to define rationalizability. But Pearce (1984) uses mixed strategy to eliminate the never-best-responses.

[6]This characterization is related to Basu and Weibull (1991) where a set of strategy profiles is said to be *closed under rational behavior (curb)* if the set contains all its best responses. The abbreviation *curb* will be used throughout the paper.

[7]See, for example, Brandenburger and Dekel (1987), Dekel et al. (2007), and so on.

state is a singleton, then the best responses of all players form a Nash equilibrium.

I characterize point-rationalizability in large games by considering three different formulations of societal responses. The first game model takes distributions of all the actions as societal interdependence: the payoffs to each player depend on both his or her own action and the proportion of other players who play each action. For example, in voting games,[8] the payoff of each voter depends on the electoral outcomes which in turn depends on proportion of the electorate that votes for each candidate. The second model formulates averages of all players' actions as societal responses: each player's payoff depends on both his or her own action and the mean of the aggregate strategy profiles. For example, in the rational expectation model,[9] the profit of each farmer depends on his or her own supply and the aggregation supply of all farmers. Finally, the third game model formulates transformed summary statistics: players' payoffs depend on their own actions and the mean of the strategy profiles under a general transformation. For example, in monopolistic competition models with summary statistics,[10] the payoff of each firm depends its own action and some summary statistics of the aggregate strategy profile.

This paper is most closely related to the work of Jara-Moroni (2008) who characterizes point-rationalizability similarly, but within a framework of large games considered in Rath (1992) in which societal responses are set as averages of actions, while the action set is a compact subset of a finite dimensional Euclidean space. The first formulation of societal responses in the paper works on distributions as considered in Khan and Sun (1995) and Keisler and Sun (2009) instead, and the second formulation addresses this issue for the characterization of point-rationalizability in games with averages as societal responses, but with an infinite dimensional action set. Moreover, note that the tools for the analysis in Jara-Moroni (2008) are based on the integration theorem from Aumann (1965), which yields results on the convexity, hemicontinuity and compactness of the integral in finite dimensional settings that are no longer valid for general infinite settings, while at the same time the infinite dimensional action set is widely used in the literature for large games and mathematical economics. The third formulation generalizes that of Jara-Moroni (2008), by allowing a general compact metric space and a general transformation.

The rest of the paper is organized as follows: Section 2 presents the model and results on large games with societal responses formulated as distributions; Section 3 shows the corresponding results when societal responses are formulated as averages and the action

[8] See, for example, Banks and Duggan (2006).

[9] See, for example, Guesnerie (1992).

[10] See, for example, Rauh (2003, 1997).

set is infinite dimensional; and Section 4 gives characterizations of point-rationalizability in the large games with transformed summary statistics. Section 5 concludes. Some technical proofs are collected in Appendix A, and the basic results on distributions and integrations of correspondences are given in Appendix B.

2 Games with Distributions as Societal Responses

Let I be the set of players, $(I, \mathcal{I}, \lambda)$ an atomless probability space[11] representing the space of player names, and A a non-empty compact metric space[12] which represents the action space. Each player $i \in I$ choose her or his own action from the action set A. Let $\mathcal{M}(A)$ be the space of probability measures on A endowed with the weak topology and $\mathcal{U}_A$ be the space of real-valued continuous functions on $A \times \mathcal{M}(A)$, endowed with its sup-norm topology and with $\mathcal{B}(\mathcal{U}_A)$ its Borel σ-algebra.

A *large non-anonymous game* $\mathcal{G}$ is a measurable function from I to $\mathcal{U}_A$. And a *strategy profile* of the game is a measurable function $f : I \to A$ which specifies a strategy for each player. A *Nash equilibrium* of a game $\mathcal{G}$ is a strategy profile $f^* : I \to A$, such that for λ-almost all $i \in I$,

$$u_i(f^*(i), \lambda f^{*-1}) \geq u_i(a, \lambda f^{*-1})$$

for all $a \in A$, where $u_i = \mathcal{G}(i)$.

In addition, if the action set A of a game is countable, there exists a Nash equilibrium of the game.[13] One can consider the general action set too when name space is saturated as in Keisler and Sun (2009). Note that the general existence result on of Nash equilibrium for large games with any cardinality of action set A involves the basic definition of saturation. For the convenience of the reader, the following definition is taken from Hoover and Keisler (1984, Definition 5.1).

Definition 1. *A probability space $(I, \mathcal{I}, \lambda)$ is said to be saturated if for any two Polish spaces X and Y, any Borel probability measure $\tau \in \mathcal{M}(X \times Y)$ with marginal probability measure τ_X on X, and any measurable mapping g from $(I, \mathcal{I}, \lambda)$ to X with distribution τ_X, there exists a measurable mapping $h : (I, \mathcal{I}, \lambda) \longrightarrow Y$ such that the measurable mapping $(g, h) : (I, \mathcal{I}, \lambda) \longrightarrow X \times Y$ has distribution τ.*

[11] For example, one can take $(I, \mathcal{I}, \lambda)$ as the unit interval $[0, 1]$ with Lebesgue measure.

[12] Note that in any sequential model, even if the action choices are finite in every period, in general the set of all actions is infinite, and thus I do not limit the action set in the model to be finite.

[13] The existence result of Nash equilibrium is shown in Schmeidler (1973), Khan and Sun (1995), and Yu and Zhang (2007) for such games with different assumptions of action sets.

A saturated probability space admits more measurable functions to capture a richer variety of non-cooperative behavior. Note that the Lebesgue unit interval is not a saturated probability space. In comparison, any atomless Loeb probability space is saturated. I can now present the following proposition[14] which is summarized from Khan and Sun (1995) and Keisler and Sun (2009) on the existence of Nash equilibrium in $\mathcal{G}$.

Proposition 1. *Every large game $\mathcal{G}$ has a Nash equilibrium if either of the following two (sufficient) conditions hold:*

(i) A is countable,

(ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.

I now consider point-rationalizability in $\mathcal{G}$. For any given strategy profile, the payoff function of each player depends on the player's own strategy and the distribution induced by some plausible strategy profile. Unlike games with finite players in which one could work on the strategy profile directly, in the large games, it is intuitively that all the players will use the societal responses they face as their forecast. As a result, when one deals with point-rationalizability, one could eliminate certain societal response which is never realized. More formally, for any $\tau \in \mathcal{M}(A)$ and any $i \in I$, let $B(i, \tau)$ (also, denoted by $B_i(\tau)$) be player i's best response when he or she faces τ. That is,

$$B(i, \tau) = \arg\max_{a \in A} u_i(a, \tau).$$

For each $i \in I$ and a set $\mathcal{D} \subseteq \mathcal{M}(A)$, the image of $\mathcal{D}$ under $B(i, \cdot)$ is the set

$$B(i, \mathcal{D}) = \bigcup_{\tau \in \mathcal{D}} B(i, \tau).$$

This can be interpreted as all the possible best responses of agent i facing the set $\mathcal{D}$ which is a subset of all action distributions.

Given $\mathcal{D} \subseteq \mathcal{M}(A)$, consider the set of all the measurable selections chosen from the correspondence $B(\cdot, \mathcal{D}) : I \twoheadrightarrow A$. Define the mapping $\tilde{Pr} : 2^{\mathcal{M}(A)} \to 2^{\mathcal{M}(A)}$, so that for any subset $\mathcal{D} \subseteq \mathcal{M}(A)$,

$$\tilde{Pr}(\mathcal{D}) = \{\lambda f^{-1} : f \text{ is a measurable selection of } B(\cdot, \mathcal{D})\}.$$

[14] I refer to results previously available in the literature as "propositions" throughout the paper.

To show $\tilde{Pr}(\mathcal{D})$ is well-defined, I need to show that the measurable selection of $B(\cdot, \mathcal{D})$ is not vacuous. Thus, it is necessary to check the measurability of the correspondence of $B(\cdot, \mathcal{D})$ for any subset $\mathcal{D}$ of $\mathcal{M}(A)$ to ensure that a measurable selection exists. This is guaranteed by the following Lemma.[15]

Lemma 1. *For any nonempty closed set $\mathcal{D}$ of $\mathcal{M}(A)$, $B(\cdot, \mathcal{D})$ is measurable, closed-valued.*[16] *and nonempty valued.*

I now am ready to define the set which captures the idea of iterative elimination of those action distributions which never happen by assuming everyone will play the best response to his or her societal response and common knowledge of this.

Definition 2. *The set of point-rationalizable distributions*[17] *$\mathbb{P}_{\mathcal{G}}$ of $\mathcal{G}$ is $\bigcap_{t=0}^{+\infty} \tilde{Pr}^t(\mathcal{M}(A))$, where*

$$\tilde{Pr}^0(\mathcal{M}(A)) = \mathcal{M}(A)$$

$$\tilde{Pr}^{t+1}(\mathcal{M}(A)) = \tilde{Pr}(\tilde{Pr}^t(\mathcal{M}(A))), \text{ for all } t \geq 1.$$

Thus, the set $\mathbb{P}_{\mathcal{G}}$ is constructed as follows: Beginning from the set of all possible action distributions $\mathcal{M}(A)$, delete any strategies which are not best responses to at least one element in $\mathcal{M}(A)$ for each player i. Then consider the set of all the distributions induced by all the measurable selections from the correspondence $B(\cdot, \mathcal{M}(A))$. Infinite repetition of this process generates the set $\mathbb{P}_{\mathcal{G}}$. The point-rationalizable strategy profile of $\mathcal{G}$ is thus a measurable selection of $B(\cdot, \mathbb{P}_{\mathcal{G}})$. It is easy to see that Nash equilibrium is a point-rationalizable strategy profile. Moreover, one can observe that $\tilde{Pr}^{t+1}(\mathcal{M}(A)) \subseteq \tilde{Pr}^t(\mathcal{M}(A))$. I now ready to show $\mathbb{P}_{\mathcal{G}}$ is well-defined under certain conditions.

Theorem 1. *The set of point-rationalizable distributions of $\mathcal{G}$ is non-empty, compact and convex if either of the following two conditions hold:*

(i) A is countable,

(ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.

Proof: Note that the societal responses generated by any Nash equilibrium can never be eliminated by this process of elimination of non-reachable societal responses. Thus $\mathbb{P}_{\mathcal{G}} \neq \emptyset$

[15] I shall defer all proofs to Lemmas in Appendix A.

[16] Hence, $B(\cdot, \mathcal{D})$ is also compact-valued since A is compact.

[17] The intuitive idea of a "societal response" is formalized by "distribution" in this section, by "average" in the next, and by a "transformed statistic" in Section 4.

follows from the fact that Nash equilibrium of Γ exists given (i) or (ii) holds by Proposition 1.

To show that $\mathbb{P}_{\mathcal{G}}$ is also convex and compact, consider the following sequence $F^t : I \twoheadrightarrow A, t \geq 0$ given by

$$F^0(i) = A, \text{ for all } i \in I$$

$$F^t(i) = B\left(i, \tilde{Pr}^{t-1}(\mathcal{M}(A))\right), \text{ for all } i \in I, \text{ if } t \geq 1,$$

where

$$\tilde{Pr}^{t}(\mathcal{M}(A)) = \{\lambda f^{-1} : f \text{ is a measurable selection of } F^t\}.$$

By Berge's maximum theorem (see, for example, Berge (1959, Section III.6)), for each $i \in I$, the joint continuity of u_i on $A \times \mathcal{M}(A)$ implies the upper hemicontinuity of $B(i, \cdot)$ on $\mathcal{M}(A)$. By Aliprantis and Border (1999, Lemma 17.8), $B(i, X)$ is compact for any compact subset $X \subseteq \mathcal{M}(A)$. Furthermore, note that A is compact, so is $\mathcal{M}(A)$. By Lemma 1, F^1 is measurable and compact-valued. By induction over t, together with Proposition 3 in Appendix A, $\tilde{Pr}^{t-1}(\mathcal{M}(A))$ is also compact and convex for all $t \geq 1$. Hence $\mathbb{P}_{\mathcal{G}}$ is also convex and compact as the intersection of the nested family of nonempty compact convex sets $\{\tilde{Pr}^{t}(\mathcal{M}(A))\}$. ∎

I now connect the point-rationalizability to curb[18] set.

Definition 3. *A set of action distributions $\mathcal{D}$ is said to be point-curb in $\mathcal{G}$ if $\mathcal{D} \subseteq \tilde{Pr}(\mathcal{D})$.*

By definition, a set of action distributions $\mathcal{D}$ is point-curb if all players have the point-belief that any strategy profile which induces an action distribution lies outside $\mathcal{D}$ will not be played. This simply implies that such strategies in the strategy profile shall indeed not be played by almost all players who are rational.

A point-curb set of distributions $\mathcal{D}$ is said to be *tight* if $\mathcal{D} = \tilde{Pr}(\mathcal{D})$. Bernheim (1984) shows that in a finite game with continuous payoffs and compact strategy sets, a *maximal tight point-curb* set of strategy profiles[19] coincides with the set of point-rationalizable strategy profiles. I am now ready to establish the equivalence of the maximum tight point-curb set and point-rationalizability in terms of action distributions in large games.

Theorem 2. *A set of action distributions in any large non-anonymous game $\mathcal{G}$ is a maximum*

[18] See footnote 6. I shall use a special case of curb set by considering on payoff maximizing given societal responses, instead of expected utility maximizing given any subjective beliefs about societal responses. Thus, I use "point-curb" for this restriction.

[19] It is trivial to see that a set of strategy profiles F is point-curb only if $F \subseteq B(\cdot, \mathcal{D}_F)$ where $\mathcal{D}_F = \{\lambda f^{-1} : f \text{ is a measurable selection of } F\}$ in the context of large games.

tight point-curb set if and only if it is the set of point-rationalizable distributions of the same game, providing one of the following two conditions holds:

(i) A is countable,

(ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.

Proof: I first show $\mathbb{P}_{\mathcal{G}}$ is a tight point-curb set of distributions, namely, $\tilde{Pr}(\mathbb{P}_{\mathcal{G}}) \subseteq \mathbb{P}_{\mathcal{G}}$. Towards this end, for any $\tau \in \tilde{Pr}(\mathbb{P}_{\mathcal{G}})$, there exists a measurable selection $f : I \to S$ of $B(\cdot, \mathbb{P}_{\mathcal{G}})$, such that $\tau = \lambda f^{-1}$ by the definition of $\tilde{Pr}$. Moreover, $\mathbb{P}_{\mathcal{G}} \subseteq \tilde{Pr}^t(\mathcal{M}(A))$, for all $t \geq 0$. Hence, $B(i, \mathbb{P}_{\mathcal{G}}) \subseteq B(i, \tilde{Pr}^t(\mathcal{M}(A)))$, for any $t \geq 0$ and for all $i \in I$. Thus, f is a selection of $B(\cdot, \tilde{Pr}^t(\mathcal{M}(A)))$. This implies for any t, $\tau \in \tilde{Pr}^t(\mathcal{M}(A))$, which is to say, $\tau \in \mathbb{P}_{\mathcal{G}}$. Hence, $\tilde{Pr}(\mathbb{P}_{\mathcal{G}}) \subseteq \mathbb{P}_{\mathcal{G}}$.

I still need to show $\mathbb{P}_{\mathcal{G}} \subseteq \tilde{Pr}(\mathbb{P}_{\mathcal{G}})$ to show $\mathbb{P}_{\mathcal{G}}$ is tight. To do so, consider the same sequence $\{F^t\}$ which is constructed in the proof of Theorem 1. It is easy to see F^t is measurable and compact-valued. Now consider the correspondence $F : I \twoheadrightarrow A$ defined as

$$F(i) = cl - Lim\{F^t(i)\}.^{20}$$

Take a point $\tau \in \mathbb{P}_{\mathcal{G}}$. It is clear that $\tau \in \mathcal{D}_{F_t}$, for all $t \geq 0$. This gives a sequence of measurable selections $\{f^t\}_{t \in \mathbb{N}}$, such that $\tau_t = \lambda f^{t^{-1}} = \tau$, for all $t \geq 0$. It is trivial that $\{\tau_t\}$ converges weakly to τ. According to Proposition 3 in Appendix B, $\tau \in \mathcal{D}_F$. I now only need to show that $F(i) \subseteq B(i, \mathbb{P}_{\mathcal{G}})$ so that I can obtain that $\tau \in \mathcal{D}_F \subseteq \tilde{Pr}(\mathbb{P}_{\mathcal{G}})(= \mathcal{D}_{B(\cdot, \mathbb{P}_{\mathcal{G}})})$. This is implied by the upper hemicontinuity of $B(i, \cdot)$: For any $y \in F$, according to the construction of F, one can find $y^t \in F^t$ for each $t \geq 0$, such that $y = cl - Lim\{y^t\}$. That is to say, fix any i, there is a sequence $\{\tau^k\}$ such that $\tau^k \in \tilde{Pr}^{t_k - 1}(\mathbb{P}_{\mathcal{G}})$, $y^{t_k}(i) \in B(i, \tau^k)$ and $lim_k y^{t_k}(i) = y(i)$. Thus, one can obtain that the limit of $\{y^{t_k}(i)\}$ must belong to $B(i, \mathbb{P}_{\mathcal{G}})$ since all weakly limit points of $\{\tau_k\}$ are in $\mathbb{P}_{\mathcal{G}}$, being the intersection of a nested family of compact sets. Thus $F(i) \subseteq B(i, \mathbb{P}_{\mathcal{G}})$. Thus, $\mathbb{P}_{\mathcal{G}}$ is a tight point-curb set.

Let the maximal tight point-curb set of distributions of $\mathcal{G}$ be $\mathbb{C}_{\mathcal{G}}$. I am now ready show that $\mathbb{P}_{\mathcal{G}} = \mathbb{C}_{\mathcal{G}}$. First it is easy to check for any two sets $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{M}(A)$, $\tilde{Pr}(\mathcal{D}_1) \cup \tilde{Pr}(\mathcal{D}_2) \subseteq \tilde{Pr}(\mathcal{D}_1 \cup \mathcal{D}_2)$. If $\mathbb{C}_{\mathcal{G}} \neq \mathbb{P}_{\mathcal{G}}$, there must be $\mathcal{D} \subseteq \mathcal{M}(A)$ for which $\tilde{Pr}(\mathcal{D}) = \mathcal{D}$, and some number k such that $\mathcal{D} \cap \tilde{Pr}^k(\mathcal{M}(A)) = \mathcal{D}$, i.e., $\mathcal{D} \cup \tilde{Pr}^k(\mathcal{M}(A)) = \tilde{Pr}^k(\mathcal{M}(A))$, but $\mathcal{D} \cap \tilde{Pr}^{k+1}(\mathcal{M}(A)) \subset \mathcal{D}$. Note that $\tilde{Pr}(\mathcal{D}) \cup \tilde{Pr}(\tilde{Pr}^k(\mathcal{M}(A))) \subseteq \tilde{Pr}(\mathcal{D} \cup \tilde{Pr}^k(\mathcal{M}(A))) =$

[20]For any sequence $\{x_t\}$ in a topological space, denote $cl - Lim\{x_t\}$ the set of its limit points. For any sequence of sets $\{A_t\}$ in a topology space, denote $cl - Lim\{A_t\}$ the union of all such $cl - Lim\{x_t\}$ with $x_t \in A_t$ for all t.

$\tilde{Pr}(\tilde{Pr}^k(\mathcal{M}(A)))$. Intersecting left and right sides with $\mathcal{D}$, one can get $\mathcal{D} \subseteq \mathcal{D} \cap \tilde{Pr}^{k+1}(\mathcal{M}(A))$, which is a contradiction. This completes the proof. ∎

The main technical challenge in the proofs of Theorem 1 and 2 is to identify the adequate convergence concept for the elimination process and to ensure the valid strategy profile selection in each step of elimination. Such problem does not apply in the case of finite player games considered in Bernheim (1984) or is covered in the case of a continuum of players with the strategy (action) spaces is finite dimensional and the societal responses as a convex combinations of actions considered in Jara-Moroni (2008). I now conclude this section with the following two remarks.

Remark 1. The framework in this section is phrased in terms of distributions. The most relevant mathematical tools revolve around the distribution of a correspondence from an atomless probability space to a metric space. When A is countable, the theory is developed in Khan and Sun (1995) based on the Bollobás-Varopoulos Bollobás and Varopoulos (1974) extension of the marriage lemma. If A is uncountable, one has to impose the saturation property upon the name space so that one can use the distribution of correspondences in Keisler and Sun (2009). The compactness of the action set can be relaxed by assuming instead that (i) the common action space is a complete metric space, and (ii) each player may choose a heterogenous compact subset for the common action set. For details, one can refer to Yu and Zhang (2007).

Remark 2. It is worth to point out that it is interesting when the set of point-rationalizable societal response $\mathbb{P}_{\mathcal{G}}$ of a game $\mathcal{G}$ is a singleton. As shown in Proposition 1, there always exists a Nash equilibrium in the games described in this section. If $\mathbb{P}_{\mathcal{G}}$ is a singleton, it must be the distribution induced by the Nash equilibrium. This observation is also true for the next two sections.

3 Games with Averages as Societal Responses

I now present the characterization results on point-rationalizability in large games where societal responses are considered as averages or mean of everyone's strategy. There is a natural question as to what are the assumptions on the parameters that ensure the measurability of the function summarizing the responses taken in a non-cooperative setting. This measurability problem does not arise for a game with a finite set of players. In particular, I deal with an infinite set of strategies which cannot be expressed as a linear combination of a finite

number of elements.[21] As we show in earlier section, one needs compactness, upper hemicontinuity, convexity of correspondences which indeed hinge on whether the point-rationalizable responses are not eliminated by the processes under the assumption of rationality and common knowledge of point-belief of rationality. Throughout this section, I still fix an atomless probability measure space $(I, \mathcal{I}, \lambda)$ as the space of players.

3.1 A Benchmark: Games with Action Set in $\mathbb{R}^\infty$

In order to define a large non-anonymous game and the corresponding point-rationalizable societal response, I have to specify the notion of "societal response," as a notion of an integral or an average.

let $\mathbb{R}^\infty$ be the space of infinite sequences endowed with the product topology. Let $\{e_n\}_{n\geq 1}$ be the standard vectors in $\mathbb{R}^\infty$ and $e_0 = 0$. Let $A = \{e_n\}_{n\in\mathbb{N}}$ be the action set and $\overline{con}\,(A)$ the closed convex hull of A. Before defining a game and point-rationalizability, I first provide the mathematical preliminaries.

For any function $f : I \longrightarrow \mathbb{R}^\infty; f(i) \in \{e_n\}_{n\in\mathbb{N}}$, f is measurable if $f^{-1}(e_n) \in \mathcal{I}$ for any $n \in \mathbb{N}$. The integral of any measurable function f over I is thus given by

$$\int_I f d\lambda \ = \sum_{n\in\mathbb{N}} e_n \lambda(f^{-1}).$$

By the standard definition of the integral of a correspondence which is due to Aumann (1965), the integral of a correspondence F from I into A is,

$$\left\{ \int_I f d\lambda : f \text{ is an integrable selection of } F \right\}.$$

Let $\mathcal{C}_A$ be the space of continuous functions on $A \times \overline{con}\,(A)$. Since $\overline{con}\,(A)$ is compact in the topology of coordinatewise convergence, so is $A \times \overline{con}\,(A)$. Thus, $\mathcal{C}_A$ can be endowed with the sup-norm topology with its Borel σ-algebra $\mathcal{B}(\mathcal{C}_A)$.

A *large non-anonymous game* $\mathcal{G}^C$ is a measurable function from I to $\mathcal{C}_A$. And a *strategy profile* of such a game is a measurable function $f : I \to A$. A *Nash equilibrium* of a game $\mathcal{G}^C$ is a $f^* : I \to A$, such that for λ-almost all $i \in I$,

$$u_i(f^*(i), \int_I f^* d\lambda) \geq u_i(a, \int_I f^* d\lambda)$$

[21] Note that in Jara-Moroni (2008), the analysis for point-rationalizability is given for societal responses formulated as averages but with the action set as a compact subset of finite dimensional space.

for all $a \in A$, where $u_i = \mathcal{G}^C(i)$.

In a game described above, payoff function of any player depend on each player's own strategy and the societal response which is formulated as the integration of everyone's strategies. The following proposition, which is a restatement of Theorem 1 in Khan et al. (1997), shows the existence of Nash equilibrium of such games.

Proposition 2. *There exists an Nash equilibrium in $\mathcal{G}^C$, which is a measurable function from I to $\mathcal{C}_A$.*

I now define point-rationalizability to the model above. For any strategy profile f, its integral over I, $\int_I f d\lambda \in \overline{con}\,(A)$. According to rationality, for any $i \in I$, and any fixed $\iota \in \overline{con}\,(A)$ for which player i perceives as societal response, player i will only choose action from $B(i, \iota)$, where

$$B(i, \iota) = \arg\max_{a \in A} u_i(a, \iota).$$

For any $D \subseteq \overline{con}\,(A)$, the possible actions of agent i should be in the set:

$$B(i, D) = \bigcup_{\iota \in D} B_i(\iota).$$

I have to show the measurability of the correspondence $B(i, D)$ for any subset D of $\overline{con}\,(A)$ as well.

Lemma 2. *$B(\cdot, D)$ is measurable and compact-valued for any closed set D of $\overline{con}\,(A)$.*

If all players know that the possible societal responses are restricted to some subset, then the averages that are generated by all possible strategy profiles due to the rationality can be characterized by the correspondence $\tilde{Pr} : 2^{\overline{con}\,(A)} \to 2^{\overline{con}\,(A)}$ so that for any subset $X \subseteq \overline{con}\,(A)$,

$$\tilde{Pr}(X) = \int_I B(i, X) di = \left\{ \int_I f di : f \text{ is a measurable selection of } B(\cdot, X) \right\}.$$

Similar to the previous sections, I am ready to define both the point-rationalizability and maximal tight point-curb set when societal responses are averages.

Definition 4. *The set of all point-rationalizable averages of $\mathcal{G}^C$ is $\bigcap_{t=0}^{+\infty} \tilde{Pr}^t(\overline{con}\,(A))$, where*

$$\tilde{Pr}^0(\overline{con}\,(A)) = \overline{con}\,(A),$$

$$\tilde{Pr}^{t+1}(\overline{con}\,(A)) = \tilde{Pr}(\tilde{Pr}^t(\overline{con}\,(A))), \text{ for all } t \geq 1.$$

Definition 5. *The maximal tight curb set of averages of the game $\mathcal{G}^C$ is the maximal set $D \subseteq \overline{con}\,(A)$ satisfying*

$$D = \tilde{Pr}(D).$$

Similar to the results in last section, the next theorem characterizes point-rationalizability in game $\mathcal{G}^C$.

Theorem 3. *Let $\mathcal{G}^C$ be a measurable function from I to $\mathcal{C}_A$, the set of point-rationalizable averages of $\mathcal{G}^C$ is nonempty, convex and compact. Moreover, it is equivalent to the maximal tight point-curb set of the same game.*

Proof: The proofs of Theorems 1 and 2 apply directly here other than the fact that Proposition 2 and the relevant analogues of the convexity, compactness, convergence and upper hemicontinuous results are used. Such analogues are reported as Proposition 4 in Appendix B. ∎

3.2 General Games

I have shown the characterization of the point-rationalizability with the societal response formalized as an "average" over all of the players' strategies in the game when action set is $\{e_n\}_{n\in N}$. Note that any compact (infinite) set A can be embedded in a Banach space $(X, \|\cdot\|)$, where X is assumed to be separable. To discuss point-rationalizability in a more general setting, I now consider societal response as the Bochner integral of a measurable function.[22].

Towards this, let A be a weakly compact set in a separable Banach space $(X, \|\cdot\|)$. Let $\mathcal{C}_A^B$ be the space of weakly continuous real-valued functions on $A \times \overline{con}\,(A)$ endowed with the sup-norm topology with its Borel σ-algebra $\mathcal{B}(\mathcal{C}_A^B)$. The next theorem shows if in addition (i) A is countable; or (ii) $(I, \mathcal{I}, \lambda)$ is saturated, one can characterize the point-rationalizability as well. I first provide a lemma with respect to the existence of Nash equilibrium to guarantee that the point-rationalizable states in this structure is not vacuous.

Lemma 3. *Proposition 2 is valid with A a nonempty weakly compact subset of a separable Banach space, $\mathcal{C}_A$ replaced by $\mathcal{C}_A^B$, and the integrals interpreted as Bochner integrals, if one of the following two conditions hold:*

(i) A is countable,

(ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.

[22]Such a formalization is well-established in the literature, see Khan and Yannelis (1991) and its references.

I now can present the characterization of point-rationalizability in general games as follows.

Theorem 4. *Theorem 3 is valid with A a nonempty weakly compact subset of a separable Banach space, $\mathcal{C}_A$ replaced by $\mathcal{C}_A^B$, and the integrals interpreted as Bochner integrals, if either of the following two conditions hold:*

(i) A is countable,

(ii) $(I,\mathcal{I},\lambda)$ is a saturated probability space.

Proof: One can follow the proof of Theorem 3 by using Lemma 3 and Proposition 5 in Appendix B instead of Proposition 2 and Proposition 4. ∎

Note that norm continuous functions on norm compact sets are weakly continuous too. Together with the fact that if A is norm compact, $\overline{con}\,(A)$ is also norm compact, I now consider the space of payoffs as $\mathcal{C}_{\mathcal{N}A}^{\mathcal{B}}$, the space of norm continuous real-valued functions on $A \times \overline{con}\,(A)$ endowed with the sup-norm topology with its Borel σ-algebra by this topology. Then the following result if a direct corollary of Theorem 4.

Corollary 1. *Theorem 4 is valid with A a norm compact set and $\mathcal{C}_A^B$ substituted with $\mathcal{C}_{\mathcal{N}A}^{\mathcal{B}}$.*

I next explore the alternative definition of measurability and integrability of averages. Consider the dual X^* of a separable Banach space X. Let A be a weak* compact subset of X^* and B the weak* closed convex hull of A. Note that B is still weak* compact. Let $\mathcal{C}_A^{B^*}$ be the space of weak* continuous real-valued functions on $A \times B$ endowed with the sup-norm topology and its Borel σ-algebra $\mathcal{B}(\mathcal{C}_A^{B^*})$ generated by this topology. All integrals are to interpreted as Gel′fand integrals.

Theorem 5. *Lemma 3 and Theorem 4 are valid with the space $\mathcal{C}_A^{B^*}$ substituting $\mathcal{C}_A^B$ and with the integrals all interpreted as Gel′fand integrals.*

Proof: All one needs is Proposition 6 reported in Appendix B. ∎

Note that Theorem 5 covers the case of the norm topology in X^* too provided that the Gel′fand integral is used. Thus, consider an action set A^m consisting of probability measures on a compact set A. Let $\mathcal{C}_A^m$ be the space of weakly continous real-valued functions on $A^m \times \overline{con}\ (A^m)$ endowed with the weak* topology and with the Borel σ-algebra $\mathcal{B}(\mathcal{U}_A^m)$ generated by this topology. I now collude this section with the following observation.

Corollary 2. *Theorem 5 is valid by replacing $\mathcal{C}_A^B$ with $\mathcal{C}_A^m$, and A with A^m.*

4 Games with Transformed Statistics as Societal Responses

I have now also provided the characterization for point-rationalizability for large games with societal responses as averages of the strategies to cover cases when A is infinite dimensional. There are some certain games in the literature to deal with societal responses where only some general transformations of players' strategies may be available. For example, in the study of monopolistic competitions (see, for example, Rauh (1997) and Vives (1999)), it is assumed that the players' payoffs depend on their own actions and the transformed summary statistics of the aggregate strategy profiles in terms of the moments of the distributions of players' actions. The existence of Nash equilibrium has been studied in Rauh (1997) and Yu and Zhu (2005) for such games where societal responses formulated as transformed statistics. I now discuss the point-rationalizability for such games.

As previous sections, let an atomless probability space $(I, \mathcal{I}, \lambda)$ represent the space of player names, a nonempty compact metric space A be the set of actions. Each player $i \in I$ chooses a strategy from A. A *strategy profile* is a measurable function $f : I \to A$, which specifies a strategy for each player.

Let s be a continuous function from A to the n-dimensional Euclidean space $\mathbb{R}^n$, and C the range of s. The continuity of s and compactness of A imply that C is also compact. Let Σ be the convex hull of C. Thus, for any given strategy profile f, let $\sigma_f = \int_I (s \circ f) d\lambda \in \Sigma$. The mean σ_f of $s \circ f$ is a transformed summary statistics of the society which the players can observe. A characteristics (or payoff) function for a player is a real-valued continuous function defined on $A \times \Sigma$, which means that any player's payoff depends on his or her own action $a \in A$ and a transformed summary statistics $\sigma \in \Sigma$. Let $\mathcal{P}$ denote the space of all continuous payoff functions with the supremum norm.

A *game* is thus a measurable function $\Gamma : I \to \mathcal{P}$. And its *equilibrium* for such a game is a strategy profile $f^* : I \to A$ such that each player plays a best response against the induced vector of summary statistics, i.e.,

$$\Gamma(i)(f^*(i), \sigma_{f^*}) \geq \Gamma(i)(a, \sigma_{f^*})$$

for λ-almost all $i \in I$ and $a \in A$.

The model deals with general compact metric action set but with finite dimensional target space of summary statistics. As noted in Yu and Zhu (2005), the finiteness of the target space is justifiable. For instance, it is natural to assume that firms' payoffs depend on their own quantities along the time and the summary statistics of the society. If one set time to be $[0, T]$. A continuum of firms $[0, 1]$ take actions from action set A, where A is taken

to be a bounded closed subset of $L^\infty([0,T],\mathbb{R})$ with topology $\sigma(L^\infty([0,T],\mathbb{R}),L^1([0,T],\mathbb{R}))$. Let D be an upper bound for A and $s : A \to \mathbb{R}^n$ a projection at n epoches: for $f \in P$, $s(f) = (f(\tau_1),\ldots,f(\tau_n))$, where $(\tau_1,\ldots,\tau_n)$ are n fixed sampling times. The set of summary statistics Σ can be taken as $[0,D]^n$. The payoff function for a firm is a real-valued continuous function defined on $A \times \Sigma$.

The existence of Nash equilibrium for such games has been shown in Yu and Zhu (2005). In this type of game, societal responses is formulated as transformed summary statistics described above. All players have forecasts over the set of summary statistics, since given any $\sigma \in \Sigma$, the relevant information that each player makes decision (or chooses action) is just the value of σ. For any player $i \in I$, for any given summary statistics $\sigma \in \Sigma$, let

$$B(i,\sigma) = \arg\max_{a\in A}\Gamma(i)(a,\sigma).$$

The mapping $B(i,\cdot) : \Sigma \twoheadrightarrow A$ gives player $i \in I$ a rule to choose his or her action rationally based on point-belief if his or her perception of societal response is σ. Thus, facing a subset $X \subseteq \Sigma$, i.e., when i believes that transformed statistics in X is possible, the possible actions of agent i should be in the set:

$$B_i(X) = \bigcup_{\sigma\in\Sigma} B_i(\sigma).$$

It is necessary to check the measurability of $B(\cdot,X)$ for any subset X of Σ.

Lemma 4. *The correspondence $B(\cdot,X)$ is measurable and compact-valued for any closed set X of Σ.*

Now, focus on the process of elimination of non-reachable or non-generated summary statistics. Given that all players know that the forecasts are restricted to X, they know that the actual summary statistics must satisfy certain conditions. In order to characterize this, we define the mapping $\tilde{Pr} : 2^\Sigma \to 2^\Sigma$ so that for any subset $X \in \Sigma$,

$$\tilde{Pr}(X) = \left\{ \int_I s\circ f(i)d\lambda, f \text{ is a measurable selection of } B(\cdot,X) \right\}.$$

Let $G : I \times \Sigma \twoheadrightarrow \Sigma$ to be the correspondence defined by $G(i,\sigma) = s(B(i,\sigma))$. Thus, one may rewrite $\tilde{Pr}(X)$ as the integral [23] of a set valued mapping

$$\tilde{Pr}(X) = \int_I G(i,X)d\lambda.$$

[23] See Section 3.1 or Aumann (1965) for the integral of a correspondence.

Similar to the previous sections, I now define the point-rationalizable summary statistics which captures the idea of iterative elimination of those transformed statistics which never happen by assuming everyone will play the best response to certain societal responses and common knowledge of this. Starting from Σ, consider the following mappings:

$$\tilde{Pr}^0(\Sigma) = \Sigma,$$

$$\tilde{Pr}^{t+1}(\Sigma) = \tilde{Pr}(\tilde{Pr}^t(\Sigma)), \text{ for all } t \geq 1.$$

Definition 6. *The set of all point-rationalizable transformed statistics of* Γ *is*

$$\mathbb{P}_\Gamma = \bigcap_{t=0}^{+\infty} \tilde{Pr}^t(\Sigma).$$

With the mapping $\tilde{Pr}$, I could define the maximal tight curb set of transformed statistics in game Γ as follows:

Definition 7. *The maximal tight curb set of summary statistics of* Γ *is the maximal set* $X \subseteq \Sigma$ *satisfying*

$$X = \tilde{Pr}(X).$$

I am now ready to present the last result of this paper for games, which characterizes point-rationalizability for large non-anonymous games with transformed summary statistics.

Theorem 6. *In any game with transformed statistics* Γ*, its set of point-rationalizable summary statistics* $\mathbb{P}_\Gamma$ *is equivalent to its maximum tight point-curb set. Moreover,* $\mathbb{P}_\Gamma$ *is nonempty, convex and compact.*

Proof: I only show that $\mathbb{P}_\Gamma \subseteq \tilde{Pr}(\mathbb{P}_\Gamma)$. Other steps are similar to the proofs of Theorem 1 and 2. Towards this end, consider the sequence $\{F^t\}$ defined as follows:

$$F^0(i) = C, \text{ for all } i \in I$$

$$F^t(i) = G(i, \tilde{Pr}^{t-1}(\Sigma)), \text{ for all } i \in I \text{ and for all } t \geq 1,$$

where $\tilde{Pr}^t(\Sigma) = \int_I F^t d\lambda$. Similar to the proof of the Theorem in Yu and Zhu (2005), one can show that $\tilde{Pr}^0(\cdot)$ is nonempty-valued, convex-valued and upper hemicontinuous. Furthermore, sine A is compact, C is also compact. Thus, $\tilde{Pr}^0(\Sigma)$ is nonempty, convex and also compact. By Lemma 4 and the fact the s is contiunous, F^1 is compact-valued. By

induction over t, one can also assert $\tilde{Pr}^t(\Sigma)$ is also nonempty, compact and convex for any $t \geq 1$. Hence, F^t is measurable and compact valued for all $t \in \mathbb{N}$.

Take any point $\sigma \in \mathbb{P}_\Gamma$. Consider the correspondence $F : I \twoheadrightarrow \Sigma$ defined as $F(i) = cl - Lim\{F^t(i)\}$. It is clear that $\sigma \in \int_I F_t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{f^t\}_{t\in\mathbb{N}}$, such that $\sigma = \int_I s \circ f^t$, for all $t \geq 0$. Let $g^t = s \circ f^t$, for all $t \geq 0$. According to the Lemma in Aumann (1976a), one gets $\sigma \in \int_I F$ as all the limit points of $\{g^t(i)\}$ are in $F(i)$ for all $i \in I$. Moreover, by Berge's maximum theorem, for each $i \in I$, the joint continuity of u_i on $A \times \Sigma$ implies the upper hemicontinuity of $B(i, \cdot)$ on Σ. It is easy to see that $G(i, \cdot)$ is upper hemincontinuous by Aliprantis and Border (1999, Theorem 14.22) as s is continuous. At the same time, for any $y \in F$, $y = cl - Lim\{y^t\}$, where $y^t \in F^t$ for all $t \in \mathbb{N}$. This is to say, for any fixed i, there is a sequence $\{\sigma^k\}$ of Σ such that $\sigma^k \in \tilde{Pr}^{t_k-1}(\mathbb{P}_\Gamma)$, $y^{t_k}(i) \in G(i, \sigma^k)$ and $lim_k y^{t_k}(i) = y(i)$. Moreover, because s is continuous, s^{-1} is weakly measurable. This implies that one can find a Borel measurable selector h of s^{-1} by Kuratowski-Ryll-Nardzewski selection theorem.[24] Let $f = h \circ y$. It is clear that f is measurable. Thus, one can obtain $y(i) \in G(i, \mathbb{P}_\Gamma)$ by the upper hemicontinuity of $G(i, \cdot)$, since all cluster points of $\{\sigma^k\}$ are in $\mathbb{P}_\Gamma$, being the intersection of a nested family of compact sets, and $y = s \circ f$, where f is measurable. Thus we have $F(i) \subseteq G(i, \mathbb{P}_\Gamma)$ for all $i \in I$. Hence, $\sigma \in \tilde{Pr}(\mathbb{P}_\Gamma)$. ∎

Remark 3. The game model used to characterize point-rationalizability in Jara-Moroni (2008) is a special case of the model considered in this section. To see this, one can take A as a subset of $\mathbb{R}^n$ and s as an identity map to cover the case considered in Jara-Moroni (2008). However, the target space of transformed summary statistics can only be finite-dimensional in general because I use the integration theorem in Aumann (1965) and the theorem is on the integration of the correspondence in $\mathbb{R}^n$. It can be checked if one uses an saturated probability space of players, the main result in this section may still hold for infinite-dimensional target spaces.

5 Conclusions

In this paper, I have formally examined the concept of point-rationalizability in the three different settings emphasized in the theory of large games. The set of point-rationalizable societal responses which survives from the process of eliminating non-best replies in all three cases has been defined respectively. In addition, the set is shown to be well-defined and equivalent to the maximum tight point-curb set of the same game.

[24] See, for example, Aliprantis and Border (1999, Theorem 18.13).

A natural question is if one could could strengthen point-rationalizability to rationalizability. As shown in Bernheim (1984) and Jara-Moroni (2008), the proofs for characterizing point-rationalizability and rationalizability are analogues of each other.[25] The tools that we developed and used should also work for characterizing "correlated" rationalizability.[26] But the difficulty here is how to capture the idea of the independence of players' behavior in large games, and thus to define an independent rationalizability. This is non-trivial and an open question due to the measurability problem[27] of the independence of a continuum of random variables.

The concept of the point-rationalizability can be justified by the epistemic assumption that it is common knowledge that only best responses are ever chosen. Thus, another question to ask is under which general conditions that one can relate equilibrium to point-rationalizability so that one can provide a plausible justification for reaching equilibria. Moving beyond the questions of analysis, one can ask how to apply the notion of point-rationalizability (as in Guesnerie and Jara-Moroni (2011)) to a broad spectrum of situations: monopolistic competition, as in Rauh (2003), financial markets, as in Constantinides and Rosenthal (1984), restaurant pricing and the economics of "social influences," as in Karni and Levin (1994), and many other papers that deal with a continuum of players. We hope to return to both questions in subsequent work.

Appendix A

I now provide proofs for a series of technical lemmas.

Lemma 5. *Let X be a compact metric space, S a measurable and closed-valued correspondence from an atomless probability space $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$. For each $i \in I$ consider a upper hemicontinuous and closed-valued mapping $M(i, \cdot) : \mathcal{M}(X) \twoheadrightarrow X$. Let $G(i)$ be the graph mapping $Gr(M(i, \cdot))$. If G is measurable, then the correspondence $M(\cdot, S(\cdot)) : I \twoheadrightarrow X$ is measurable and closed-valued.*

Proof: Let $X(i) = X$, $R(i) = M(i, S(i))$, $Q(i) = (S(i) \times Y(i)) \cap G(i)$ for all $i \in I$. It is easy to see that $R = P \circ Q$, where P be the projection from $\mathcal{M}(X) \times X$ to X. Thus, given the fact that the extreme points of $\mathcal{M}(X)$ is identified with X under the embedding $x \mapsto \delta_x$, the correspondence $S(\cdot) \times X(\cdot) : I \to \mathcal{M}(X) \times X$ is measurable by Aliprantis and Border (1999, Theorem 18.10 and Lemma 18.4 (2)) and its intersection Q with G is also measurable by

[25] One can refer to Jara-Moroni (2008) for more details.

[26] See Brandenburger and Dekel (1987) for the discussion on correlated and independent rationalizability.

[27] See Sun (2006) and its references on this measurability problem.

Aliprantis and Border (1999, Lemma 18.4 (3)). Since $M(i,\cdot)$ is upper hemicontinuous and closed valued, $G(i)$ is closed by Aliprantis and Border (1999, Theorem 17.11). Thus, Q is also closed-valued. As P is continuous, this completes the proof. ∎

Proof of Lemma 1: For any fixed nonempty closed set $\mathcal{D}$, let $\mathcal{D}(i) = \mathcal{D}$ for all $i \in I$. It is clear that $\mathcal{D}(\cdot)$ is a measurable and closed-valued correspondence. By Berge's maximum theorem, for each $i \in I$, the joint continuity of u_i on $A \times \mathcal{M}(A)$ implies that $B(i,\cdot)$ is upper hemicontinuous and has nonempty compact values on $\mathcal{M}(A)$. Thus, by Lemma 5, I only need to show $G : I \twoheadrightarrow \mathcal{M}(A) \times A$ is measurable where $G(i) = Gr(B(i,\cdot))$. According to the definition of measurable correspondence, it is sufficient to show for any closed $C \subseteq A \times \mathcal{M}(A)$, the lower inverse of G defined by $G^l(C) = \{i \in I : C \cap G(i) \neq \emptyset\}$ is measurable.

Towards this end, for any closed set $C \subseteq A \times \mathcal{M}(A)$, let $U_C = \{g \in \mathcal{U}_A :$ there is some $(a,\tau) \in C,$ such that, $g(a,\tau) \geq g(a',\tau),$ for all $a' \in A\}$. It is easy to see that $\mathcal{G}^{-1}(U_C) = G^l(C)$. Given $\mathcal{G}$ is measurable, it is sufficient to prove that U_C is closed. Thus, I only need to show that the uniform limit g^* of any uniformly convergent sequence $\{g^n\}$ in U_C is an element of U_C as well. By Aliprantis and Border (1999, Theorem 2.65), $g^* \in \mathcal{U}_A$. Thus, the continuity of g^* implies that for any convergent sequence $\{(a^n,\tau^n)\}$ with limit (a^*,τ^*), there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$, $|g^*(a^n,\tau^n) - g^*(a^*,\tau^*)| < \varepsilon/2$. At the same time, the uniform convergence of g^n implies that there exists $N_2 \in \mathbb{N}$ such that for any $n > N_2$, and for all $(a,\tau) \in A \times \mathcal{U}_A$, $|g^n(a,\tau) - g^*(a,\tau)| < \varepsilon/2$. Thus, for any $n > N$, where $N = \max\{N_1, N_2\}$,

$$|g^n(a^n,\tau^n) - g^*(a^*,\tau^*)| \leq |g^n(a^n,\tau^n) - g^*(a^n,\tau^n)| + |g^*(a^n,\tau^n) - g^*(a^*,\tau^*)| < \varepsilon.$$

Thus, the sequence $\{g^n(a^n,\tau^n)\}_{n\in N}$ converges to $g^*(a^*,\tau^*)$. To show that $g^* \in U_C$, it is sufficient to show that there exists a pair $(a,\tau) \in C$ such that $g^*(a,\tau) \geq g^*(a',\tau)$, for all $a' \in A$. Since $g^n \in U$, it is clear that for each $n \in \mathbb{N}$, $(a^n,\tau^n) \in C$, $g^n(a^n,\tau^n) \geq g^n(a',\tau^n), \forall a' \in A$. Because C is closed, one can pick (a^*,τ^*) as the limit of a convergent subsequence of $\{(a^n,\tau^n)\}$. Without loss of generality, one can take the subsequence to be the same sequence. The limit (a^*,τ^*) is the point which satisfies $g^*(a^*,\tau^*) \geq g^*(a',\tau^*)$ for any fixed $a' \in A$. Hence, $g^* \in U_C$. The proof is now completed. ∎

Lemma 6. *Let X be a compact set in some vector space, S a measurable and closed-valued correspondence from an atomless probability space $(I,\mathcal{I},\lambda)$ to $\overline{con}(X)$. For each $i \in I$ consider a upper hemicontinuous and closed-valued mapping $M(i,\cdot) : \overline{con}(X) \twoheadrightarrow X$. Let $G(i)$ be the graph mapping $Gr(M(i,\cdot))$. If G is measurable, then the correspondence $M(\cdot,S(\cdot)) : I \twoheadrightarrow X$ is measurable and closed-valued.*

Proof: Similar to the proof of Lemma 5, the result holds here providing the fact that $X \subseteq \overline{con}(A)$. ∎

Proofs of Lemma 2 and Lemma 4: The proofs modify the proof of Lemma 1 above by replacing Lemma 5 with Lemma 6, and use the corresponding notation in their sections. ∎

Proof of Lemma 3: The proof of Proposition 2 which is provided in Khan et al. (1997) follows here other than the fact that we have to use the relevant analogues in Proposition 5 of the convexity, compactness, convergence and upper hemicontinuous results to guarantee the existence of Nash equilibrium of the game. ∎

6 Appendix B

The following result is culled from the theorems in Yu and Zhang (2007) and Keisler and Sun (2009).

Proposition 3. *Let X be a compact metric space and $(\Omega, \mathcal{A}, P)$ an atomless probability space. Then the following results are valid if, in addition, (i) X is a countable, or (ii) $(\Omega, \mathcal{A}, P)$ is a saturated probability space.*

P0: Let $\{f_n\}$ be a sequence of measurable functions from Ω to X such that $\tau_n = Pf_n{}^{-1}$ converges weakly to $\tau \in \mathcal{M}(X)$ as $n \to \infty$. Let $D(\omega) = cl - Lim\{f_n(\omega)\}$. Then, $D(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of D such that $Pf^{-1} = \tau$.

P1: For any correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F = \{Pf^{-1} : f$ is a measurable selection of $F\}$ is convex.

P2: For any compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is compact.

In the following, we collect the existing results on the integration of correspondences. The next result is a collection of results in Khan et al. (1997) and Khan and Sun (1996). Note that the integral is defined as in Section 3.1.

Proposition 4. *Let $(\Omega, \mathcal{A}, P)$ an atomless probability space, then the following results hold:*

P0: Let $\{f_n\}$ be a sequence of measurable functions from Ω to $\mathbb{R}^\infty$, such that $\int f_n dP$ converges to ι as $n \to \infty$. Let $D(\omega) = cl - Lim\{f_n(\omega)\}$. Then, $D(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of D such that $\int f dP = \iota$.

P1: For any correspondence F from $(\Omega, \mathcal{A}, P)$ into $\mathbb{R}^\infty$, then the integral $\int_I F dP$ of F is

convex.

P2: For any compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ into $\mathbb{R}^\infty$, then the integral $\int_I F dP$ of F is compact.

The next two propositions are modified from Khan et al. (1997) and Sun (1997). Note that as shown in Sun and Yannelis (2008), the properties in Sun (1997) for Loeb space also hold for any saturated probability space. Proposition 4 is based on Bochner integral, while the last one on the Gel′fand integral.

Proposition 5. *Let $(\Omega, \mathcal{A}, P)$ be an atomless probability space and X a separable Banach space. The following results are valid if, in addition, (i) X is a countable, or (ii) $(\Omega, \mathcal{A}, P)$ is a saturated probability space. then the following results hold:*

P0: Let $\{f_n\}$ be a sequence of measurable functions from Ω to X such that $\int f_n dP$ converges to ι as $n \to \infty$. Let $D(\omega) = w - cl - Lim\{f_n(\omega)\}$. Then, $D(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of D such that $\int f dP = \iota$.

P1: For any correspondence F from $(\Omega, \mathcal{A}, P)$ into X, $\int_\Omega F dP$ is convex.

P2: For any integrably bounded, weakly compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\int_\Omega F dP$ is weakly compact.

P3: Let Y be a metric space and F a correspondence from $\Omega \times Y$ to X such that for each fixed $y \in Y$, $F(\cdot, y)$ is a measurable, weakly compact-valued correspondence from Ω to X. If $F(\omega, y)$ is upper hemicontinuous on Y for each fixed i, then $\int_\Omega F(\omega, y) dP$ is weakly upper hemicontinuous on Y.

Proposition 6. *Proposition 5 holds with X standing for the dual of a separable Banach space, "weakly" replaced by weak*, and with the integrals all interpreted as Gel′fand integrals.*

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