Pricing Rule in a Clock Auction

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Abstract

We analyze a discrete clock auction with lowest-accepted bid (LAB) pricing and provisional winners, as adopted by India for its 3G spectrum auction. In a perfect Bayesian equilibrium, the provisional winner shades her bid while provisional losers do not. Such differential shading leads to inefficiency. The size of the inefficiency declines with smaller bid increments. An auction with highest-rejected bid (HRB) pricing and exit bids is strategically simple, has no bid shading, and is fully efficient. In addition, it has higher revenues than the LAB auction, assuming profit maximizing bidders. The bid shading in the LAB auction exposes bidders to the possibility of losing the auction at a price below the bidder's value. Thus, fear of losing may cause bidders in the LAB auction to bid more aggressively than predicted assuming profit-maximizing bidders. We extend the model by adding an anticipated loser's regret to the payoff function. Revenue from the LAB auction yields higher expected revenue than the HRB auction when bidders' fear of losing at profitable prices is sufficiently strong. This would provide one explanation why India, with an expressed objective of revenue maximization, adopted the LAB auction for its upcoming 3G spectrum auction, rather than the seemingly superior HRB auction.

(JEL D44, C78, L96. Keywords: auctions, clock auctions, spectrum auctions, behavioral economics, market design.)

1 Introduction

In 2008, the government of India announced that radio spectrum for 3G mobile services would be auctioned in 2009. The 2.1 GHz spectrum is to be sold as paired spectrum (2x5 MHz blocks) in each of 22 telecommunication *circles* covering India. There are one to four lots available in each circle and each bidder can obtain at most one lot per circle. The government's stated objective for the auction emphasizes revenue maximization rather than efficiency. In December 2008, the government announced the chosen auction design: a discrete clock auction with lowest-accepted bid (LAB) pricing and provisional winners (India 2008). Here we examine the equilibrium properties of such an auction in a simplified setting. As a comparison, we analyze a discrete clock auction with highest-rejected bid (HRB) pricing and exit bids. This format is often used in high-stake auctions in practice (Ausubel and Cramton 2004).

Interestingly, with profit maximizing bidders, we find the HRB auction dominates the LAB auction in both efficiency and revenues. The HRB with exit bids is fully efficient, since it is a dominant strategy to bid up to one's valuation. In contrast, LAB with provisional winners has differential shading, since a

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provisional winner shades her bid, whereas a provisional loser does not. This differential shading creates an inefficiency and reduces revenues. Given this strong theoretical result, it may seem odd that India chose the LAB format.

One potential explanation comes from behavioral economics. If bidders anticipate the regret of losing at a profitable price, they may be reluctant shade bids as a provisional winner. This fear of losing has been shown to explain overbidding in first-price sealed-bid auctions (Filiz-Ozbay and Ozbay 2007, Delgado et al. 2008). When we extend the standard theory to include a fear of losing, we find that if bidders' fear of losing is sufficiently strong, then the LAB auction revenue dominates the HRB auction. This result provides an explanation for India's selection of a seemingly inferior auction format.

Most theoretical papers on clock auctions assume a continuous clock for convenience. In practice, clock auctions use a discrete price clock, since these auctions typically are conducted on the internet, and communication is not sufficiently reliable to bind bidders to higher prices with the continuous passage of time. With discrete bid levels, the two predominant pricing rules, lowest-accepted bid and highest-rejected bid, are distinct, and the auction designer must select a pricing rule as well as other elements of the design. Currently, there is little literature for the auction designer to turn to for help with this issue.

There are a limited number of papers investigating auctions with discrete bid levels. These papers focus on explaining bidding behavior. Chwe (1989) studied the first-price auction with discrete bid levels and showed that the expected revenue is less than its continuous counterpart. Mathews and Sengupta (2008) analyzed a sealed-bid second-price auction with discrete bids.

More closely related is the work that considers choices of bid levels in ascending auctions. Rothkopf and Harstad (1994) is an important early contribution, determining optimal bid levels that maximize expected revenue in an oral auction. The paper also introduces the trade-off between auction duration and bid increments. David et al. (2007) extend the model of Rothkopf and Harstad and find that decreasing bid increments are optimal. Although the pricing rule in our paper is the same as in Rothkopf and Harstad, the auction formats have important differences, which result in significantly different bidding behavior.

We consider a discrete clock auction with two pricing rules: highest-rejected bid and lowest-accepted bid. Bidders have independent private values and unit demands. We first analyze bidding behavior in an HRB auction. This is our benchmark for comparing performance with the LAB auction. The HRB auction is a useful benchmark because of its simplicity, its desirable properties (efficiency and truth dominance), and its use in practice. In contrast, the LAB auction forces bidders to engage in difficult tradeoffs. We are only able to solve for equilibrium bidding behavior in a simplified setting. Nonetheless, we show that an LAB auction is generally inefficient. Despite this inefficiency, the LAB auction can yield higher revenues if bidders anticipate the regret of losing at profitable prices, and therefore engage in less bid shading than a bidder focused solely on profit maximization.

2 Discrete clock auction with unit demand

There are *K* identical items for sale to *N* risk-neutral bidders, where $K < N$. The seller values the items at zero. Each bidder *i* demands at most one item and bidder *i*'s private valuation for the item is x_i. Each x_i is independently drawn from the distribution F_i with positive density function f_i on $[\underline{x}, \overline{x}]$,

which we normalize to [0,1]. F_i and f_i are common knowledge. Bidder *i*'s payoff if she wins the item is $x_i - m_i$ where m_i is bidder *i*'s payment and zero otherwise.

Before the auction starts, the seller announces a vector of bid levels, $\mathbf{P} = (P_0, P_1, \ldots, P_{T-1})$ where P_t is the price at round *t* and *T* is the number of bid levels. The clock price increases every round so that $P_0 < P_1 < \ldots < P_{T-1}$. Define the bid increment in round t as $\Delta_t = P_t - P_{t-1}$, $t = 1,2,\ldots,T-1$. Assume that $P_0 = 0$ and $P_{T-1} = 1$. The auction begins in round one at a price P_1. In each round *t*, a bidder chooses either to bid at the current clock price, $q_{it}(P_t) = 1$, or to drop out, $q_{it}(P_t) = 0$. Once a bidder drops out she cannot bid again. Let $Q_t = \sum_{i=1}^{N} q_{it}(P_t)$ be total demand in round *t*. If there is excess demand, $Q_t > K$, the auction proceeds to the next round. The auction ends in the round *t** where $Q_{t^*} \leq K$.

2.1 Highest-rejected bid with exit bids

In the HRB format (highest-rejected bid with exit bids), if bidder *i* drops out in round *t*, the bidder can submit an exit bid—a price between P_{t-1} and P_t at which she wants to drop out. The final price is determined by the highest-rejected exit bid. Specifically, in round *t** such that $Q_{t^*} = K$, the highest exit bid among the bidders dropping out determines the final price. If $Q_{t^*} = K' < K$, the *K-K'+1*st-highest exit bid determines the final price.

Proposition 1. *In the HRB auction, truthful bidding (bidding up to one's valuation) is a weakly-dominant strategy. Therefore, the HRB auction yields an efficient allocation. Each of the K-highest bidders wins and pays the K+1*st*-highest valuation*.

This result follows immediately from the unit demand setting, and the ability to submit exit bids at actual valuations. Unit demand guarantees that each winner pays the Vickrey price, thereby inducing truthful bidding. Since all bidders bid truthfully, the items are awarded to the bidders who value them most.

2.2 Lowest-accepted bid with provisional winners

In the LAB format (lowest-accepted bid with provisional winners), if there is excess demand in any round, the auction proceeds to the next round. The seller then randomly selects *K* provisional winners for the next round from active bidders and ranks them from *K* to one (highest rank is given priority). Other remaining active bidders are designated as provisional losers (a rank of 0) for the next round. The auction ends if there is no excess demand. The items are awarded to the active bidders in the current round and the remaining items are awarded to an inactive provisional winner with the highest rank and so on until all items are sold. Let $\theta = \{0,1,\ldots,K\}$ be a set of all possible ranks. Let $R_{it} \in \theta$ denote bidder *i*'s rank in round *t*. Let $\mathbf{H}_{it} = (R_{i1}, R_{i2}, \ldots, R_{it})$ be a vector of bidder *i*'s ranking history from round one to round *t* and $\mathbf{H}_t = (\mathbf{H}_{i1}, \mathbf{H}_{i2}, \ldots, \mathbf{H}_{it})$. Assume that ranking history is common knowledge.

The final price is uniform and determined by the K^{th}-highest winner's bid. Therefore, if $Q_t = K$, the final price is P_t and if $Q_t < K$, the final price is P_{t-1}. Notice that the ranking is relevant in determining the allocation only when $Q_t < K$, since then there is excess demand at the final price P_{t-1}.

One important difference between the LAB auction described here and an ascending bid auction with the same pricing rule is that in the LAB auction, a provisional winner must keep topping her own bid in order to be eligible to bid in subsequent rounds. In contrast, in an ascending bid auction with LAB, being a provisional winner is counted as being active. Therefore, a provisional winner does not need to bid in order to be eligible to bid in the subsequent rounds.

To better illustrate the LAB pricing rule with ranking, consider an auction with three bidders and two items for sale. All bidders bid in round $t-1$ and bidder 2 and 3 are selected as provisional winners with ranks of 1 and 2, respectively, in round t while Bidder 1 is a provisional loser. There are eight possible outcomes as shown in Table 1. If all bidders bid, the auction proceeds to round $t+1$ with a price of P_{t+1}. Regardless of ranking, if there are exactly two active bidders, they win the items at the current round price P_t. If there is one active bidder, the item is awarded to the active bidder and an inactive provisional winner with highest rank at the previous round price P_{t-1}. Finally, if there is no active bidder, bidder 2 and 3 win at the previous round price P_{t-1}.

Table 1. LAB auction with two bidders and one item in round t with a price of P_t

Case	Bidder 1 (Rank=0)	Bidder 2 (Rank=1)	Bidder 3 (Rank=2)	Outcome
1	Bid	Bid	Bid	Proceed to round $t+1$ and provisional winners are reselected
2	Bid	Bid	Drop	Bidder 1 and 2 win and the final price is P_t
3	Bid	Drop	Bid	Bidder 1 and 3 win and the final price is P_t
4	Drop	Bid	Bid	Bidder 2 and 2 win and the final price is P_t
5	Bid	Drop	Drop	Bidder 1 and 3 win and the final price is P_{t-1}
6	Drop	Bid	Drop	Bidder 2 and 3 win and the final price is P_{t-1}
7	Drop	Drop	Bid	Bidder 2 and 3 win and the final price is P_{t-1}
8	Drop	Drop	Drop	Bidder 2 and 3 win and the final price is P_{t-1}

Lemma 1. *For any K and any* $\mathbf{H}_t$*, in round s, a bidder with valuation* $x \in [P_t, P_{t+1})$ *drops out if* $P_s > P_{t+1}$.

If the auction ends, the final price can be either the current round price or the previous round price. However, when the previous round price is strictly above a bidder's valuation, regardless of ranking, it is a weakly-dominant strategy to drop out immediately since bidding yields a payoff that cannot be positive.

Lemma 2. *For any K and for any* $\mathbf{H}_t$*, a provisional loser in round s with valuation* x *bids if* $x \geq P_s$*. For any* $K > 1$ *and for some* $\mathbf{H}_t$*, a provisional loser in round t with valuation* $x \in [P_t, P_{t+1})$ *bids in round* $t+1$ *if* x *is sufficiently close to* P_{t+1}*.*

When a provisional loser drops out, her payoff is zero but, as long as her valuation is above the current round price, bidding yields positive payoff. Thus, a provisional loser never shades her bid and may even bid above her valuation. For some ranking history, a provisional loser may bid at one price level above her valuation since there is a positive probability that she wins the item at the previous round price which is below her valuation. However, such strategy entails a risk of winning the item at a

price higher than her valuation. For instance, consider the example in Table 1. Suppose bidder 1 has a valuation $x \in [P_{t-1}, P_t)$. If she bids and bidder 2 and 3 drop out as in Case 5, which has a positive probability of occurrence, bidder 1 wins the item at P_{t-1} gaining a positive payoff.

Lemma 3. *For any K, there exists a round $t' \leq t$ in which, for some history $\mathbf{H}_{t'}$, a provisionally winning bidder i with a valuation $x_i \in [P_t, P_{t+1})$ drops out.*

Intuitively, a provisional winner faces the first-price incentive—an incentive to keep the price low—which results in bid shading. A provisional winner can win the item at a lower price if she drops out with a risk of losing. In contrast, if she stays in the auction, her probability of winning the item increases but so does the expected price she pays. In addition, a provisional winner with higher rank faces less risk of losing when shading since a greater number of new bids are required to displace her provisional winning bid. Consequently, a provisional winner with higher rank has a stronger incentive to shade her bid.

Proposition 2. *The LAB auction is inefficient.*

There are two sources of inefficiency: discrete bid levels and asymmetric bidding behavior. First, discrete bid levels may prevent bidders from expressing a difference in their valuations. Any auction with discrete bid levels is generally inefficient. As the number of bid levels go to infinity, then we converge to a continuous clock auction. In the limit, the LAB pricing rule is equivalent to the Vickrey pricing and hence the outcome is efficient. Second, although bidders and their bidding strategy are symmetric, asymmetric bidding behavior is introduced by naming and ranking provisional winners as discussed in Lemma 2 and 3. A provisional winner shades her bid while a provisional loser with the same valuation does not. Differential bid shading is exhibited among provisional winners and bidding above valuation is found among provisional losers. These differences are a separate source of inefficiency.

Exit bids in the HRB auction overcome both sources of inefficiency of the LAB auction. In the HRB auction, there is no bid shading and all value differences are expressed.

3 Equilibrium characterization with one item and two bidders

In order to further characterize equilibrium behavior in the LAB auction, it is necessary to simplify the setting to one item and two bidders. Although this setting is limiting, it will provide intuition for cases with more items and more bidders.

With two bidders and a single item, two ranks are possible. Each bidder is either a provisional winner or a provisional loser. The auction ends when at least one bidder drops out. If in round *t* only one bidder drops out, the active bidder receives the item and pays P_t. If both bidders drop out in round *t*, the provisional winner in round *t* gets the item at a price of P_{t-1}.

Definition *Bidder i follows a straightforward bidding strategy in round t when*

$$q_{it}(x_i; \mathbf{P}) = \begin{cases} 1 & ; x_i \geq P_t \\ 0 & ; x_i < P_t \end{cases}$$

Lemma 4 *For a provisional loser, bidding straightforwardly is a weakly-dominant strategy. For a bidder with any rank, bidding above her valuation is a weakly-dominated strategy.*

The fact that a provisional loser bids straightforwardly greatly simplifies the analysis. To construct an equilibrium, we need to solve for shading behavior of a provisional winner.

Lemma 5 *A provisional winner's payoff from dropping out is decreasing in her valuation.*

Simply put, a provisional winner's bidding strategy is monotonic in valuation. For example, consider a provisional winner with valuation *x* in round $s<t+1$ and some ranking history $\mathbf{H}_s$. Suppose the provisional winner drops out in round *s* if $x=P_t$ and bids if $x=P_{t+1}-\varepsilon$ where $\varepsilon>0$. There exists $\hat{x}\in\left[P_t,P_{t+1}\right)$ such that a provisional winner with valuation $\hat{x}$ is indifferent between bidding and dropping out in round *s*. In addition, Lemma 5 implies that a provisional winner shades if $x\in\left[P_t,\hat{x}\right]$ and bids if $x\in\left[\hat{x},P_{t+1}\right)$.

A provisional winner's bidding strategy depends not only on her valuation, but also on the history of the opponent's ranks. For instance, a provisional winner may be reluctant to dropout when a ranking history implies that the opponent has a higher valuation. In an extreme case, a provisional winner may not shade at all when an inferred provisional loser's valuation is higher than the current round price. Notice that this dependence on the history of rankings implies that even with identical value realizations different ranking histories may produce different allocations and final prices.

Define $\hat{x}_s^t\left(H_s\right)$ as a valuation of the bidder with valuation $x\in\left[P_t,P_{t+1}\right]$ who is indifferent between bidding and dropping out at round *s* given ranking history H_s and $\hat{x}_s^t:\left(\left\{0,1\right\}^s\right)\to\left[P_t,P_{t+1}\right]$. An equilibrium of a discrete clock auction with LAB is fully characterized by $\hat{x}_s^t\left(H_s\right)$ for all $t=0,1,\ldots,T-2$, $s=1,2,\ldots,T-2$ and for all $H_s\in\mathbf{\Omega}_t$ where $s\le t$ and $\mathbf{\Omega}_t$ is a set of all possible histories. Note that $\hat{x}_s^t\left(H_s\right)=P_t$ implies that a provisional winner with any valuation in $\left[P_t,P_{t+1}\right)$ bids and $\hat{x}_s^t\left(H_s\right)=P_{t+1}$ implies that a provisional winner with any valuation in $\left[P_t,P_{t+1}\right)$ drops out in round *s*.

Proposition 3. *A symmetric equilibrium of the LAB auction is given as follows. In any round s with a ranking history H_s, a provisional winner with valuation $x\in\left[P_t,P_{t+1}\right)$ bids if $x\ge\hat{x}_s^t\left(H_s\right)$ and drops out if $x<\hat{x}_s^t\left(H_s\right)$. In any round and any ranking history, a provisional loser follows a straightforward bidding strategy.*

4 Equilibrium characterization with a fixed bid increment and uniform valuations

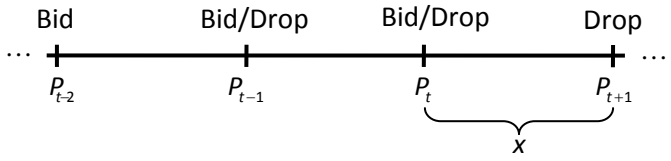
To further characterize equilibrium behavior, it is helpful to assume a fixed bid increment and that valuations are uniformly distributed on [0,1]. Thus, $\Delta_t=\Delta$ for all *t* and $F\left(x\right)=x$.

Lemma 6 *For any ranking history, a provisional winner in round s with a valuation $x\in\left[P_t,P_{t+1}\right)$ always bids if $s\le t-2$.*

Figure 1 summarizes the strategy of a bidder with valuation $x\in\left[P_t,P_{t+1}\right]$ when she is a provisional winner. According to Lemma 6, if she is a provisional winner in any round 1 to $t-2$, she will bid. She

may bid or dropout, if she is a provisional winner in round $t-2$ and $t-1$ depending on her valuation and ranking history. By Lemma 4, she will drop out in round $t+1$ regardless.

Figure 1. Strategy of a provisional winner with valuation $x \in [P_t, P_{t+1})$



Proposition 4. *Inefficiency in the LAB auction can be reduced by smaller bid increments.*

In this setting, efficiency suffers from discrete bid levels and differential bid shading. However, according to Lemma 6, the amount of bid shading is limited to only two bid levels. Therefore, smaller bid increments reduce the absolute amount of bid shading as well as inefficiency from discrete bid levels.

5 An example with four bid levels

To get a better sense of the equilibrium derivation, consider an example with four bid levels: $\mathbf{P} = (0, 1/3, 2/3, 1)$ and $\Delta = 1/3$. Bidding strategy of the provisional winner is shown in Table 2. Here, we omit a ranking history of the other bidder since it is just a complement of a provisional winner's ranking history.

Table 2. Bidding strategy of a provisional winner in an LAB auction with four bid levels

Round	1	2		3
Price	P_1	P_2	P_2	P_3
Valuation\Ranking history	(1)	$(1,1)$	$(0,1)$	Any
$[P_0, P_1)$	Drop	-	-	-
$[P_1, P_2)$	Drop	Drop	Drop	-
$[P_2, P_2 + \frac{2}{5}\Delta)$	Drop	Drop	Bid	Drop
$[P_2 + \frac{2}{5}\Delta, P_3)$	Bid	Drop	Bid	Drop

A bidder never bids above her valuation. As $\hat{x}_1^1(1) = P_1$ and $\hat{x}_1^2(1) = P_2 + \frac{2}{5}\Delta$, a provisional winner with $x \in [P_1, P_2 + \frac{2}{5}\Delta)$ shades in round 1. To show how the ranking history affects bidding behavior, consider a provisional winner with $x \in [P_1, P_3)$. In round 2, a ranking history of $(1,1)$ implies that a provisional loser has a valuation in $[P_1, P_3]$ and thus dropping out is a weakly-dominant strategy. In contrast, a ranking history of $(0,1)$, says a provisional loser in round 2 was a provisional winner and bid in round 1, implies that her valuation is greater than P_2 and that she will bid with certainty. Therefore, a weakly-dominant strategy for a provisional winner is to bid if her valuation is above the current round price. As a result, a provisional winner bids in round 2 if her valuation is in $[P_2, P_3)$ and drops out if her valuation is in $[P_1, P_2]$. Intriguingly, a bidder with valuation in $[P_2, P_2 + \frac{2}{5}\Delta)$ will drop in round 1 if she is a

provisional winner but if the history is $(0,1)$, she will bid in round 2 once she learns more about her opponent's valuation.

Thus, ranking history affects a provisional winner's bidding strategy. Being a provisional winner and bidding reveals some information about the bidder's valuation so that the opponent must adjust her bidding strategy accordingly. A bidder who was a provisional loser in previous rounds can better assess her opponent's valuation. It is this history dependence that makes the equilibrium calculation so difficult.

With four bid levels, the expected revenue is 0.278 which is lower than that of the HRB auction, which is equal to 1/3.

6 Expected revenue with profit maximizing bidders

In this section, we continue to assume that there is a single item and two bidders whose valuations are uniformly distributed on $[0,1]$. There are *T* bid levels, equally spaced.

Since all bidders bid truthfully in the HRB auction, the expected revenue is $R_{HRB} = 1/3$.

Calculating an expected revenue of the LAB auction is tedious since all possible histories and associated outcomes have to be considered. Thus, the problem grows exponentially in the number of bid levels, making derivation of an equilibrium with an arbitrary number of bid levels impossible. We however can calculate expected revenue for small *T*. We also can calculate revenue in the limit as the number of bid levels goes to infinity. The limit result comes from the limit of an upper bound on revenue and a lower bound on revenue. Since both limits converge to the same thing, the equilibrium expected revenue must converge as well.

Lemma 7. *Suppose all bidders follow the straightforward strategy. For any* $T \geq 3$*, the expected revenue is given by*

$$R_{LAB}^{SB}(T) = \frac{(T-2)(2T+3)}{6(T-1)^2}$$

In addition, $\lim_{T\to\infty} R_{LAB}^{SB}(T) = R_{HRB}$.

Next, consider a maximum shading strategy—a provisional winner does not imply a provisional loser's valuation from a ranking history. That is, in any round *t*, a provisional winner infers that a provisional winner's valuation is greater than or equal to the last round price. As the name suggests, a provisional winner who maximizes her profit will not shade by an amount more than the one who follows a maximum shading strategy. Specifically, with the maximum shading strategy, $\hat{x}_{t-1}^{t}(H_{t-1}) = P_t + \frac{2}{5}\Delta$ for any $t \geq 2$ and for any $H_{t-1} \in \mathbf{\Omega}_t$.

Lemma 8. *Suppose a provisional winner follows maximum shading strategy and a provisional loser follows a straightforward bidding strategy. For any* $T \geq 3$*, the expected revenue is given by*

$$R_{LAB}^{MS}(T)=\frac{T\left(397+5T\left(10T-39\right)\right)-549}{150\left(T-1\right)^{3}}.$$

In addition, $\lim_{T\to\infty} R_{LAB}^{MS}(T)=R_{HRB}$.

Proposition 5. *The expected revenue of the LAB auction converges to that of an HRB auction as T goes to infinity*.

The expected revenue of the LAB auction when bidders follow straightforward bidding strategy is an upper bound on expected revenues and the expected revenue of the LAB auction when a provisional winner follows a maximum shading strategy as a lower bound. Since the upper and lower bounds converge to 1/3, the expected revenue of the LAB auction converges to 1/3 as well.

Calculating the expected revenue of the LAB auction is tedious, since all ranking histories have to be considered. Expected revenues with 4-7 bid levels are as follows:

T	4	5	6	7	∞
E(Revenue)	0.278	0.301	0.311	0.317	0.333

We see that the expected revenue is increasing in the number of bid levels at a decreasing rate. We conjecture that the LAB auction always yields lower expected revenue than the HRB auction.

7 Extension to bidders with a fear of losing at profitable prices

In this section, we extend the model to include the possibility that bidders' anticipated regret of losing at profitable prices will cause them to shade bids less than under pure profit maximization. There is strong support for this view in sealed-bid first price auctions (Filiz-Ozbay and Ozbay 2007, Delgado et al. 2008). It seems plausible that the same behavioral bias—caring more about the negative emotion coming from losing at profitable prices than the positive emotion of extra profit from successful bid shading—may exist in the dynamic context. In a separate paper with Emel Filiz-Ozbay and Erkut Ozbay, we examine and confirm this possibility in the experimental laboratory (Cramton et al. 2009). Here we present the basic theory.

The behavioral theory posits that bidders systematically put too much weight on profits lost from losing at profitable prices and too little weight on profits gained from successful bid shading. In our context it is the bids of a provisional winner that are affected by this asymmetric treatment of profits, since only the provisional winner has an incentive to shade her bids.

Define $\tilde{\pi}_{it}\left(q_{it},q_{jt},R_{it}\right)$ as bidder *i*'s payoff if the auction ends given that bidder *i* submits q_{it}, bidder *j* submits q_{jt} and R_{it} is bidder *i*'s rank in round *t* where $q_{it},q_{jt}\in\{bid,drop\}$. Recognizing anticipated loser's regret, we define bidder *i*'s payoff function as

$$\tilde{\pi}_{it}\left(q_{it}=bid,q_{jt}=drop,R_{it}=1\right)=x_i-P_t$$
$$\tilde{\pi}_{it}\left(q_{it}=drop,q_{jt}=bid,R_{it}=1\right)=-\alpha\max\left\{0,E_t\left[\pi_{i,t+1}\left(q_{it}=bid,q_{jt}=bid,R_{it}=1\right)\right]\right\}$$
$$\tilde{\pi}_{it}\left(q_{it}=drop,q_{jt}=drop,R_{it}=1\right)=x_i-P_{t-1}$$
$$\tilde{\pi}_{it}\left(q_{it}=bid,q_{jt}=drop,R_{it}=0\right)=x_i-P_t$$
$$\tilde{\pi}_{it}\left(q_{it}=drop,q_{jt}=bid,R_{it}=0\right)=-\alpha\max\left\{0,E_t\left[\pi_{i,t+1}\left(q_{it}=bid,q_{jt}=bid,R_{it}=0\right)\right]\right\}$$
$$\tilde{\pi}_{it}\left(q_{it}=drop,q_{jt}=drop,R_{it}=0\right)=-\alpha\max\left\{0,E_t\left[\pi_{i,t+1}\left(q_{it}=bid,q_{jt}=drop,R_{it}=0\right)\right]\right\}$$

where α is the regret coefficient indicating the strength of anticipated regret. In this section, assume that $0\le\alpha<1$. Note that the payoffs are the same as in the no-regret case (α = 0), except when a provisional winner shades and loses.

It is still a weakly dominant strategy for a provisional loser to bid straightforwardly. Consider a provisional winner with a valuation $x\in\left[P_t,P_{t+1}\right)$. In round $t+1$, whether the provisional winner bids or drops out, by bidding, the provisional loser could win only at an unprofitable price. Hence, she does not have a fear of losing so that she drops out. Therefore, it is a weakly-dominated strategy to bid above her valuation. Moreover, dropping out below her valuation is a weakly-dominated strategy as well. For a provisional loser with a fear of losing, shading penalizes the bidder even more than in the no-regret case resulting in a negative expected payoff; whereas, bidding yields a positive expected payoff.

Again we consider the case with two bidders and a single item. Valuations are uniformly distributed on [0,1] with *T* equally spaced bid levels.

Lemma 9. *Suppose a provisional winner follows a maximum shading strategy and a provisional loser follows straightforward bidding strategy. For any* $T\ge3$*, the expected revenue is given by*

$$\tilde{R}_{LAB}^{MS}\left(T,\alpha\right)=\frac{50T^3-195T^2+397T-549+2\alpha\left(50T^3-150T^2+163T-171\right)+\alpha^2\left(50T^3-105T^2-23T-9\right)}{150\left(1+\alpha\right)^2\left(T-1\right)^3}.$$

In addition, $\lim_{T\to\infty}\tilde{R}_{LAB}^{MS}\left(T,\alpha\right)=R_{HRB}$ *for any* $0\le\alpha<1$.

As in the no-regret case, $\tilde{R}_{LAB}^{MS}\left(T,\alpha\right)$ is a lower bound of the expected revenue of the LAB auction with fear of losing. The upper bound is the same as the no-regret case in Lemma 7. Therefore, the expected revenue of the LAB auction with fear of losing converges to 1/3.

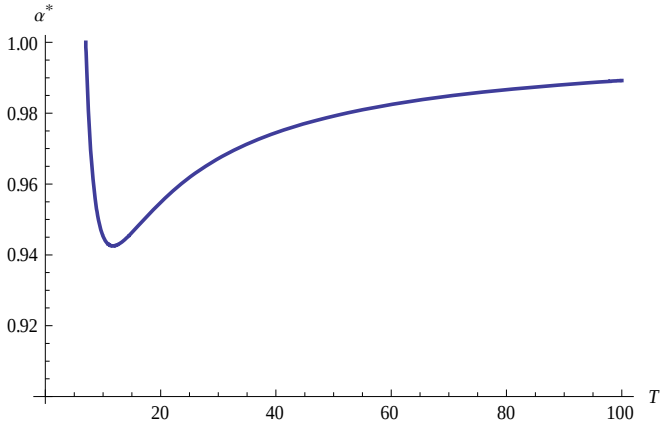
Nonetheless, in the interesting case of finite T as in any spectrum auction, if the regret coefficient is sufficiently large, the LAB auction yields *higher* expected revenue than the HRB auction. The necessary condition is given in the Proposition 6.

Proposition 6. *Suppose a provisional winner follows maximum shading strategy and a provisional loser follows straightforward bidding strategy. If* $T\ge7$ *and*

$$\frac{121-13T}{45T^2-173T+41}+5\frac{\sqrt{3\left(27T^4-252T^3+896T^2-1328T+468\right)}}{45T^2-173T-41}\le\alpha<1,\ \textit{then}\ \tilde{R}_{LAB}^{MS}\left(T,\alpha\right)\ge R_{HRB}.$$

Define α^* as the critical regret coefficient such that $\tilde{R}_{LAB}^{MS}(T,\alpha^*)=R_{HRB}$. Figure 2 shows the relationship between the critical regret coefficient and the number of bid levels. With six bid levels a regret coefficient of one guarantees that the LAB auction yields higher revenue than the HRB auction. If $T\geq 7$ and $\alpha\geq 1$, we can be certain that the expected revenue of the LAB auction is higher. In Section 8, we show that the actual regret coefficient (as opposed to the upper bound α^*) with six bid levels that makes the LAB and RHB auctions have equal revenues is 0.66, not 1.

Figure 2. Relationship between the critical regret coefficient and the number of bid levels



8 An example with four and six bid levels and fear of losing

As before, we can explicitly calculate a perfect Bayesian equilibrium with fear of losing for a small number of bid levels. First, consider the case of four bid levels. By using a similar solution technique we find, a provisional winner's bidding strategies when $0\leq\alpha<1$ and $\alpha\geq 1$ are as shown in Table 2 and 3, respectively.

Table 2. Bidding strategy of a provisional winner with fear of losing, $0\leq\alpha<1$

Round	1	2		3
Price	P_1	P_2	P_2	P_3
Valuation\Ranking history	(1)	$(1,1)$	$(0,1)$	Any
$[P_0,P_1)$	Drop	-	-	-
$[P_1,P_2)$	Drop	Drop	Drop	-
$\left[P_2,P_2+\frac{2(1-\alpha)}{5(1+\alpha)}\Delta\right)$	Drop	Drop	Bid	Drop
$\left[P_2++\frac{2(1-\alpha)}{5(1+\alpha)}\Delta,P_3\right)$	Bid	Drop	Bid	Drop

Table 3. Bidding strategy of a provisional winner with fear of losing, $\alpha \geq 1$

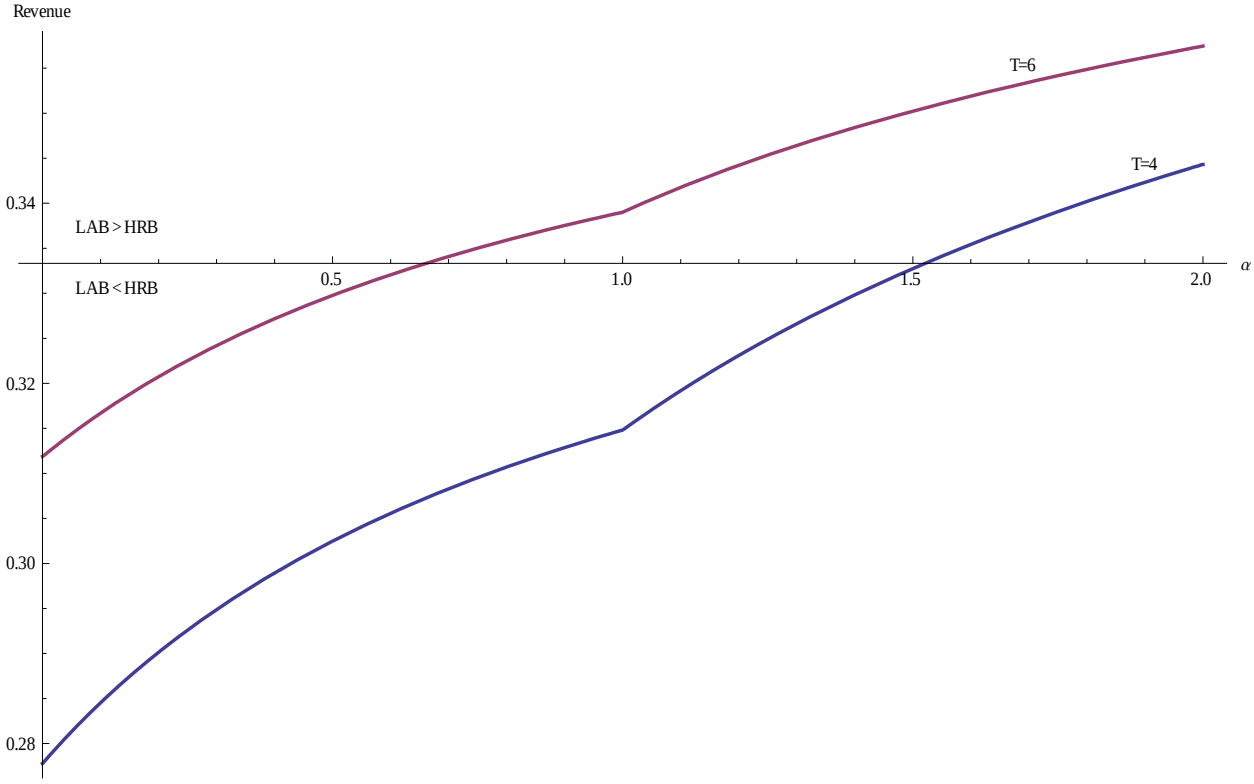
Round	1	2		3
Price	P_1	P_2	P_2	P_3
Valuation\Ranking history	(1)	$(1,1)$	$(0,1)$	Any
$[P_0, P_1)$	Drop	-	-	-
$[P_1, P_1 + \frac{2}{1+\alpha}\Delta)$	Drop	Drop	Drop	-
$[P_1 + \frac{2}{1+\alpha}\Delta, P_2)$	Bid	Drop	Drop	-
$[P_2, P_3)$	Bid	Drop	Bid	Drop

Notice that the equilibrium strategy with $\alpha < 1$ is similar to the no-regret case but the critical valuation is instead a function of the regret coefficient. If $\alpha \geq 1$, in round 1 a provisional winner with any valuation in $[P_2, P_3)$ always bids and one with valuation $[P_1 + \frac{2\Delta}{1+\alpha})$ now bids. In both cases, the larger is the regret coefficient, the lesser is the amount of shading. Expected revenues when $\alpha < 1$ and $\alpha \geq 1$ are

$$\tilde{R}_{LAB}(4,\alpha) = \begin{cases} \dfrac{15+19\alpha}{54(1+\alpha)} & ;\alpha < 1 \\ \dfrac{5+\alpha(25+\alpha(27+11\alpha))}{27(1+\alpha)^3} & ;\alpha \geq 1 \end{cases}.$$

Similar calculations can be done for the case of six bid levels. Figure 3 plots the expected revenue from the LAB auction with T = 4 (blue) and 6 (red) as a function of the regret coefficient. Revenue increases with the fear of losing, and exceeds the revenue of 1/3 from the HRB auction if the fear of losing is sufficiently strong. The x-axis is drawn at 1/3, so points above the axis are instances where the LAB auction yields higher revenues than the HRB auction. With four bid levels, the LAB auction yields higher expected revenue than the HRB auction if $\alpha > 1.52$; with six bid levels, the LAB auction yields higher expected revenue than the HBR auction if $\alpha > 0.66$. Since six or more bid levels is typical in practice, we conclude that the LAB auction may yield higher revenues than the HRB auction with plausible levels of fear of losing.

Figure 3. Expected revenue of the LAB auction with fear of losing, *T* = 4 and 6



9 Lowest-accepted bid with exit bids

Our version of lowest-accepted bid is motivated from India's 3G auction. A variation, which may be preferable is the lowest-accepted bid with exit bids (LABx). This is identical to HRB in that exit bids are allowed, but the price is set by the lowest-accepted bid, so there is a first-price incentive to shade one's bid.[2]

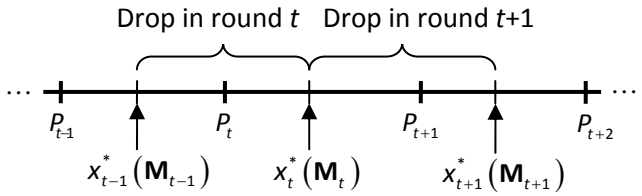
Assume that there are *N* bidders whose valuations are uniformly distributed on an interval $[0,1]$. The LABx format is identical to the HRB auction except the final price is determined from the lowest-accepted bid. If there is an excess demand in any round, the auction proceeds to the next round. If a bidder drops out in round *t*, the bidder can submit an exit bid—a price between P_{t-1} and P_t. The final price is determined by the lowest-accepted exit bid. That is, in round *t** such that $Q_{t^*} = K$, the auction ends and the final price is P_{t^*} and the items are awarded to active bidders. If $Q_{t^*} = K' < K$, the $\left(K - K'\right)^{th}$ highest exit bid determines the final price and the items are awarded to the active bidders and exiting bidders with highest $K - K'$ exit bids. Let $\mathrm{N} = \{2,\ldots,N\}$ be a set of all possible number of active bidders. Let $\mathbf{M}_t = \left(M_1, M_2, \ldots, M_t\right)$ be a vector of history of a number of active bidders from round one to round *t*

[2] Another variation is highest-rejected bid with provisional winners, but this case has been analyzed in Rothkopf and Harstad (1994) and David et al. (2007), so we do not repeat the analysis here. Moreover, highest-rejected bid with provisional winners is dominated in both revenue and efficiency by highest-rejected bid with exit bids, regardless of whether bidders fear losing.

where $M_t \in \mathrm{N}$ is a number of active bidders in round *t* for all *t* and $M_1 \le M_2 \le \ldots \le M_t$. Let $\mathbf{\Theta}_t$ be a vector of possible active bidder history at round *t*.

Proposition 7. *A symmetric equilibrium is fully characterized by* $x_t^*(\alpha, \mathbf{M}_t)$ *and* $B_t(x, \alpha, \mathbf{M}_t)$ *for all round t and for any* $\mathbf{M}_t \in \mathbf{\Theta}_t$. *Define* $x_0^*(\alpha, (\,)) = 0$. *In any round t and for any* $\mathbf{M}_t \in \mathbf{\Theta}_t$, *a bidder with valuation* $x \in \left[x_{t-1}^*(\alpha, \mathbf{M}_{t-1}), x_t^*(\alpha, \mathbf{M}_t) \right)$ *drops out in round t and submits an exit bid equal to* $B_t(x, \alpha, \mathbf{M}_t)$. $x_t^*(\alpha, \mathbf{M}_t)$ *is decreasing in* α *and* $B_t(x, \alpha, \mathbf{M}_t)$ *is increasing in x and* α.

Figure 4. Bidding strategy in the LABx auction



In each round, given an active bidder history, each bidder has a belief on valuations of the active bidders and chooses an exit bid accordingly. If the optimal exit bid is lower than the current round price, the bidder drops out. Otherwise, the bidder stays in the auction, since bidding at the current round price is more profitable than exiting at any price below. In the next round, as the bidder learns a number of active bidders in the previous round, she updates her belief and reselects an exit bid. As shown in Figure 4, in round *t* with an active bidder history $\mathbf{M}_t$, if a bidder chooses an exit bid above P_t, she bids at P_t and learns that the remaining bidders have valuations above $x_t^*(\mathbf{M}_t)$.

$x_t^*(\alpha, \mathbf{M}_t)$ *and* $B_t(x, \alpha, \mathbf{M}_t)$ can be derived iteratively from round 1 to round *T*. In any round *t* and any $\mathbf{M}_t \in \mathbf{\Theta}_t$, bidders infer that their opponents' valuations are in $\left[x_{t-1}^*(\alpha, \mathbf{M}_{t-1}), 1 \right]$. The exit bid function is

$$\tilde{B}_t(x; \alpha, z_t, M_t) = \frac{1}{G(s \mid z_t)^{1-\alpha}} \int_{z_t}^{x} s \, dG(s \mid z_t)^{1-\alpha}$$

where $z_t = x_{t-1}^*(\alpha, \mathbf{M}_{t-1})$ and $G(s \mid z_t) = F(s \mid z_t)^{M_t - 1}$. $x_t^*(\alpha, \mathbf{M}_t) \equiv x^*$ can be obtained by solving the condition $\tilde{B}_t(x^*; \alpha, x_{t-1}^*(\alpha, \mathbf{M}_{t-1}), M_t) = P_t$ for x^*. Note that the equilibrium exit bid function can be rewritten as $B_t(x, \alpha, \mathbf{M}_t) \equiv \tilde{B}_t(x, \alpha, x_{t-1}^*(\alpha, \mathbf{M}_{t-1}), M_t)$.

Proposition 8. *The LABx auction is efficient.*

Because the bidding strategy is monotonic in valuation and symmetric, the allocation is efficient. The use of exit bids overcomes both the inefficiency arising from discrete bid levels and the inefficiency arising from the asymmetry created by provisional winners.

Proposition 9. *Revenue equivalence between HRB and LABx auctions holds if* $\alpha = 0$ *. If* $\alpha > 0$ *, the LABx auction yields higher revenue than the HRB auction*.

Since the allocation rules and expected payoffs of the lowest-valuation bidder of the HRB and LABx auctions are the same, revenue equivalence immediately follows when $\alpha = 0$. In contrast to the HRB auction, in the LABx auction, bidders submit exit bids below their valuations. Fear of losing therefore impacts the bidding strategy in the LABx auction. Similar to the first-price auction, a fear of losing reduces amount of shading. Higher exit bids, relative to the $\alpha = 0$ case, imply higher expected revenue of LABx when $\alpha > 0$.

10 An example of the LABx auction with four bid levels and fear of losing

Assume that there are two bidders and each bidder's valuation is uniformly distributed over an interval $[0,1]$. In a two-bidder case, an active bidder history is irrelevant since the auction ends when any bidder drops out. A bidding strategy is given in Table 4.

Table 4. Bidding strategy of the LABx auction with four bid levels and fear of losing

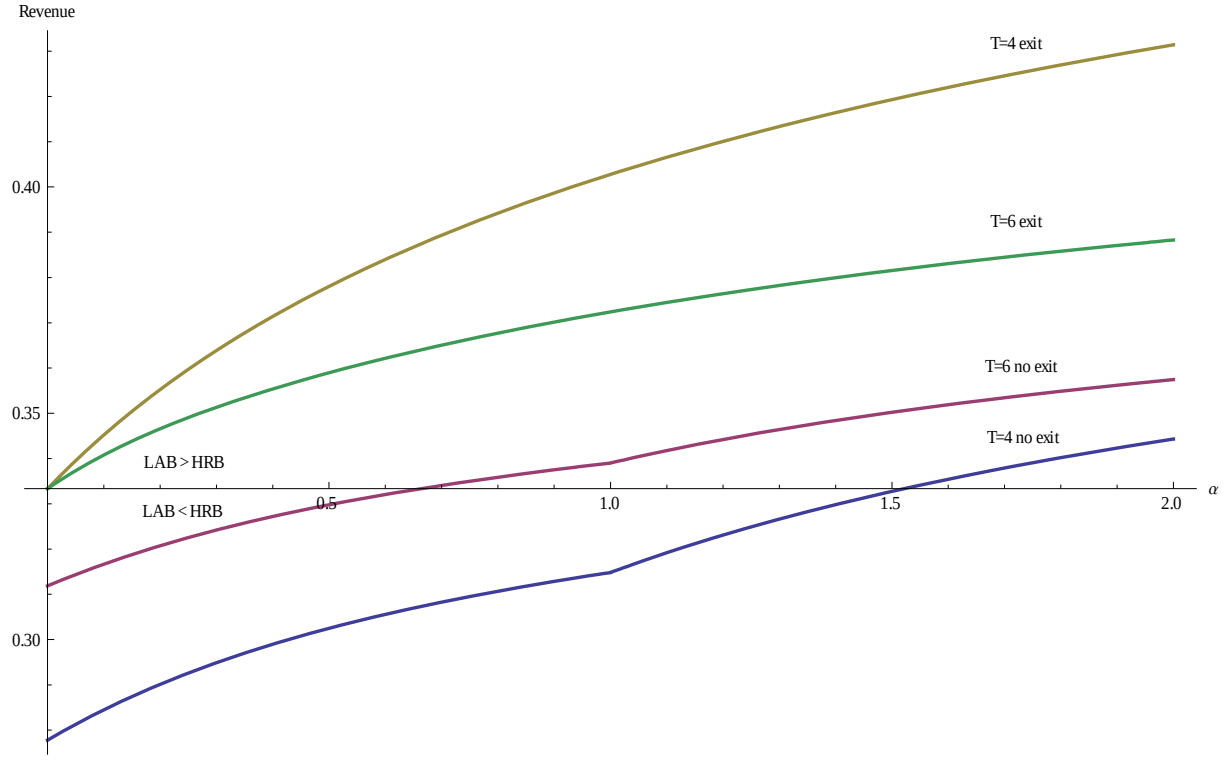
Valuation	Drop round	Exit bid
$\left[0, \frac{2+\alpha}{1+\alpha}\Delta\right)$	1	$\frac{1+\alpha}{2+\alpha}x$
$\left[\frac{2+\alpha}{1+\alpha}\Delta, \frac{(2+\alpha)(1+2\alpha)}{(1+\alpha)^2}\Delta\right)$	2	$\frac{\Delta}{1+\alpha}+\frac{1+\alpha}{2+\alpha}x$
$\left[\frac{(2+\alpha)(1+2\alpha)}{(1+\alpha)^2}\Delta, 1\right]$	3	$\frac{(1+2\alpha)\Delta}{1+\alpha}+\frac{1+\alpha}{2+\alpha}x$

If $\alpha = 0$ and both bidders bid in round 1, it implies that both bidders have valuations in $\left[\frac{(2+\alpha)(1+2\alpha)}{(1+\alpha)^2}\Delta, 1\right]$ and thus no bidder drops in round 2. The expected revenue is given as follows.

$$\tilde{R}_{LABx}(4,\alpha) = \frac{18+124\alpha+361\alpha^2+580\alpha^3+553\alpha^4+314\alpha^5+98\alpha^6+13\alpha^7}{27(2+\alpha)(1+\alpha)^6}$$

Expected revenues of the LAB and LABx auctions with fear of losing with four and six bid levels are shown in Figure 5. As the number of bid levels increases, the expected revenue when $\alpha > 0$ decreases. This is because LABx converges to HRB as the number of bid levels goes to infinity. The LABx auction achieves maximum revenues when the number of bid levels is set to two; that is, the first-price sealed-bid auction achieves the maximum revenues when bidders fear losing.

Figure 5. Expected revenue of the LAB and LABx auctions with fear of losing, *T* = 4 and 6



11 Conclusion

The pricing rule is of fundamental importance in practical auction design. It is now well understood that the pricing rule impacts both the efficiency and the revenues of the auction. Although there is an immense literature on the pricing rule in static (sealed-bid) auctions—first-price vs. second-price in single unit auctions and pay-as-bid vs. uniform-price in multi-unit auctions—little is known about alternative pricing rules in dynamic auctions. This paper begins to fill that gap.

We find that the highest-rejected-bid auction with exit bids (HRB auction) is superior in both efficiency and revenues to the lowest-accepted-bid with provisional winners (LAB auction) when bidders seek to maximize profits. Given this, it may seem odd that India, with a stated objective of revenue maximization, chose the LAB auction.

Behavioral economics provides one plausible explanation for the choice. With the LAB auction, profit maximizing bidders engage in bid shading and therefore face the risk of losing at profitable prices. Bidders who fear losing at profitable prices reduce their bid shading in order to lessen this risk. Provided the fear of losing is sufficiently high, the LAB auction revenues exceed those of the HRB auction. Thus, the LAB auction may achieve India's primary objective of maximizing revenues. However, the lowest-accepted bid with exit bids (LABx) is strictly superior to LAB in all cases in both revenue and efficiency. The use of exit bids eliminates both sources of inefficiency in the LAB auction—discrete bid levels and asymmetric bidding strategies caused by provisional winners.

Together with Emel Filiz-Ozbay and Erkut Ozbay, we conducted laboratory experiments to test the theory (Cramton et al. 2009). The experiments confirm bidding behavior consistent with a significant loser's regret coefficient. Both the LAB and LABx auction achieve significantly higher revenues than the

HRB auction. Consistent with the theory, bidders in the HRB auction tend to bid true values and the efficiency and revenues is as predicted by the theory. With LAB and LABx auctions, the bidders engage in much less bid shading than is seen in the standard theory without a fear of losing. This accounts for the significantly higher revenues under the lowest-accepted bid pricing rule.

Despite the possibility of higher revenues from the LAB format, we would recommend against its use in India or elsewhere. If there are revenue gains, the gains likely are modest (and tightly bounded as we have shown). Offsetting these potential revenue gains are inefficiencies. One source of inefficiency, which we have ignored so far, is bidder participation costs. As we have seen, bidding strategy in the LAB auction is incredibly complex even in the simplest cases. In sharp contrast, bidding strategy in the HRB auction is simple in simple settings: a bidder of modest size and with additive values across circles, which is often a good first approximation, can bid straightforwardly, raising the bid on each circle until the bidder's value is reached. The great complexity of bidding strategy of the LAB format is an important reason to favor the strategically simpler and more efficient HRB format.

Our view is that India would be better off in the long-run if it focused on efficient auctions. Efficient auctions are much simpler for bidders and still raise substantial revenues. The long-run revenues of the state are apt to be highest from a policy that promotes the rapid and efficient development of wireless communications. Auction revenues are only one piece of the revenue puzzle. For a country like India, the much more important piece is the promise of long-term sustainable growth. The rapid and efficient development of wireless communications will play a big role in achieving this growth.

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Appendix

Proof of Proposition 1. A bidder's maximization problem is to choose an optimal exit bid. Consider a bidder with a valuation $x \in [P_t, P_{t+1})$. Suppose that $\tilde{x}$ is the K^{th} highest competing bid among bidders other than bidder *i*. First, it is a weakly-dominated strategy to submit an exit bid $x' < x$. Suppose, if $\tilde{x} \le x' < x$, her payoff is equal the one when she bid her valuation. If $x' < \tilde{x} \le x$, she loses is better off by bidding her valuation. If $x' < x < \tilde{x}$, she loses regardless. Second, it is also a weakly-dominated strategy to submit an exit bid $x' > x$. If $\tilde{x} < x < x'$, she receives the same payoff as submitting an exit bid of x. If $x < \tilde{x} < x'$, she wins the item at price above her valuation and receives a negative payoff. Finally, if $x < x' < \tilde{x}$, she loses regardless. Therefore, it is a weakly-dominant strategy to submit an exit bid equal to the bidder's valuation.

Proof of Lemma 1. Suppose the probability that fewer than *K* bids are submitted in round *s* is equal to z. A provisional winner's expected payoff if she drops out in round *s* is $z(x - P_s)$. If she bids, her expected payoff is $z(x - P_{s+1}) + (1-z)\lambda$ where λ is the expected payoff if the auction proceeds to the next round. This expected payoff is strictly negative. Therefore, the expected payoff of dropping out exceeds that of bidding. Next, consider a provisional loser. Dropping out yields payoff of zero while bidding yields a strictly negative payoff since there is strictly positive probability that she wins the item at a price above her valuation.

Proof of Lemma 2. In any round $t+1$, a provisional loser has a positive probability of winning the item at P_t if fewer than $K-1$ bids are submitted. However, she may win the item at P_{t+1} which is above her valuation. If she is a provisional winner in round $t+2$, she will certainly drop out immediately according to Lemma 1. She will win the item at P_{t+1} when several conditions are met: (1) there are $K-1$ bids or more submitted in round $t+1$, (2) she is selected to be a provisional winner of rank *L* and (3) less than *L* bids are submitted in round $t+2$. Since dropping out in round $t+1$ yields a payoff of zero, she will bid when the expected payoff of bidding is positive. The expected payoff of bidding is $(x_i - P_t)\Pr(\text{win at } P_t \,|\, \mathbf{H}_{t+1}) + (x_i - P_{t+1})\Pr(\text{win at } P_{t+1} \,|\, \mathbf{H}_{t+1})$. For some $\mathbf{H}_{t+1}$, the first term is strictly positive while the second term is strictly negative. As x_i goes to P_t, the first term increases and the second term converges zero. Therefore, for x_i sufficiently close to P_{t+1}, bidder *i*'s expected payoff of bidding is positive. Note that for some $\mathbf{H}_{t+1}$, $\Pr(\text{win at } P_t \,|\, \mathbf{H}_{t+1})$ is zero and thus bidding yields strictly negative profit.

Next, it is simple to show that it is a weakly-dominant strategy for a provisional loser to bid when the current clock price is less than her valuation. If a provisional loser drops out, she receives a payoff of zero but if she bids, there is a positive probability that she is selected as a provisional winner in the subsequent round resulting in a positive expected payoff.

Proof of Lemma 3. To show this, consider a provisional winner in round t with valuation $x \in [P_t, P_{t+1})$. It is optimal to drop out in round t if

$$\begin{aligned}\pi_{it}\left(drop,\mathbf{H}_t\right) &\equiv \left(x_i - P_{t-1}\right)\Pr\left(\text{fewer than } K \text{ bids submitted in round } t|\mathbf{H}_t\right)\\ &> \left(x_i - P_t\right)\Pr\left(\text{fewer than } K \text{ bids submitted in round } t|\mathbf{H}_t\right)\\ &\quad + \left(x_i - P_t\right)\Pr\left(\text{bidder } i \text{ is a provisional winner in round } t+1 \text{ and wins}|\mathbf{H}_t\right)\\ &\equiv \pi_{it}\left(bid,\mathbf{H}_t\right)\end{aligned}$$

Or, equivalently,

$$\begin{aligned}&\Delta_{t-1}\Pr\left(\text{fewer than } K \text{ bids submitted in round } t|\mathbf{H}_t\right)\\ &> \left(x - P_t\right)\Pr\left(\text{bidder } i \text{ is a provisional winner in round } t+1 \text{ and wins}|\mathbf{H}_t\right)\end{aligned}$$

Since both probabilities are strictly positive for some $\mathbf{H}_t$, as x_i goes to P_t, the left-handed side of the inequality remains unchanged while the right-handed side goes to zero. In such scenario, it is optimal to drop out in round *t*.

Proof of Proposition 2. According to Lemma 2, provisional losers may bid above their valuations but never shade their bids and, according to Lemma 3, there is differential bid shading among provisional winners. Bidders with the same valuation bid differently and some bidders with higher valuations may drop out earlier than other bidders with lower valuations. Such asymmetric bidding leads to inefficient allocation with positive probability.

Proof of Lemma 4. Consider a bidder *i* with a valuation $x_i \in \left[P_t, P_{t+1}\right)$. First, when she is a provisional loser, it is a weakly-dominated strategy to drop out at round $t' < t$ according to Lemma 2. Second, regardless of rank, any bidder *i* has a strictly negative expected utility if she bids above her valuation since the final price will be at least P_{t+1} in the one item case. So, bidding above one's valuation is a weakly-dominated strategy.

Proof of Lemma 5. Consider a provisional winner *i* in round *t*. Using Lemma 1, the gain from dropping out in round *t* is

$$\pi_t\left(drop,\mathbf{H}_t\right) - \pi_t\left(bid,\mathbf{H}_t\right) = \Delta_{t-1}\Pr\left(\text{bidder } j \neq i \text{ drops out in round } t|\mathbf{H}_t\right) - \mathrm{E}_t\left[\Pi_{i,t+1}\left(x_i\right)|\mathbf{H}_t\right]$$

where $\mathrm{E}_t\left[\Pi_{i,t+1}\left(x_i\right)|\mathbf{H}_t\right]$ is bidder *i*'s expected payoff in round $t+1$. $\mathrm{E}_t\left[\Pi_{i,t+1}\left(x_i\right)|\mathbf{H}_t\right]$ is increasing in x_i and, therefore, the gain from dropping out in round *t* is decreasing in x_i.

Proof of Proposition 3. Weakly-dominant strategies are defined in Lemma 4 and 5.

Proof of Lemma 6. We consider a scenario that is most susceptible to bid shading. Specifically, when a provisional winner infers that a provisional loser's valuation is in $\left[P_{t-3}, 1\right]$. In round $t-2$, a provisional winner *i* will drop if $\pi_{i,t-2}\left(drop,\mathbf{H}_{t-2}\right) \leq \pi_{i,t-2}\left(bid;\mathbf{H}_{t-2}\right)$. That is,

$$\begin{aligned}\left(x_i - p_{t-3}\right)\frac{F\left(P_{t-2}\right) - F\left(P_{t-3}\right)}{1 - F\left(P_{t-3}\right)} &\leq \left(x_i - p_{t-2}\right)\frac{F\left(P_{t-2}\right) - F\left(P_{t-3}\right)}{1 - F\left(P_{t-3}\right)} + \frac{1}{2}\left(x_i - p_{t-2}\right)\frac{F\left(P_{t-1}\right) - F\left(P_{t-2}\right)}{1 - F\left(P_{t-3}\right)}\\ &\quad + \frac{1}{2}\mathrm{E}_{t-2}\left[\Pi_{i,t-1}\left(x_i\right)|\mathbf{H}_{t-2} \cup \left(R_{i,t-1} = 0\right)\right]\end{aligned}$$

Or, equivalently,

$$\frac{\Delta^2}{1-P_{t-3}} \le \frac{1}{2}\left(x_i - p_{t-2}\right)\frac{\Delta}{1-P_{t-3}} + \frac{1}{2}\mathrm{E}_{t-2}\left[\Pi_{i,t-1}\left(x_i\right)|\mathbf{H}_{t-2} \cup \left(R_{i,t-1}=0\right)\right]$$

Since $x_i - p_{t-2} \ge 2\Delta^2$,

$$\frac{\Delta^2}{1-P_{t-3}} \le \frac{\Delta^2}{1-P_{t-3}} + \frac{1}{2}\mathrm{E}_{t-2}\left[\Pi_{i,t-1}\left(x_i\right)|\mathbf{H}_{t-2} \cup \left(R_{i,t-1}=0\right)\right]$$

$$0 \le \frac{1}{2}\mathrm{E}_{t-2}\left[\Pi_{i,t-1}\left(x_i\right)|\mathbf{H}_{t-2} \cup \left(R_{i,t-1}=0\right)\right]$$

Proof of Proposition 4. This proposition follows immediately from Lemma 6.

Proof of Lemma 7. We have to calculate a probability that the final price is P_t for $t=0,1,\ldots T-1$. The final price is P_0 only if both bidders have valuations in $\left[P_0,P_1\right)$. For $t>0$, there are two possible cases: (1) both bidders have valuations in $\left[P_t,P_{t+1}\right)$ and (2) one bidder has a valuation in $\left[P_{t-1},P_t\right)$ and the other bidder has a valuation in $\left[P_t,1\right]$.

$$\begin{aligned} R_{LAB}^{SB}\left(T\right) &= P_0\left(F\left(P_1\right)-F\left(P_0\right)\right)^2 + \sum_{t=1}^{T-1} P_t\left[\left(F\left(P_{t+1}\right)-F\left(P_t\right)\right)^2 + 2\left(1-F\left(P_t\right)\right)\left(F\left(P_t\right)-F\left(P_{t-1}\right)\right)\right] \\ &= \frac{\left(T-2\right)\left(2T+3\right)}{6\left(T-1\right)^2} \end{aligned}$$

Proof of Lemma 8. The strategies are given as follows. A provisional winner does not infer the opponent's valuation from the ranking history. That is, in any round t, she maximizes her payoff given that the opponent has a valuation in $\left[P_{t-1},1\right]$. Therefore, we can find that $\hat{x}_{t-1}^t\left(H_{t-1}\right)=P_t+\frac{2}{5}\Delta$ for any $t\ge 2$ and for any $H_{t-1}\in\mathbf{\Omega}_t$. If the final price is P_t, there are three possible combination of bids determining the final price: (1) only a provisional winner drops out in round t, (2) only a provisional loser drops out in round t and (3) both of them drop out in round $t+1$. Since a provisional winner never shades more than two bid levels below her valuation, a ranking history of only three rounds before the final round is relevant. Suppose bidder *i* is a provisional winner at the final round. Hence, there are eight possible cases with a final price of P_t for $t=3,4,\ldots,T-4$ as shown in Table 5.

Table 5. Possible scenarios in which the final price is P_t for $t=3,4,\ldots,T-4$

Case	Ranking history	Valuation		Dropping bidder
		Provisional winner	Provisional loser	
1	$\left(H_{i,t-3},H_{i,t-2},H_{i,t-1},H_{i,t}\right)=\left(1,1,1,1\right)$	$\left[P_t+\frac{2}{5},P_{t+1}+\frac{2}{5}\right)$	$\left[P_t,1\right]$	Provisional winner
2	$\left(H_{i,t-3},H_{i,t-2},H_{i,t-1},H_{i,t}\right)=\left(1,1,1,1\right)$	$\left[P_{t+1}+\frac{2}{5},1\right]$	$\left[P_{t-1},P_t\right)$	Provisional loser
3	$\left(H_{i,t-3},H_{i,t-2},H_{i,t-1},H_{i,t}\right)=\left(1,0,1,1\right)$	$\left[P_{t-1}+\frac{2}{5},P_{t+1}+\frac{2}{5}\right)$	$\left[P_t+\frac{2}{5},1\right]$	Provisional winner
4	$\left(H_{i,t-3},H_{i,t-2},H_{i,t-1},H_{i,t}\right)=\left(0,1,1,1\right)$	$\left[P_t+\frac{2}{5},P_{t+1}+\frac{2}{5}\right)$	$\left[P_t,1\right]$	Provisional winner

5	$(H_{i,t-3},H_{i,t-2},H_{i,t-1},H_{i,t})=(0,1,1,1)$	$\left[P_{t+1}+\frac{2}{5},1\right]$	$\left[P_{t-1},P_{t+1}+\frac{2}{5}\right)$	Provisional loser
6	$(H_{i,t-3},H_{i,t-2},H_{i,t-1},H_{i,t})=(0,0,1,1)$	$\left[P_{t-1},P_{t+1}+\frac{2}{5}\right)$	$\left[P_{t}+\frac{2}{5},1\right]$	Provisional winner
7	$(H_{i,t-2},H_{i,t-1},H_{i,t},H_{i,t+1})=(1,1,1,1)$	$\left[P_{t+1}+\frac{2}{5},P_{t+2}+\frac{2}{5}\right)$	$\left[P_{t},P_{t+1}\right)$	Both
8	$(H_{i,t-2},H_{i,t-1},H_{i,t},H_{i,t+1})=(0,0,1,1)$	$\left[P_{t+1}+\frac{2}{5},P_{t+2}+\frac{2}{5}\right)$	$\left[P_{t}+\frac{2}{5},P_{t+1}\right)$	Both

We have to calculate a probability that the auction ends at P_0, P_1, P_2, P_{T-3} and P_{T-2} separately because their associated probabilities are different as shown in Table 6.

Table 6. Possible scenarios of which the final price are P_0, P_1, P_2, P_{T-3} and P_{T-2}

Final price	Ranking history	Valuation		Dropping bidder
		Provisional winner	Provisional loser	
P_0	$(H_{i1})=(1)$	$\left[P_0,P_2+\frac{2}{5}\right)$	$\left[P_0,1\right]$	Both
P_1	$(H_{i1})=(1)$	$\left[P_0,P_2+\frac{2}{5}\right)$	$\left[P_1,1\right]$	Provisional winner
P_1	$(H_{i1})=(1)$	$\left[P_2+\frac{2}{5}\Delta,1\right]$	$\left[P_0,P_1\right)$	Provisional loser
P_1	$(H_{i1},H_{i2})=(1,1)$	$\left[P_2+\frac{2}{5},P_3+\frac{2}{5}\right)$	$\left[P_1,P_2\right)$	Both
P_2	$(H_{i1},H_{i2})=(1,1)$	$\left[P_2+\frac{2}{5},P_3+\frac{2}{5}\right)$	$\left[P_2,1\right]$	Provisional winner
P_2	$(H_{i1},H_{i2})=(1,1)$	$\left[P_3+\frac{2}{5}\Delta,1\right]$	$\left[P_1,P_2\right)$	Provisional loser
P_2	$(H_{i1},H_{i2})=(0,1)$	$\left[P_1,P_3+\frac{2}{5}\right)$	$\left[P_2+\frac{2}{5}\Delta,1\right]$	Provisional winner
P_2	$(H_{i1},H_{i2},H_{i3})=(1,1,1)$	$\left[P_3+\frac{2}{5},P_4+\frac{2}{5}\right)$	$\left[P_2,P_3\right)$	Both
P_2	$(H_{i1},H_{i2},H_{i3})=(0,1,1)$	$\left[P_3+\frac{2}{5},P_4+\frac{2}{5}\right)$	$\left[P_2+\frac{2}{5},P_3\right)$	Both
P_{T-2}	$(H_{i,T-6},H_{i,T-5},H_{i,T-4},H_{i,T-3})=(1,1,1,1)$	$\left[P_{T-3}+\frac{2}{5},P_{T-2}+\frac{2}{5}\right)$	$\left[P_{T-3},1\right]$	Provisional winner
P_{T-2}	$(H_{i,T-6},H_{i,T-5},H_{i,T-4},H_{i,T-3})=(1,1,1,1)$	$\left[P_{T-2}+\frac{2}{5},1\right]$	$\left[P_{T-4},P_{T-3}\right)$	Provisional loser
P_{T-2}	$(H_{i,T-6},H_{i,T-5},H_{i,T-4},H_{i,T-3})=(1,0,1,1)$	$\left[P_{T-4}+\frac{2}{5},P_{T-2}+\frac{2}{5}\right)$	$\left[P_{T-3}+\frac{2}{5},1\right]$	Provisional winner
P_{T-2}	$(H_{i,T-6},H_{i,T-5},H_{i,T-4},H_{i,T-3})=(0,1,1,1)$	$\left[P_{T-3}+\frac{2}{5},P_{T-2}+\frac{2}{5}\right)$	$\left[P_{T-3},1\right]$	Provisional winner
P_{T-2}	$(H_{i,T-6},H_{i,T-5},H_{i,T-4},H_{i,T-3})=(0,1,1,1)$	$\left[P_{T-2}+\frac{2}{5},1\right]$	$\left[P_{T-4},P_{T-2}+\frac{2}{5}\right)$	Provisional loser
P_{T-2}	$(H_{i,T-6},H_{i,T-5},H_{i,T-4},H_{i,T-3})=(0,0,1,1)$	$\left[P_{T-4},P_{T-2}+\frac{2}{5}\right)$	$\left[P_{T-3}+\frac{2}{5},1\right]$	Provisional winner
P_{T-2}	$(H_{i,T-5},H_{i,T-4},H_{i,T-3},H_{i,T-2})=(1,1,1,1)$	$\left[P_{T-2}+\frac{2}{5},1\right]$	$\left[P_{T-3},P_{T-2}\right)$	Both
P_{T-2}	$(H_{i,T-5},H_{i,T-4},H_{i,T-3},H_{i,T-2})=(0,0,1,1)$	$\left[P_{T-2}+\frac{2}{5},1\right]$	$\left[P_{T-3}+\frac{2}{5},P_{T-2}\right)$	Both
P_{T-1}	$(H_{i,T-5},H_{i,T-4},H_{i,T-3},H_{i,T-2})=(1,1,1,1)$	$\left[P_{T-2}+\frac{2}{5},1\right]$	$\left[P_{T-2},1\right]$	Provisional winner

P_{T-1}	$\left(H_{i,T-5},H_{i,T-4},H_{i,T-3},H_{i,T-2}\right)=(1,0,1,1)$	$\left[P_{T-3}+\frac{2}{5},1\right]$	$\left[P_{T-2}+\frac{2}{5},1\right]$	Provisional winner
P_{T-1}	$\left(H_{i,T-5},H_{i,T-4},H_{i,T-3},H_{i,T-2}\right)=(0,1,1,1)$	$\left[P_{T-2}+\frac{2}{5},1\right]$	$\left[P_{T-2},1\right]$	Provisional winner
P_{T-1}	$\left(H_{i,T-5},H_{i,T-4},H_{i,T-3},H_{i,T-2}\right)=(0,0,1,1)$	$\left[P_{T-3},1\right]$	$\left[P_{T-2}+\frac{2}{5},1\right]$	Provisional winner

Using Table 4 and 5, we can calculate the expected revenue of the LAB auction with maximum shading strategy as

$$
\begin{aligned}
R_{LAB}^{MS}(T)=&P_0\left[\frac{1}{2}\Delta\left(\frac{12}{5}\Delta\right)\right]+P_1\left[\frac{1}{2}\Delta\left(1-\frac{12}{5}\Delta\right)+\frac{1}{2}(1-\Delta)\left(\frac{12}{5}\Delta\right)+\frac{1}{4}\Delta^2\right]\\
&+P_2\left[\frac{1}{4}\left(\frac{12}{5}\Delta\right)\left(1-\frac{12}{5}\Delta\right)+\frac{1}{4}\Delta\left(1-\frac{17}{5}\Delta\right)+\frac{1}{4}\Delta(1-2\Delta)+\frac{1}{8}\Delta^2+\frac{1}{8}\Delta\left(\frac{3}{5}\Delta\right)\right]\\
&+\sum_{t=3}^{T-4}\frac{1}{8}P_t\left[\Delta(1-t\Delta)+\Delta\left(1-(t+1)\Delta-\frac{2}{5}\Delta\right)+2\Delta\left(1-t\Delta-\frac{2}{5}\Delta\right)+\Delta(1-t\Delta)\right.\\
&\left.+\left(1-(t+1)\Delta-\frac{2}{5}\Delta\right)\left(\frac{3}{5}\Delta\right)+\left(\frac{12}{5}\Delta\right)\left(1-t\Delta-\frac{2}{5}\Delta\right)+\Delta^2+\Delta\left(\frac{3}{5}\Delta\right)\right]\\
&+\frac{1}{8}P_{T-3}\left[\Delta(1-(T-3)\Delta)+\Delta\left(1-(T-2)\Delta-\frac{2}{5}\Delta\right)+2\Delta\left(1-(T-3)\Delta-\frac{2}{5}\Delta\right)\right.\\
&\left.+\Delta(1-(T-3)\Delta)+\left(1-(T-2)\Delta-\frac{2}{5}\Delta\right)\left(\frac{3}{5}\Delta\right)+\left(\frac{12}{5}\Delta\right)\left(1-(T-3)\Delta-\frac{2}{5}\Delta\right)+\Delta\left(\frac{3}{5}\Delta\right)+\left(\frac{3}{5}\Delta\right)^2\right]\\
&+\frac{1}{8}P_{T-2}\left[(1-(T-2)\Delta)\left(\frac{3}{5}\Delta\right)+\left(\frac{8}{5}\Delta\right)\left(1-(T-2)\Delta-\frac{2}{5}\Delta\right)+(1-(T-2)\Delta)\left(\frac{3}{5}\Delta\right)+(2\Delta)\left(1-(T-2)\Delta-\frac{2}{5}\Delta\right)\right]
\end{aligned}
$$

Replacing $\Delta=\frac{1}{T-1}$ and $P_t=\frac{t}{T-1}$ yields Lemma 8.

Proof of Proposition 5. According to Lemma 7 and 8, the lower bound and upper bound of the expected revenue converge to 1/3.

Proof of Lemma 9. The calculation is similar to the proof of Lemma 8.

Proof of Proposition 6. Using the lower bound defined in Lemma 9 we solve for $\alpha\in[0,1)$ such that $\tilde{R}_{LAB}^{MS}(T,\alpha)\geq 1/3$.

Proof of Proposition 7. Consider bidder *i* with valuation x. Suppose in some round *t*, she has a belief that the remaining bidders have valuation in $[z_t,1]$ where $z_t\geq P_{t-1}$. Consider a symmetric exit bid function $\tilde{B}(x)$. Her payoff function is

$$\pi_t(x;\alpha,z_t,M_t)=\int_{z_t}^{\tilde{B}^{-1}(b)}x-b\;dG(s\mid z_t)-\alpha\int_{\tilde{B}^{-1}(b)}^{\tilde{B}^{-1}(x)}x-\tilde{B}(s)\;dG(s\mid z_t)$$

where $G(s \mid z_t) = F(s \mid z_t)^{M_t - 1}$. By differentiating the payoff function and solving the first-order condition, the symmetric bidding function can be written as follows

$$\tilde{B}_t(x;\alpha, z_t, M_t) = \frac{1}{G(x \mid z_t)^{1-\alpha}} \int_{z_t}^{x} s \, dG(s \mid z_t)^{1-\alpha}.$$

$\tilde{B}_t(x;\alpha, z_t, M_t)$ is increasing in all parameters. Since $z_t = x_{t-1}^*(\alpha, \mathbf{M}_{t-1})$, the exit bid function can be rewritten as $B_t(x, \alpha, \mathbf{M}_t)$ and it is increasing in *x* and α as well.

The next step is to prove that if the optimal exit bid is higher than the current round price, it is optimal to bid at the current round price. Consider a bidder with a valuation *x* who bids as if she has a valuation x'. Her payoff function is

$$\pi_t(x', x, z) = xG(x' \mid z) - \tilde{B}(x', \alpha, z) G(x' \mid z) - \alpha \int_{x'}^{\tilde{B}^{-1}(x')} x - \tilde{B}(s, \alpha, z) \, dG(s \mid z).$$

Differentiating the payoff function with respect to x' yields

$$\begin{aligned}
\frac{\partial}{\partial x'} \pi_t(x', x, z) &= (1+\alpha)\left(x - B(x', \alpha, z)\right) g(x' \mid z) - B'(x', \alpha, z) G(x' \mid z) \\
&= (1+\alpha)(x - x') g(x' \mid z) + (1+\alpha)\left(x' - B(x', \alpha, z)\right) g(x' \mid z) - B'(x', \alpha, z) G(x' \mid z) \\
&= (1+\alpha)(x - x') g(x' \mid z)
\end{aligned}$$

Consider a bidder with valuation *x* such that $B(x, \alpha, z) > P_t$. She is constrained to bid either at the current round price or submit and an exit bid between P_{t-1} and P_t. Since $\pi_t(x', x, z)$ is increasing for all $\tilde{x} < x$, it is optimal to bid at the current round price.

Proof of Proposition 8. This follows immediately from Proposition 9.

Proof of Proposition 9. The allocation rules and expected payoffs of the lowest-valuation bidder of the HRB and LABx auctions are the same since the exit bid function is increasing in valuation. Therefore, revenue equivalence holds when $\alpha = 0$. Since bidders shade their bids in the LABx auction, fear of losing therefore impacts the bidding strategy. In contrast, bidders bid truthfully in the HRB auction so that the strategy is not affected by fear of losing. As the exit bid function is increasing in α, the expected revenue of the LABx auction is increasing in α as well.

Solving for equilibrium of the uniform distribution, fixed increment and four bid levels

Define $\pi_{it}(q_{it}, \mathbf{H}_{it})$ as bidder *i*'s expected payoff if she submits $q_{it} \in \{bid, drop\}$ given bidder *i*'s ranking history $\mathbf{H}_{it}$. Note that we can omit the other bidder ranking history because it is a complement of bidder *i*'s ranking history. We will solve for a bidding strategy of a bidder with the lowest valuation first.

First, consider a bidder i with valuation $x_i \in [P_0, P_1)$. If she is a provisional winner in round 1, according to Lemma 3, she will drop out.

Next, consider a bidder *i* with valuation $x_i \in [P_1, P_2)$. If she is a provisional winner in round 2, she will drop out. When she is a provisional winner in round 1, her expected payoff of dropping out and bidding are

$$\pi_{i1}(drop,(1)) = (x - P_0)\frac{F(P_1) - F(P_0)}{1 - F(P_0)} = (x - P_0)\Delta$$

$$\pi_{i1}(bid,(1)) = (x - P_1)\frac{F(P_1) - F(P_0)}{1 - F(P_0)} + \frac{1}{2}(x - P_1)\frac{F(P_2) - F(P_1)}{1 - F(P_0)} = \frac{3}{2}(x - P_1)\Delta$$

Since $\pi_{i1}(drop,(1)) > \pi_{i1}(bid,(1))$, she will drop out in round 1.

Finally, consider a bidder *i* with valuation $x_i \in [P_2, P_3)$. If she is a provisional winner in round 3, she will drop out. When she is a provisional winner in round 2 and $H_{i2} = (1,1)$ which implies that the provisional winner has a valuation in $[P_1, 1]$, she will drop out since

$$\begin{aligned}\pi_{i2}(drop,(1,1)) &= (x - P_1)\frac{F(P_2) - F(P_1)}{1 - F(P_1)} = (x - P_1)\frac{\Delta}{1 - F(P_1)} \\ &> (x - P_2)\frac{F(P_2) - F(P_1)}{1 - F(P_1)} + \frac{1}{2}(x - P_1)\frac{F(P_2) - F(P_1)}{1 - F(P_1)} = \frac{3}{2}(x - P_2)\frac{\Delta}{1 - F(P_1)} = \pi_{i2}(bid,(1,1))\end{aligned}$$

When she is a provisional winner in round 2 and $H_{i2} = (0,1)$ which implies that the provisional winner has a valuation in $[P_2, 1]$, she will bid.

If she is a provisional winner in round 1, we can calculate $\hat{x}_1^2((1)) \equiv \hat{x}$ as follows.

$$\begin{aligned}\pi_{i1}(drop,(1)) &= \pi_{i1}(bid,(1)) \\ (x - P_0)\frac{F(P_1) - F(P_0)}{1 - F(P_0)} &= (\hat{x} - P_1)\frac{F(P_1) - F(P_0)}{1 - F(P_0)} + \frac{1}{2}(\hat{x} - P_1)\frac{F(P_1) - F(P_0)}{1 - F(P_0)} \\ &\quad + \frac{1}{2}\left((\hat{x} - P_2)\frac{F(P_2) - F(P_1)}{1 - F(P_0)} + \frac{1}{2}(\hat{x} - P_2)\frac{F(P_2) - F(P_1)}{1 - F(P_0)}\right) \\ (\hat{x} - P_0)\frac{\Delta}{1 - P_0} &= (\hat{x} - P_1)\frac{\Delta}{1 - P_0} + \frac{1}{2}(\hat{x} - P_1)\frac{\Delta}{1 - P_0} + \frac{1}{2}\left((\hat{x} - P_2)\frac{\Delta}{1 - P_0} + \frac{1}{2}(\hat{x} - P_2)\frac{\Delta}{1 - P_0}\right) \\ \hat{x} &= P_2 + \frac{2}{5}\Delta\end{aligned}$$

To calculate the expected revenue, we consider all possible realizations of valuations and outcomes as shown in Table 7 to 9.

Table 7. Possible outcomes for all realizations in round 1 and $H_{11}=(1)$.

	Bidder 2's value			
Bidder 1's value	$[P_0,P_1)$	$[P_1,P_2)$	$[P_2,P_2+\frac{2}{5}\Delta)$	$[P_2+\frac{2}{5}\Delta,P_3)$
$[P_0,P_1)$	P_0	P_1	P_1	P_1
$[P_1,P_2)$	P_0	P_1	P_1	P_1
$[P_2,P_2+\frac{2}{5}\Delta)$	P_0	P_1	P_1	P_1
$[P_2+\frac{2}{5}\Delta,P_3)$	P_1	Cont	Cont	Cont

Table 8. Possible outcomes for all realizations in round 2 and $H_{12}=(1,1)$.

	Bidder 2's value			
Bidder 1's value	$[P_0,P_1)$	$[P_1,P_2)$	$[P_2,P_2+\frac{2}{5}\Delta)$	$[P_2+\frac{2}{5}\Delta,P_3)$
$[P_0,P_1)$	-	-	-	-
$[P_1,P_2)$	-	-	-	-
$[P_2,P_2+\frac{2}{5}\Delta)$	-	-	-	-
$[P_2+\frac{2}{5}\Delta,P_3)$	-	P_1	P_2	P_2

Table 9. Possible outcomes for all realizations in round 2 and $H_{12}=(1,0)$.

	Bidder 2's value			
Bidder 1's value	$[P_0,P_1)$	$[P_1,P_2)$	$[P_2,P_2+\frac{2}{5}\Delta)$	$[P_2+\frac{2}{5}\Delta,P_3)$
$[P_0,P_1)$	-	-	-	-
$[P_1,P_2)$	-	-	-	-
$[P_2,P_2+\frac{2}{5}\Delta)$	-	-	-	-
$[P_2+\frac{2}{5}\Delta,P_3)$	-	P_2	P_2	P_2

Outcomes when ranking histories are $H_{11}=(0)$, $H_{12}=(0,0)$ and $H_{12}=(0,1)$ are similar to Table 7 to 9, respectively, with bidder 1 and 2 swapped. Therefore,

$$R_{LAB}(4)=2\left[\frac{1}{2}P_0\Delta\left(\frac{12}{5}\Delta\right)+\frac{1}{2}P_1\left(\Delta\left(\frac{3}{5}\Delta\right)+\frac{1}{2}(2\Delta)\left(\frac{12}{5}\Delta\right)+\frac{1}{4}\Delta\left(\frac{3}{5}\Delta\right)\right)+P_2\left(\frac{1}{4}\Delta\left(\frac{3}{5}\Delta\right)+\frac{1}{4}(2\Delta)\left(\frac{3}{5}\Delta\right)\right)\right]$$

Substituting $(P_0,P_1,P_2,P_3)=(0,\frac{1}{3},\frac{2}{3},1)$ and $\Delta=\frac{1}{3}$ yields an expected revenue of 0.2778.