Testing macroeconomic models by indirect inference on unfiltered data*

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Abstract

We extend the method of indirect inference testing to data that is not filtered and so may be non-stationary. We apply the method to an open economy real businss cycle model on UK data. We review the method using a Monte Carlo experiment and find that it performs accurately and has good power.

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In recent work Le et al (2011) have developed a method, based on indirect inference, for testing a calibrated or already estimated DSGE macroeconomic model where the data are stationary. As macro data may be non-stationary, it is usual to filter the data to make it stationary before calibrating or estimating the model. In this paper we extend their discussion of testing to when the data are non-stationary but not made stationary.

The null hypothesis is that the model to be tested, which has already been estimated or calibrated, is correct even if the equation residuals are non-stationary. The test procedure is based on comparing the properties of an auxiliary model estimated on actual data with those obtained using simulated data from the given model. On the null hypothesis that the given model is correct, the properties of the two sets of estimates of the auxiliary model should be the same. In this test procedure the residuals of the given model, which are calculated from the given estimates and may be stationary or non-stationary, are treated as observable. The critical factor in performing this test is that the auxiliary model is chosen in such a way that the distribution of the test statistic has good size and power characteristics.

We begin by describing the testing procedure we and various coauthors have developed for testing models on stationary data before going on to explain how this might be extended to non-stationary data. Since these methods are numerical in nature, they can only be explored in application. A testing method has to have power but not excessive power. Thus it is not practically useful if either a) it rejects everything that is slightly untrue or b) it rejects nothing at all. To determine the power of such a method therefore requires experience in application.We have developed a substantial amount of experience with our coauthors on this method applied to models of stationary data; we have found that it does have substantial power but that models can be found that pass the test for features of the data that are of major interest for policymakers — such as the volatility and interrelations of the major macro variables like GDP, inflation and interest rates. We would like to know whether corresponding results hold for models based on non-stationary data. We explore this issue using a Real Business Cycle model of the UK. Montecarlo experiments show that our test procedure performs well; it has both good size and power. We complete the paper by drawing some provisional conclusions.

1 Model evaluation by indirect inference

Indirect inference provides a classical statistical inferential framework for judging a calibrated or already, but possibly, partially estimated model whilst maintaining the basic idea employed in the evaluation of the early RBC models of comparing the moments generated by data simulated from the model with actual data. Using moments for the comparison is a distribution free approach. Instead, we posit a general but simple formal model (an auxiliary model) — in effect the conditional mean of the distribution of the data — and base the comparison on features of this model estimated from simulated and actual data.

Indirect inference on structural models may be distinguished from indirect estimation of structural models. Indirect estimation has been widely used for some time, see Smith (1993), Gregory and Smith (1991,1993), Gourieroux et al. (1993), Gourieroux and Monfort (1995) and Canova (2005). In estimation the parameters of the structural model are chosen so that when this model is simulated it generates estimates of the auxiliary model similar to those obtained from actual data. In the use of indirect inference for model evaluation the parameters of the structural model are taken as given. The aim is to compare the performance of the auxiliary model estimated on simulated data derived from the given estimates of a structural model — which is taken as the true model of the economy, the null hypothesis — with the performance of the auxiliary model when estimated from actual data. If the structural model is correct then its predictions about the impulse responses, moments and time series properties of the data should match those based on actual data. The comparison is based on the distributions of the two sets of parameter estimates of the auxiliary model, or of functions of these estimates.

Le et al (2011) discuss issues that arise in the choice of a VAR as the auxiliary model and in the comparison of a DSGE model with it — see also Canova (2005), Dave and DeJong (2007), Del Negro and Schorfheide (2004, 2006) and Del Negro et al (2007a,b) together with the comments by Christiano (2007), Gallant (2007), Sims (2007), Faust (2007) and Kilian (2007). The *a priori* structural restrictions of the DSGE model impose restrictions on the VAR; see Canova and Sala (2009) for an example of lack of identification, however DSGE models are generally over-identified via the cross-equation restrictions implied by rational expectations — see Minford and Peel (2002, pp.436–7).

A formal statement of the inferential problem is as follows. Using the notation of Canova (2005) which was designed for indirect estimation, we define y_t an $m \times 1$ vector of observed data $(t = 1, ..., T)$, $x_t(\theta)$ an $m \times 1$ vector of simulated time series of S observations generated from the structural macroeconomic model, θ a $k \times 1$ vector of the parameters of the macroeconomic model. $x_t(\theta)$ and y_t are assumed to

be stationary and ergodic. We set $S = T$ since we require that the actual data sample be regarded as a potential replication from the population of bootstrapped samples. The auxiliary model is $f[y_t, \alpha]$; an example is the $VAR(p)$ $y_t = \Sigma_{i=1}^{p} A_i y_{t-i} + \eta_t$ where α is a vector comprising elements of the A_i and of the covariance matrix of y_t. On the null hypothesis H_0: $\theta = \theta_0$, the stated values of θ whether obtained by calibration or estimation; the auxiliary model is then $f[x_t(\theta_0), \alpha(\theta_0)] = f[y_t, \alpha]$. We wish to test the null hypothesis through the $q \times 1$ vector of continuous functions $g(\alpha)$. Such a formulation includes impulse response functions. On H_0 $g(\alpha) = g[\alpha(\theta_0)]$.

Let a_T denote the estimator of α using actual data and $a_S(\theta_0)$ the estimator of α based on simulated data for θ_0. We may therefore obtain $g(a_T)$ and $g[a_S(\theta_0)]$. Using N independent sets of simulated data obtained using the bootstrap we can also define the bootstrap mean of the $g[a_S(\theta)]$, $\overline{g[a_S(\theta_0)]} = \frac{1}{N}\Sigma_{k=1}^{N} g_k[a_S(\theta_0)]$. The Wald test statistic is based on the distribution of $g(a_T) - \overline{g[a_S(\theta_0)]}$ where we assume that $g(a_T) - \overline{g[a_S(\theta_0)]} \xrightarrow{p} 0$. The resulting Wald statistic may be written as

$$WS = (g(a_T) - \overline{g[a_S(\theta_0)]})' W(\theta_0)^{-1} (g(a_T) - \overline{g[a_S(\theta_0)]}) \tag{1}$$

where $W(\theta_0)^{-1}$ is the inverse of the variance-covariance matrix of the distribution of $g(a_T) - \overline{g[a_S(\theta_0)]}$.

$W(\theta_0)$ can be obtained from the asymptotic distribution of $g(a_T) - \overline{g[a_S(\theta_0)]}$ and the asymptotic distribution of the Wald statistic would then be chi-squared. Instead, we obtain the empirical distribution of the Wald statistic by bootstrap methods based on defining $g(\alpha)$ as a vector consisting of the VAR coefficients and the variances of the data or the VAR disturbances.

The following steps summarise our implementation of the Wald test by bootstrapping:

Step 1: Estimate the errors of the economic model conditional on the observed data and θ_0.

Estimate the structural errors ε_t of the DSGE macroeconomic model, $x_t(\theta_0)$, given the stated values θ_0 and the observed data. The number of independent structural errors is taken to be less than or equal to the number of endogenous variables. The errors are not assumed to be normally distributed. Where the equations contain no expectations the errors can simply be backed out of the equation and the data. Where there are expectations estimation is required for the expectations; here we carry this out using the robust instrumental variables methods of McCallum (1976) and Wickens (1982), with the lagged endogenous data as instruments — thus effectively we use the auxiliary model VAR.

Step 2: Derive the simulated data

On the null hypothesis the $\{\varepsilon_t\}_{t=1}^{T}$ are the structural errors. The simulated disturbances are drawn from these errors. In some DSGE models, including the model here, many of the structural errors are assumed to be generated by autoregressive processes rather than being serially independent. If they are then, under our method, we need to estimate them. We derive the simulated data by drawing the bootstrapped disturbances by time vector to preserve any simultaneity between them, and solving the resulting model using a projection method due to Minford et al. (1984, 1986) and similar to Fair and Taylor (1983). To obtain the N bootstrapped simulations we repeat this drawing each sample independently. We set $N = 1000$.

Step 3: Compute the Wald statistic

We estimate the auxiliary model — a VAR(1) — using both the actual data and the N samples of simulated data to obtain estimates a_T and $a_S(\theta_0)$ of the vector α. The distribution of $a_T - \overline{a_S(\theta_0)}$ and its covariance matrix $W(\theta_0)$ are found by estimating the auxiliary VAR on each of the boostrapped simulations from Step 2, thus obtaining N values of $a_S(\theta_0)$; we obtain the covariance of the simulated variables directly from the bootstrap samples. The resulting set of a_k vectors $(k = 1, ..., N)$ represents the sampling variation implied by the structural model from which estimates of its mean, covariance matrix and confidence bounds may be calculated directly. Thus, the estimate of $W(\theta_0)$ is

$$\frac{1}{N}\Sigma_{k=1}^{N}(a_k - \overline{a_k})'(a_k - \overline{a_k}) \tag{2}$$

where $\overline{a_k} = \frac{1}{N}\Sigma_{k=1}^{N} a_k$. We then calculate the Wald statistic for the data sample; we estimate the bootstrap distribution of the Wald from the N bootstrap samples.

As noted, the auxiliary model used is a VAR(1) and is for a limited number of key macro variables. By raising the lag order and increasing the number of variables, the stringency of the overall test of the

model is increased. If we find that the structural model is already rejected at order 1, we do not proceed to a more stringent test based on a higher order[1].

Rather than focus our tests on just the parameters of the auxiliary model or the impulse response functions, we also attach importance to the ability to match data variability, hence the inclusion here of the VAR residuals in α. As highlighted in the debates over the Great Moderation and the recent banking crisis, a major macroeconomic issue also concerns the scale of real and nominal volatility. In this way our test procedure is within the traditions of RBC analysis.

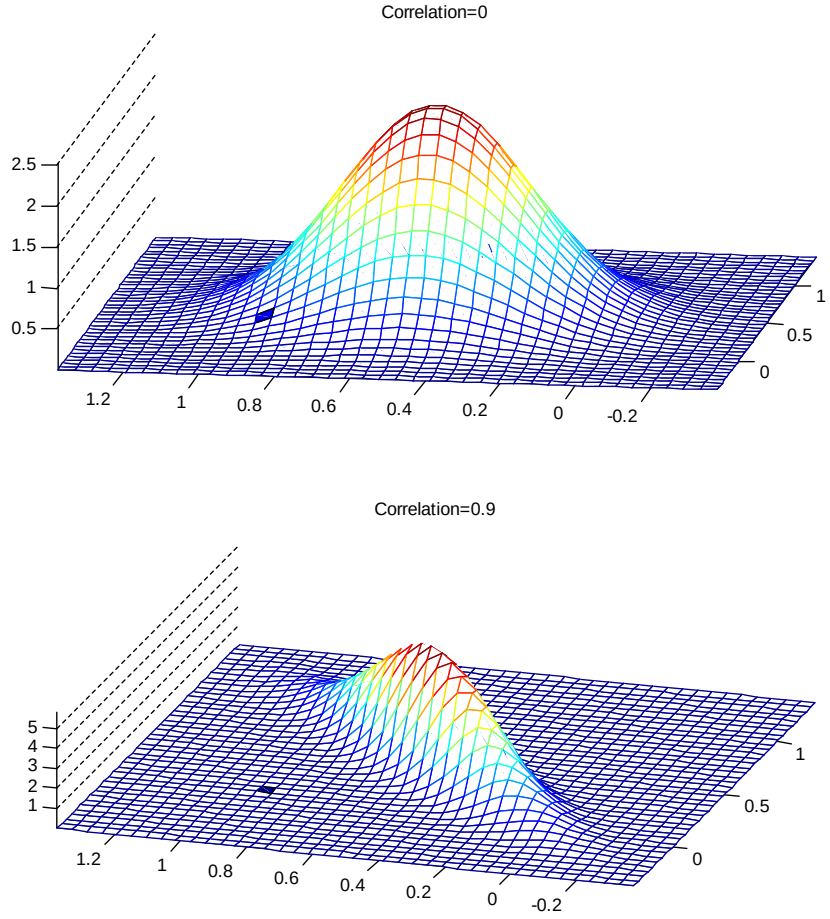


Figure 1: Bivariate Normal Distributions ($0.1, 0.9$ shaded) with correlation of 0 and 0.9.

The idea of the test can be seen usefully with a simple example, with just two VAR coefficients being assessed. Figure 1 shows the joint distribution of the two coefficients under two assumptions: the top illustrates the case where the two are uncorrelated, the bottom where they are highly correlated. Two data points are shown, one in blue, the other in red. The Wald statistic would not reject the blue point

[1] This point is illustrated in Le et al (2011) for the model dealt with in that paper with the results for varying the lag order of the VAR used there on stationary data:

	Wald stat.	Mah. Dis.
VAR(1)	100	2.8
VAR(2)	100	4.55
VAR(3)	100	5.1

Notice how the normalised Mahalanobis Distance (a transform of the Wald value — see below for the full definition) gets steadily larger, indicating a steadily worsening fit, as the lag order is increased.

In fact the general representation of a stationary loglinearised DSGE model is a VARMA, which would imply that the true VAR should be of infinite order, at least if any DSGE model is the true model. However, for the same reason that we have not raised the VECM order above one, we have also not added any MA element. As DSGE models do better in meeting the challenge this could be considered.

in the top case, but reject it in the bottom case. It would reject the red point in the top case but not in the bottom case. In the top case the covariance matrix of the coefficients has zero off-diagonal elements, whereas in the bottom case they are non-zero. It is unusual to find zero off-diagonal elements because different features of the data to be correlated across the samples generated by the DSGE model[2].

We refer to the Wald statistic based on the full set of variables as the Full Wald test; it checks whether the a vector lies within the DSGE model's implied joint distribution and is a test of the DSGE model's specification in a wide sense. We use the Mahalanobis Distance based on the same joint distribution, normalised as a t-statistic, as an overall measure of closeness between the model and the data. In effect, this conveys the same information as in the Wald test but is in the form of a t-value[3].

We also consider a second Wald test, which we refer to as a 'Directed Wald statistic'. This focuses on more limited features of the structural model. Here we seek to know how well a particular variable or limited set of variables is modelled and we use the corresponding auxiliary equations for these variables in the VAR as the basis of our test. For example, we may wish to know how well the model can reproduce the behaviour of output and inflation by creating a Wald statistic based on the VAR equation for these two variables alone.

A Directed Wald test can also be used to determine how well the structural model captures the effects of a particular set of shocks. This requires creating the joint distribution of the IRFs for these shocks alone. For example, to determine how well the model deals with supply shocks, we construct the joint distribution of the IRFs for the supply shocks and calculate a Wald statistic for this. Even if the full model is misspecified, a Directed Wald test provides information about whether the model is well-specified enough to deal with specific aspects of economic behaviour.

In this paper we focus on testing a particular specification of a DSGE model and not on how to respecify the model should the test reject it. Rejection could, of course, be due to sampling variation in the original estimates and not because the model is otherwise incorrect. This is an issue worth following up in future work. For further discussion of estimation issues see Smith (1993), Gregory and Smith (1991,1993), Gourieroux et al. (1993), Gourieroux and Monfort (1995), Canova (2005), Dridi et al (2007), Hall et al (2009), and Fukac and Pagan (2010).

1.1 Handling non-stationary data

It is common practice when estimating a DSGE model to first filter the data to ensure that they are stationary. As a result the model residuals derived from the estimated model are also stationary. It is well-known that using filtered data may distort the dynamic properties of the model in ways that are not easy to uncover. For example, the popular HP filter alters the lag dynamic structure, generating cycles where possibly none exist. Because the filter is two-sided, it also transforms the forward-looking properties of the model. These are serious defects in the estimation of a DSGE model where both the expectations structure and the impulse response functions are usually matters of considerable interest. For these reasons we prefer not to filter the data, but to use the original data.

The data generated by a DSGE model are often non-stationary. This could be either because the model structure generates non-stationarity (e.g. by making state variables functions of predetermined variables that depend on accumulated shocks, such as net foreign assets, as here), or because the model

[2] To understand why DSGE models will typically produce high covariances and so distributions like those in the bottom panel of Figure 1, we can give a simple example in the case where the two descriptors are the persistence of inflation and interest rates. If we recall the Fisher equation, we will see that the persistence of inflation and interest rates will be highly correlated. Thus in samples created by the DSGE model from its shocks where inflation is persistent, so will interest rates be; and similarly when the former is non-persistent so will the latter tend to be. Thus the two estimates of persistence under the null have a joint distribution that reflects this high correlation.

In Figure 1, we suppose that the model distribution is centred around 0.5 for each VAR coefficient; and the data-based VAR produced values (the blue ones) for their partial autocorrelations of 0.1 and 0.9 respectively for inflation and interest rates — the two VAR coefficients. We suppose too that the 95% range for each was $0 - 1.0$ (a standard deviation of 0.25) and thus each is accepted individually. If the parameters are uncorrelated across samples, then the situation is as illustrated in the top panel. They will also be jointly accepted.

Now consider the case where there is a high positive covariance between the parameter estimates across samples, as implied by the DSGE model (with its Fisher equation). The lower panel illustrates the case for a 0.9 cross-correlation between the two parameters. The effect of the high covariance is to create a ridge in the density mountain; and the joint parameter combination of $0.1, 0.9$ will be rejected even though individually the two parameters are accepted.

The red data values $(0, 0)$ analogously reject the model in the top case but do not reject it in the more usual bottom case of high correlation.

[3] The Mahalanobis Distance is the square root of the Wald value. As the square root of a chi-squared distribution, it can be converted into a t-statistic by adjusting the mean and the size. We normalise this here by ensuring that the resulting t-statistic is 1.645 at the 95% point of the distribution.

incorporates non-stationary exogenous variables such as a technology shock in the production function (an unobservable variable), or world income in the export function (an observable variable).

We assume that, after linearisation, the solution of the model can be represented by a vector error correction model. If there are unobservable non-stationary variables, such as a technology shock, then the number of cointegrating vectors will be less than the number of endogenous variables. Put another way, one or more of the long-run structural equations will have a non-stationary residual. As we have estimates of all of the coefficients of the model we can construct these residuals from the data. If we treat these residuals as observable variables then we would have as many cointegrating relations as endogenous variables. This allows us to represent the solution of the estimated model as a VECM in which the non-stationary residuals appear as observable variables, and to use an unrestricted version of this VECM as our auxiliary model.

In order to use this method we must include in the auxiliary model these nonstationary residuals derived from the DSGE model. If we did not then the auxiliary model will not contain key variables required for cointegration; thus there would not be cointegration and the VECM would not be stationary after allowing for error correction. It follows that the auxiliary model is partly conditioned by the DSGE model, this latter being the null hypothesis. We can express this as saying that the VECM is constructed under the null hypothesis. This by no means implies non-rejection by the data-generated VECM of the DSGE model because the data picks a variety of parameters that may well be inconsistent with the DSGE model. All that the constraint of the null does is to guarantee that the VECM achieves cointegration under the null. The residual assumption guarantess that the DSGE model achieves cointegration.

It might seem attractive to test for the existence of cointegration for each equation of the DSGE model. However, as we have seen, this is not possible because a) any non-stationary residual is treated as a legitimate cointegrating variable b) with a lack of cointegration the DSGE model would not have a solution and hence no simulation and Wald test would be possible. The testing we carry out here in effect imposes cointegration and tests at the later stage of the model's simulation performance.

1.1.1 The auxiliary equation

After log-linearisation a DSGE model can usually be written in the form

$$A(L)y_t = BE_t y_{t+1} + C(L)x_t + D(L)e_t \tag{3}$$

where y_t are p endogenous variables and x_t are q exogenous variables which we assume are driven by

$$\Delta x_t = a(L)\Delta x_{t-1} + d + c(L)\epsilon_t. \tag{4}$$

The exogenous variables may contain both observable and unobservable variables such as a technology shock. The disturbances e_t and ϵ_t are both iid variables with zero means. It follows that both y_t and x_t are non-stationary. L denotes the lag operator $z_{t-s} = L^s z_t$ and $A(L)$, $B(L)$ etc are polynomial functions with roots outside the unit circle.

The general solution of y_t is

$$y_t = G(L)y_{t-1} + H(L)x_t + f + M(L)e_t + N(L)\epsilon_t. \tag{5}$$

where the polynomial functions have roots outside the unit circle. As y_t and x_t are non-stationary, the solution has the p cointegration relations

$$\begin{aligned} y_t &= [I - G(1)]^{-1}[H(1)x_t + f] \\ &= \Pi x_t + g. \end{aligned} \tag{6}$$

The long-run solution to the model is

$$\begin{aligned} \overline{y}_t &= \Pi \overline{x}_t + g \\ \overline{x}_t &= [1 - a(1)]^{-1}[dt + c(1)\xi_t] \\ \xi_t &= \Sigma_{i=0}^{t-1}\epsilon_{t-s} \end{aligned}$$

Hence the long-run solution to x_t, namely, $\overline{x}_t = \overline{x}_t^D + \overline{x}_t^S$ has a deterministic trend $\overline{x}_t^D = [1 - a(1)]^{-1}dt$ and a stochastic trend $\overline{x}_t^S = [1 - a(1)]^{-1}c(1)\xi_t$.

The solution for y_t can therefore be re-written as the VECM

$$\begin{aligned}\Delta y_t &= -[I-G(1)](y_{t-1}-\Pi x_{t-1})+P(L)\Delta y_{t-1}+Q(L)\Delta x_t+f+M(L)e_t+N(L)\epsilon_t \\ &= -[I-G(1)](y_{t-1}-\Pi x_{t-1})+P(L)\Delta y_{t-1}+Q(L)\Delta x_t+f+\omega_t \qquad (7)\\ \omega_t &= M(L)e_t+N(L)\epsilon_t\end{aligned}$$

Hence, in general, the disturbance ω_t is a mixed moving average process. This suggests that the VECM can be approximated by the VARX

$$\Delta y_t = K(y_{t-1}-\Pi x_{t-1})+R(L)\Delta y_{t-1}+S(L)\Delta x_t+g+\zeta_t \qquad (8)$$

where ζ_t is an iid zero-mean process.

As

$$\overline{x}_t = \overline{x}_{t-1}+[1-a(1)]^{-1}[d+\epsilon_t]$$

the VECM can also be written as

$$\Delta y_t = K[(y_{t-1}-\overline{y}_{t-1})-\Pi(x_{t-1}-\overline{x}_{t-1})]+R(L)\Delta y_{t-1}+S(L)\Delta x_t+h+\zeta_t \qquad (9)$$

Either equations (8) or (9) can act as the auxiliary model. Here we focus on (9); this distinguishes between the effect of the trend element in x and the temporary deviation from its trend. In our models these two elements have different effects and so should be distinguished in the data to allow the greatest test discrimination.

It is possible to estimate (9) in one stage by OLS. Even though there are other methods that may achieve more accurate VECM parameter estimates[4], this method is simple and may be effective in the test procedure; this is an aspect we review below.

2 The Model

Consider a home economy populated by identical infinitely lived agents who produce a single good as output and use it both for consumption and investment; all variables are in per capita terms. It coexists with another, foreign, economy (the rest of the world) in which equivalent choices are made; however because this other country is assumed to be large relative to the home economy we treat its income as unaffected by developments in the home economy. We assume that there are no market imperfections. At the beginning of each period t, the representative agent chooses (a) the commodity bundle necessary for consumption, (b) the total amount of leisure that it would like to enjoy, and (c) the total amount of factor inputs necessary to carry out production. All of these choices are constrained by the fixed amount of time available and the aggregate resource constraint that agents face. During period t, the model economy is influenced by various random shocks.

In an open economy goods can be traded but for simplicity it is assumed that these do not enter in the production process but are only exchanged as final goods. The consumption, C_t in the utility function below, is composite per capita consumption, made up of agents consumption of domestic goods, C_t^d and their consumption of imported goods, C_t^f. We treat the consumption bundle as the numeraire so that all prices are expressed relative to the general price level, P_t. The composite consumption utility index can be represented as an Armington (1969) aggregator of the form

$$C_t = \left[\omega\left(C_t^d\right)^{-\varrho}+(1-\omega)\,\varsigma_t\left(C_t^f\right)^{-\varrho}\right]^{\left(\frac{-1}{\varrho}\right)} \qquad (10)$$

[4] The auxiliary model could also be estimated in two stages. In the first stage we may obtain a super consistent estimate of Π using OLS on the set of cointegrating regressions (i.e. the long-run reduced form)

$$y_t = \Pi x_t + g + u_t.$$

A method that estimates Π with better small sample properties is to estimate by IV the model

$$y_t = \Pi x_t + g + \Gamma(L)\Delta y_t + \Lambda(L)\Delta x_t + u_t$$

where the instruments are y_{t-1}, Δy_{t-1}, Δy_{t-2}..... and x_t, Δx_t, Δx_{t-1}, Δx_{t-2}..... In the second stage we could use this estimate of Π to construct the cointegrating residual $v_t = y_{t-1} - \Pi x_{t-1}$ and treat this as an observable variable in the auxiliary model which can be then be estimated by OLS. This two-stage method could give gains in efficiency which in turn could increase the power of the Wald test. This is an area that can be pursued in further work.

where ω is the weight of home goods in the consumption function, σ, the elasticity of substitution is equal to $\frac{1}{1+\varrho}$ and ς_t is a preference error.

The consumer maximises this composite utility index, given that an amount $\widetilde{C_t}$ has been chosen for total expenditure, with respect to its components, C_t^d and C_t^f subject to $\widetilde{C_t} = p_t^d C_t^d + Q_t C_t^f$ where p_t^d is the domestic price level relative to the general price level and Q_t is the foreign price level in domestic currency relative to the general price level (the real exchange rate)[5]. The resulting expression for the home demand for foreign goods is

$$\frac{C_t^f}{C_t} = [(1-\omega)\varsigma_t]^{\sigma}(Q_t)^{-\sigma} \tag{11}$$

We also note that:

$$1 = \omega^{\sigma}\left(p_t^d\right)^{\sigma\varrho} + [(1-\omega)\varsigma_t]^{\sigma} Q_t^{\sigma\varrho} \tag{12}$$

Hence we can obtain the logarithmic approximation:

$$\log p_t^d = -\left(\frac{1-\omega}{\omega}\right)^{\sigma} \log\left(Q_t\right) - \frac{1}{\varrho}\left(\frac{1-\omega}{\omega}\right)^{\sigma} \log \varsigma_t + \text{constant} \tag{13}$$

In a stochastic environment a consumer is expected to maximise expected utility subject to the budget constraint. Each agent's preferences are given by

$$U = MaxE_0\left[\sum_{t=0}^{\infty} \beta^t u\left(C_t, L_t\right)\right], \qquad 0 < \beta < 1 \tag{14}$$

where β is the discount factor, C_t is consumption in period 't', L_t is the amount of leisure time consumed in period 't' and E_0 is the mathematical expectations operator. Specifically, we assume a time-separable utility function of the form

$$U\left(C_t, 1-N_t\right) = \theta_0\left(1-\rho_0\right)^{-1}\gamma_t C_t^{(1-\rho_0)} + \left(1-\theta_0\right)\left(1-\rho_2\right)^{-1}\zeta_t\left(1-N_t\right)^{(1-\rho_2)} \tag{15}$$

where $0 < \theta_0 < 1$, and ρ_0, $\rho_2 > 0$ are the substitution parameters; and γ_t, ξ_t are preference errors. This sort of functional form is common in the literature for example McCallum and Nelson (1999a). Total endowment of time is normalised to unity so that

$$N_t + L_t = 1 \text{ or } L_t = 1 - N_t \tag{16}$$

Furthermore for convenience in the logarithmic transformations we assume that approximately $L = N$ on average.

The representative agent's budget constraint is

$$C_t + \frac{b_{t+1}}{1+r_t} + \frac{Q_t b_{t+1}^f}{\left(1+r_t^f\right)} + p_t S_t^p = (v_t)N_t - T_t + b_t + Q_t b_t^f + \left(p_t + d_t\right) S_{t-1}^p \tag{17}$$

where p_t denotes the real present value of shares (in the economy's firms which they own), $v_t = \frac{W_t}{P_t}$ is the real consumer wage (w_t, the producer real wage, is the the wage relative to the domestic goods price level; so $v_t = w_t p_t^d$). Households are taxed by a lump-sum transfer, T_t; marginal tax rates are not included in the model explicitly and appear implicitly in the error term of the labour supply equation, ζ_t. b_t^f denotes foreign bonds, b_t domestic bonds, S_t^p demand for domestic shares and $Q_t = \frac{P_t^f}{P_t}$ is the real exchange rate.

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[5] We form the Lagrangean $L = \left[\omega\left(C_t^d\right)^{-\varrho} + (1-\omega)\left(C_t^f\right)^{-\varrho}\right]^{\left(\frac{-1}{\varrho}\right)} + \mu(\widetilde{C_t} - \frac{P_t^d}{P_t}C_t^d - \frac{P_t^f}{P_t}C_t^f)$. Thus $\frac{\partial L}{\partial \widetilde{C_t}} = \mu$; also at its maximum with the constraint binding $L = \widetilde{C_t}$ so that $\frac{\partial L}{\partial \widetilde{C_t}} = 1$. Thus $\mu = 1$ — the change in the utility index from a one unit rise in consumption is unity. Substituting this into the first order condition $0 = \frac{\partial L}{\partial C_t^f}$ yields equation (11) . $0 = \frac{\partial L}{\partial C_t^d}$ gives the equivalent equation: $\frac{C_t^d}{C_t} = \omega^{\sigma}(p_t^d)^{-\sigma}$ where $p_t^d = \frac{P_t^d}{P_t}$. Divide (10) through by C_t to obtain $1 = \left[\omega\left(\frac{C_t^d}{C_t}\right)^{-\varrho} + (1-\omega)\left(\frac{C_t^f}{C_t}\right)^{-\varrho}\right]^{\left(\frac{-1}{\varrho}\right)}$; substituting into this for $\frac{C_t^f}{C_t}$and $\frac{C_t^d}{C_t}$from the previous two equations gives us equation (12).

In a stochastic environment the representative agent maximizes the expected discounted stream of utility subject to the budget constraint. The first order conditions with respect to C_t, N_t, b_t, b_t^f and S_t^p are (where λ_t is the Lagrangean multiplier on the budget constraint):

$$\theta_0 \gamma_t C_t^{-\rho_0} = \lambda_t \tag{18}$$

$$(1-\theta_0)\,\zeta_t\,(1-N_t)^{-\rho_2} = \lambda_t\,(1-\tau_t)\,v_t \tag{19}$$

$$\frac{\lambda_t}{1+r_t} = \beta E_t \lambda_{t+1} \tag{20}$$

$$\frac{\lambda_t Q_t}{(1+r_t^f)} = \beta E_t \lambda_{t+1} Q_{t+1} \tag{21}$$

$$\lambda_t p_t = \beta E_t \lambda_{t+1}(p_{t+1} + d_{t+1}) \tag{22}$$

Substituting equation (20) in (18) yields :

$$(1+r_t) = \left(\frac{1}{\beta}\right) E_t \left(\frac{\gamma_t}{\gamma_{t+1}}\right) \left(\frac{C_t}{C_{t+1}}\right)^{-\rho_0} \tag{23}$$

Now substituting (18) and (20) in (19) yields

$$(1-N_t) = \left\{ \frac{\theta_0 C_t^{-\rho_0} v_t}{(1-\theta_0)\,\zeta_t} \right\}^{\frac{-1}{\rho_2}} \tag{24}$$

Substituting out for $v_t = w_t p_t^d$ and using (13) equation (24) becomes

$$(1-N_t) = \left\{ \frac{\theta_0 C_t^{-\rho_0} \left[(1-\tau_t) \;\; \exp\left(\log w_t - (\frac{1-\omega}{\omega})^{\sigma}(\log Q_t + \frac{1}{\varrho}\log \varsigma_t)\right)\right]}{(1-\theta_0)\,\zeta_t} \right\}^{\frac{-1}{\rho_2}} \tag{25}$$

Substituting (20) in (22) yields

$$p_t = \left(\frac{p_{t+1} + d_{t+1}}{1+r_t}\right) \tag{26}$$

Using the arbitrage condition and by forward substitution the above yields

$$p_t = \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r_t)^i} \tag{27}$$

i.e. the present value of a share is discounted future dividends.

To derive the uncovered interest parity condition in real terms, equation (20) is substituted into (21)

$$\left(\frac{1+r_t}{1+r_t^f}\right) = E_t \frac{Q_{t+1}}{Q_t} \tag{28}$$

In logs this yields

$$r_t = r_t^f + \log E_t \frac{Q_{t+1}}{Q_t} \tag{29}$$

Thus the real interest rate differential is equal to the expected change in the real exchange rate. Financial markets are otherwise not integrated and are incomplete, though assuming completeness makes no difference to the model's solution in this non-stationary world (see Appendix 3).

2.1 The Government

The government finances its expenditure, G_t, by collecting taxes on labour income, τ_t. Also, it issues debt, bonds (b_t) each period which pays a return next period.

The government budget constraint is:

$$G_t + b_t = T_t + \frac{b_{t+1}}{1+r_t} \tag{30}$$

where b_t is real bonds

2.2 The Representative Firm

Firms rent labour and buy capital inputs, transforming them into output according to a production technology. They sell consumption goods to households and government and capital goods to other firms. The technology available to the economy is described by a constant-returns-to-scale production function:

$$Y_t = Z_t {N_t}^{\alpha} {K_t}^{1-\alpha} \tag{31}$$

where $0 \leq \alpha \leq 1$, Y_t is aggregate output per capita, K_t is capital carried over from previous period $(t-1)$, and Z_t reflects the state of technology.

It is assumed that $f(N,K)$ is smooth and concave and it satisfies Inada-type conditions i.e. the marginal product of capital (or labour) approaches infinity as capital (or labour) goes to 0 and approaches 0 as capital (or labour) goes to infinity.

$$\begin{aligned} \lim_{K \longrightarrow 0} (F_K) &= \lim_{N \longrightarrow 0} (F_N) = \infty \\ \lim_{K \longrightarrow \infty} (F_K) &= \lim_{N \longrightarrow \infty} (F_N) = 0 \end{aligned} \tag{32}$$

The capital stock evolves according to:

$$K_t = I_t + (1-\delta) K_{t-1} \tag{33}$$

where δ is the depreciation rate and I_t is gross investment.

In a stochastic environment the firm maximizes the present discounted stream, V, of cash flows, subject to the constant-returns-to-scale production technology and quadratic adjustment costs for capital,

$$MaxV = E_t \sum_{i=0}^{T} d_{it}^{i} [Y_{t+i} - K_{t+i}(r_{t+i} + \delta + \kappa_{t+i}) - (w_{t+i} + \chi_{t+i}) N_{t+i} - 0.5\xi(\Delta K_{t+i})^2] \tag{34}$$

subject to the evolution of the capital stock in the economy, equation (33). Here r_t and w_t are the rental rates of capital and labour inputs used by the firm, both of which are taken as given by the firm. The terms κ_t and χ_t are error terms capturing the impact of excluded tax rates and other imposts or regulations on firms' use of capital and labour respectively. The firm optimally chooses capital and labour so that marginal products are equal to the price per unit of input. The first order conditions with respect to K_t and N_t^d are as follows:

$$\xi(1+d_{1t})K_t = \xi K_{t-1} + \xi d_{1t} E_t K_{t+1} + \frac{(1-\alpha) Y_t}{K_t} - (r_t + \delta + \kappa_t) \tag{35}$$

$$N_t = \frac{\alpha Y_t}{w_t + \chi_t} \tag{36}$$

2.3 The Foreign Sector

From equation (11) we can derive the import equation for our economy

$$\log C_t^f = \log IM_t = \sigma \log(1-\omega) + \log C_t - \sigma \log Q_t + \sigma \log \varsigma_t \tag{37}$$

Now there exists a corresponding equation for the foreign country which is the export equation for the home economy

$$\log EX_t = \sigma^F \log\left(1-\omega^F\right) + \log C_t^F + \sigma^F \log Q_t + \sigma^F \log \varsigma_t^F \tag{38}$$

Foreign bonds evolve over time to the balance payments according to the following equation

$$\frac{Q_t b_{t+1}^f}{(1+r_t^f)} = Q_t b_t^f + p_t^d EX_t - Q_t IM_t \tag{39}$$

Finally there is good market clearing:

$$Y_t = C_t + I_t + G_t + EX_t - IM_t \tag{40}$$

3 Calibration & Deterministic Simulation

The model is calibrated with the values familiar from earlier work and used in Meenagh et al. (2010) — see Kydland and Prescott, (1982), Obstfeld and Rogoff (1996), Orphanides (1998), Dittmar, Gavin and Kydland (1999), McCallum and Nelson (1999a, 1999b), McCallum (2001), Rudebusch and Svensson (1999), Ball (1999) and Batini and Haldane (1999); Appendix 1 gives a full listing. Thus in particular the coefficient of relative risk aversion (ρ_0) is set at 1.2 and the substitution elasticity between consumption and leisure (ρ_2) at unity. Home bias (ω, ω^F) is set high at 0.7. The substitution elasticity between home and foreign goods (σ, σ^F) is set at 1 both for exports and for imports, thus assuming that the UK's products compete but not sensitively with foreign alternatives; this is in line with studies of the UK (see for example Minford et al., 1984).

Before testing the model stochastically against macro behaviour, we examine its implications in the face of a sustained one-off rise in productivity. Figure 2 shows the model simulation of a rise of the productivity level by 12% spread over 12 quarters and occurring at 1% per quarter (the increase in the whole new path is unanticipated in the first period and from then on fully anticipated) — in other words a three-year productivity 'spurt'.

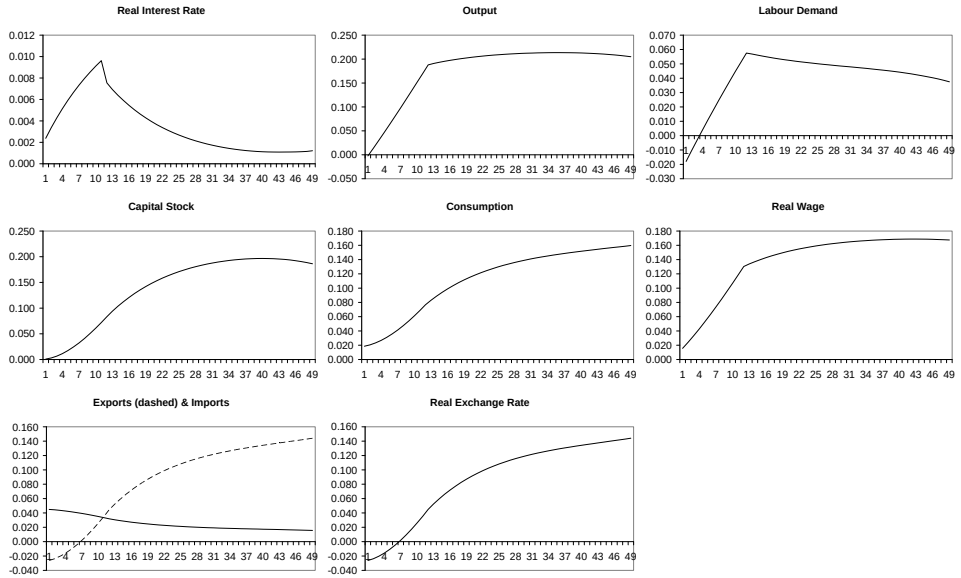


Figure 2: Plots of a 1% Productivity increase each quarter for twelve quarters

The logic behind the behaviour of the real exchange rate, Q, can be explained as follows. The productivity increase raises permanent income and also stimulates a stream of investments to raise the capital stock in line. Output however cannot be increased without increased labour supply and extra capital, which is slow to arrive. Thus the real interest rate must rise to reduce demand to the available supply while real wages rise to induce extra labour and output supply. The rising real interest rate violates Uncovered Real Interest Parity (URIP) which must be restored by a real appreciation (fall in Q) relative to the expected future value of the real exchange rate. This appreciation is made possible by the expectation that the real exchange rate will depreciate (Q will rise) steadily, so enabling URIP to be established consistently with a higher real interest rate. As real interest rates fall with the arrival on stream of sufficient capital and so output, Q also moves back to equilibrium. This equilibrium however represents a real depreciation on the previous steady state (a higher Q) since output is now higher and must be sold on world markets by lowering its price.

3.1 Stochastic processes

The model contains 8 stochastic processes: 7 shocks and 1 exogenous variable (world consumption). Of all these only one, the productivity shock, is treated as non-stationary and modelled as an $ARIMA(1,1,0)$ with a constant (the drift term, hence the deterministic trend). Since it is produced as an identity from the production function it can be directly measured. This is also true of all but two of the other shocks, which can be directly 'backed out' of their equations since they contain no expectations terms. For the

two error terms in equations containing expectations, viz consumption and the capital stock, the errors are estimated by using a robust instrumental variables estimator for the expectations due to McCallum (1976) and Wickens (1982).

Other than the productivity shock the other processes are all modelled as stationary or trend-stationary $ARMA(1,0)$ processes plus a deterministic trend. These choices cannot be rejected by the data, when they are treated as the null; however, it turns out that at the single equation level it is not easy to distinguish the two treatments, in the sense that making the alternative the null also leads to non-rejection. Hence we have used the results from the model-testing to help determine which choices to make. The choices reported here — see Table 1 — were influenced by finding that the simulated variances of key variables explode as more processes are treated as non-stationary. (Later we report the result of even treating productivity as trend-stationary; it turns out to worsen the results substantially.)

An important implication of the deterministic components of the stochastic processes is that they generate the balanced growth path (BGP) of the model. This is integrated into our simulations so that the shock elements, be they stationary or non-stationary, are added onto this basic path. In the version of the model here these deterministic components are fixed and so is therefore the BGP; of course if we were investigating policies (such as tax) that affected growth, the BGP would respond to these, however we do not do that in this paper.

Shock	Process	c	$trend$	$AR(1)$
Consumer Preference	Stationary	-0.039181^{**}		0.470434^{**}
Productivity	Non-Stationary	0.003587^{**}		0.022902
Labour Demand	Trend Stationary	0.263503^{**}	-0.002141^{**}	0.854444^{**}
Capital	Stationary	0.086334^{**}		0.870438^{**}
Labour Supply	Trend Stationary	0.717576^{**}	-0.002946^{**}	0.962092^{**}
Exports	Trend Stationary	-1.265935^{**}	0.004288^{**}	0.925119^{**}
Imports	Trend Stationary	0.007662	0.002505^{**}	0.836784^{**}
Foreign Consumption	Trend Stationary	-0.685495^{**}	0.016268^{**}	0.964308^{**}
Foreign Interest Rate	Stationary	0.002844		0.917345^{**}

Note: ** is significant at 1%, * is significant at 5%

Table 1: Error Processes

4 Testing the model

The numerical methods we use to solve the model are set out in Appendix 2 of this paper. In what follows we show how the model's simulated behaviour matches up with that of the data. Our first results are obtained using direct single stage OLS.

We note, to start with, that as usual in such studies when a wide set of variables are entered, the model is totally rejected. For example including Y, Q, C, K and r leads to a normalised Mahalanobis Distance (a t-statistic) of 7.6, massively beyond the 95% critical value of 1.645. We therefore looked for Directed Wald statistics involving smaller subsets of key variables; we wish to know if the model can replicate the behaviour of some such group, and thus define its contribution. It turns out that the model can match the behaviour of a few small subsets from among the full set. Here we show the results for the subset Y, Q and r and a summary of the subsets that get closest to the data.

Table 2 shows the results for Y, Q and r. The Wald percentile is 95.3 and the normalised distance 1.75, approximately on the 95% confidence bound; given that our method slightly over-rejects according to the Montecarlo experiment we can treat this as a borderline non-rejection. As part of the test we included the variances of the VECM residuals; these were well outside the model's 95% bounds individually but inside the joint bounds with other aspects of the data. The relationships include those with the lagged productivity trend (eYT) and with the lagged level of net foreign assets (Bf) (these being the non-stationary exogenous variables) as well as the dynamic relationships with the lagged endogenous variables, the vector of coefficients on t and the residual variances just noted. Apart from the residual variances only one individual relationship (the partial coefficient of r on Y) lies very slightly outside its 95% bound individually; good or bad individual performances do not necessarily imply that all the relationships will lie jointly within or outside the bound as this depends crucially on the covariances between the coefficients. As we see here very poor individual residual variance fits do not prevent the model overall fitting the data-estimated VECM.

	ACTUAL	LOWER	UPPER	IN/OUT
Y^Y	0.921745	0.547557	0.944052	IN
Y^Q	0.007201	−0.067114	0.189087	IN
Y^r	−0.130463	−0.123371	1.428657	OUT
Y^{eYT}	0.076943	−0.089580	0.664761	IN
Y^{trend}	0.000165	−0.000033	0.002030	IN
Y^{Bf}	−0.000001	−0.006984	0.001702	IN
Q^Y	0.075864	−0.220678	0.168833	IN
Q^Q	0.964109	0.755039	1.030719	IN
Q^r	0.540718	−0.232954	1.450536	IN
Q^{eYT}	−0.074508	−0.269644	0.499025	IN
Q^{trend}	−0.000257	−0.001527	0.000447	IN
Q^{Bf}	0.000001	−0.007092	0.003458	IN
r^Y	−0.021030	−0.028810	0.034483	IN
r^Q	−0.005892	−0.033588	0.012599	IN
r^r	0.654505	0.585070	0.886870	IN
r^{eYT}	0.033472	−0.059237	0.054454	IN
r^{trend}	0.000023	−0.000245	0.000119	IN
r^{Bf}	0.000002	−0.000755	0.000348	IN
var(Y)	0.000039	0.000356	0.002848	OUT
var(Q)	0.000327	0.000434	0.003845	OUT
var(r)	0.000008	0.000020	0.000037	OUT
Wald	95.2830			
Transformed M-dist	1.7484			

Table 2: VAR results

The table of subset results (Table 3) reveals that GDP and asset prices are well explained as we have seen but that combining these with consumption or employment leads to being rejected at 99%. Also GDP and the real exchange rate match the data when combined with either employment or consumption. Summarising one can say that the model fits the data on GDP and the two main asset prices but cannot also match the detailed behaviour of component real variables.

Subset	Wald percentile	Transformed M-dist
GDP + asset prices (+consumption or employment)		
YQr	95.3	1.77
YQC	90.4	0.89
YQCr	99.4	4.16
YQNr	99.4	3.60
GDP + Labour market bloc		
YQw	99.9	9.55
YQN	90.4	0.85
YQNw	99.9	7.58

Table 3: Table of summary results for various variable subsets

Thus the model passes well for a small set of key variables. That it fails for a broader set is a problem this model appears to share with much more elaborate structures, such as the Smets-Wouters/Christiano et al. model, with their huge efforts to include real rigidities such as habit persistence and variable capacity utilisation, as well as Calvo nominal rigidities in both wages and prices. When these are tested on stationarised data we find that invariably the inclusion of consumption wrecks the fit; however we can find a good fit to US data post-1984 for output, real interest rates and inflation taken alone. On a similar SW/CEE-style two-country model of the US and the EU, again on stationarised data, we find that it can fit output and the real exchange rate on their own but no wider set of variables.

We interpret these tests to mean that this model performs rather well in the context of model performance generally, at least in the present state of the DSGE modelling art.

4.1 Could productivity be trend-stationary?

One issue we have not so far emphasised but one that is nevertheless of empirical importance concerns our choice of error specification. Many of our error processes are not unambiguously either trend-stationary or non-stationary: that is, when we test the null of trend-stationarity we cannot reject it (at say 95% confidence) but neither can we reject the null of non-stationarity. Essentially this is because the distribution of the autoregressive coefficient is different under the two nulls. Hence in entering these errors into the DSGE model we need to make a choice that cannot be made on purely statistical grounds. The way we treat this is the same way that we treat the rest of the DSGE model specification choice where we have one; we reject one versus the other on the basis of indirect inference. We chose only to make productivity non-stationary because making the other errors non-stationary induced massively excessive variability in our key macro variables. However, this leaves the question whether even productivity should be trend-stationary, rather than non-stationary. Here we test the DSGE model under the assumption of trend-stationary productivity. Our findings are that the fit to the data worsens sharply, so that the subset of key variables that can be matched shrinks to none at all: the nearest is Y,Q,r whose Wald is 98.6 and M distance 3.65, hence rejected at 95% but accepted at 99% only. All others we looked at above are rejected at the 99% level. This gives rather clear evidence that treating productivity as non-stationary was the right choice. Thus we do not pursue this alternative representation of the model further.

Subset	Wald percentile	Transformed M-dist
GDP + asset prices (+consumption or employment)		
YQr	98.6	3.65
YQC	99.7	6.40
YQCr	99.8	8.57
YQNr	100	12.09
GDP + Labour market bloc		
YQw	100	31.70
YQN	99.9	7.33
YQNw	100	24.00

Table 4: Table of summary results for various variable subsets (Productivity trend-stationary)

4.1.1 Monte Carlo experiment testing the bootstrapping procedure for indirect inference

We now report the result of a Monte Carlo experiment on our methods, to establish their degree of accuracy and also their power. We treat the DSGE model as true and its error processes with their time-series parameters and innovations' variance, skewness and kurtosis as estimated. With 1000 replications Table 5 shows the true rejection rate at a nominal 5% confidence level is 5.7%; hence the procedure is fairly accurate.

Nominal Rejection Rate	Corresponding True Rejection Rate
10.0	10.1
5.0	5.7
1.0	1.6

Notes: The model used here was treated as the true model and the estimated residuals as the true residuals. 1000 samples of data were created by random draws from the innovations of these residuals, which were input into the model. The innovations were bootstrapped for each sample to find the Wald distribution for that sample and the Wald statistic calculated for that sample; the Table records how often the test at the chosen nominal rejection rate rejects.

Table 5: Montecarlo Rejection Rates

Table 6 shows the rejection rates at the same 5% nominal rate. We create false models by moving the parameters (of the DSGE model and of its error processes) away from their true values by + or $-x$% for alternate values; we then ask how often these false models are rejected on the data from the

true model. For this we use 10000 bootstraps on each false model. It can be seen that the method has considerable power, as it rejects 95% of the time when x is only 1% for the full set of VECM coefficients and residual variances; and 67% of the time when the coefficients on the deterministic and stochastic trends are excluded. When x reaches 5% the rejection rate reaches 100% on both. It is clear that the non-stationarity is being effectively dealt with by our VECM procedure. Thus it would appear that our simple OLS VECM procedure has desirable testing properties.

	Rejection Rate (at 5%)	
Falseness	Full VECM	VECM ex. trends
1%	95.3%	67.1%
3%	99.9%	99.6%
5%	100%	100%

Table 6: False Model Rejection Rates

5 Conclusions

In this paper we have proposed a testing procedure for macro DSGE models on non-stationary data. This procedure relies on Indirect Inference and bootstrapping to generate a Wald statistic for statsitical inference about the model's ability to fit key features of the data. To explore this procedure's practical applicability we applied it to a Real Business Cycle model of the UK. In this we found that the main shock driving the economy was a unit root productivity pocess. We found that provided we require the model only to replicate broad macro behaviour — i.e. here that of output, real interest rates and the real exchange rate — it can meet the indirect inference test rather well. Though the data is non-stationary our use of a VECM as the auxiliary equation appears to deal with the non-stationarity satisfactorily, according to Monte Carlo experiment, showing both accuracy and high power.

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6 Appendix 1: Listing of the RBC Model

6.1 Behavioural Equations

Consumption C_t ; solves for r_t:

$$\begin{aligned}(1+r_t) &= \frac{1}{\beta}E_t\left(\frac{C_t}{C_{t+1}}\right)^{-\rho_0}\left(\frac{\gamma_t}{\gamma_{t+1}}\right)\\ \log(1+r_t) &= r_t = -\rho_0(\log C_t - E_t\log C_{t+1}) + \log\gamma_t - E_t\log\gamma_{t+1} + c_0 \qquad \text{(A1_1)}\end{aligned}$$

Here we use the property that for a lognormal variable x_t, $E_t\log x_{t+1} = \log E_t x_{t+1} - 0.5\sigma^2_{\log x}$. Thus the constant c_0 contains the covariance of $(-\rho_0\log C_{t+1})$ with $(\log\gamma_{t+1})$.

UIP condition:

$$r_t = r_t^F + E_t\log Q_{t+1} - \log Q_t + c_1 \qquad \text{(A1_2)}$$

where r^F is the foreign real interest rate.

Note that equations (A1_1) and (A1_2) are combined.

Production function Y_t:

$$\begin{aligned}Y_t &= Z_t {N_t}^{\alpha} {K_t}^{1-\alpha} \qquad \text{or}\\ \log Y_t &= \alpha\log N_t + (1-\alpha)\log K_t + \log Z_t \qquad \text{(A1_3)}\end{aligned}$$

Demand for labour :

$$\begin{aligned}N_t &= \left(\frac{\alpha Y_t}{w_t(1+\chi_t)}\right) \qquad \text{or}\\ \log N_t &= c_2 + \log Y_t - \log w_t + \chi_t \qquad \text{(A1_4)}\end{aligned}$$

Capital :

$$\begin{aligned}\xi(1+d_{1t})K_t &= \xi K_{t-1} + \xi d_{1t}E_t K_{t+1} + \frac{(1-\alpha)\,Y_t}{K_t} - (r_t+\delta+\kappa_t) \text{ or}\\ \log K_t &= c_3 + \zeta_1\log K_{t-1} + \zeta_2 E_t\log K_{t+1} + (1-\zeta_1-\zeta_2)\log Y_t - \zeta_3 r_t - \zeta_3\kappa_t \qquad \text{(A1_5)}\end{aligned}$$

The producer wage is derived by equating demand for labour, N_t, to the supply of labour given by the consumer's first order conditions:

$$\begin{aligned}(1-N_t) &= \left\{\frac{\theta_0 C_t^{-\rho_0}\left[\exp\left(\log w_t - (\frac{1-\omega}{\omega})^{\sigma}(\log Q_t + \frac{1}{\rho}\log\varsigma_t)\right)\right]}{(1-\theta_0)\,\xi_t}\right\}^{\frac{-1}{\rho_2}} \qquad \text{or}\\ \log(1-N_t) &= -\log N_t = c_4 + \frac{\rho_0}{\rho_2}\log C_t - \frac{1}{\rho_2}\log w_t + \frac{1}{\rho_2}(\frac{1-\omega}{\omega})^{\sigma}\log Q_t\\ &\quad + \frac{1}{\rho_2}(\frac{1-\omega}{\omega})^{\sigma}\log\varsigma_t + \frac{1}{\rho_2}\log\xi_t \qquad \text{(A1_6)}\end{aligned}$$

where Q_t is the real exchange rate, $(1-\omega)^{\sigma}$ is the weight of domestic prices in the CPI index.

Imports IM_t:

$$\log IM_t = \sigma \log(1-\omega) + \log C_t - \sigma \log Q_t - \sigma \log \varsigma_t \tag{A1_7}$$

Exports EX_t:

$$\log EX_t = \sigma^F \log\left(1-\omega^F\right) + \log C_t^F + \sigma^F \log Q_t - \sigma^F \log \varsigma_t^F \tag{A1_8}$$

6.2 Budget constraints, market-clearing and transversality conditions:

Market-clearing condition for goods:

$$Y_t = C_t + I_t + G_t + EX_t - IM_t \tag{A1_9}$$

where investment is :

$$I_t = K_t - (1-\delta)K_{t-1}$$

and we assume the government expenditure share is an exogenous process. Loglinearised using mean GDP shares, this becomes

$$\log Y_t = 0.77 \log C_t + 6.15(\log K_t - \log K_{t-1}) + 0.3 \log G_t + 0.28 \log EX_t - 0.3 \log IM_t$$

Evolution of b_t ; government budget constraint:

$$b_{t+1} = (1+r_t)b_t + PD_t \tag{A1_10}$$

Dividends are surplus corporate cash flow:

$$\begin{aligned} d_t \overline{S_t} &= Y_t - N_t^s w_t - K_t(r_t+\delta) \\ d_t &= \frac{Y_t - N_t^s w_t - K_t(r_t+\delta)}{\overline{S_t}} \end{aligned} \tag{A1_11}$$

Market-clearing for shares, S_{t+1}^p:

$$S_{t+1}^p = \overline{S}_t \tag{A1_12}$$

Present value of share:

$$p_t = E_t \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r_t)^i} \tag{A1_13}$$

where d_t (dividend per share), p_t (present value of shares in nominal terms).

Primary deficit PD_t:

$$PD_t = G_t - T_t \tag{A1_14}$$

Tax process T_t designed to ensure convergence of government debt to transversality condition:

$$T_t = T_{t-1} + \gamma^G \frac{(PD_{t-1} + b_t r_t)}{Y_{t-1}} \tag{A1_15}$$

Evolution of foreign bonds b_t^f:

$$\frac{Q_t b_{t+1}^f}{(1+r_t^f)} = Q_t b_t^f + EX_t - Q_t IM_t \tag{A1_16}$$

Evolution of household net assets A_{t+1}:

$$A_{t+1} = (1+r_{At})A_t + Y_t - C_t - T_t - I_t \tag{A1_17}$$

where r_{At} is a weighted average of the returns on the different assets.

Household transversality condition as $T \to \infty$:

$$\Delta(\frac{A_T}{Y_T}) = 0 \tag{A1_18}$$

Government transversality condition $T \to \infty$:

$$\Delta(\frac{b_T}{Y_T}) = 0 \tag{A1_19}$$

Coefficient	Value — Single equation
α	0.70
β	0.97
δ	0.0125
ρ_0	1.20
θ_0	0.50
γ^G	0.05
ρ_2	1.00
ω	0.70
ρ	-0.50
ω^f	0.70
h	0.80
ρ_3	-0.50
σ	1
σ_F	1
ζ_1	0.5
ζ_2	0.475
ζ_3	0.25

Table 7: Model Coefficients

6.3 Values of coefficients

7 Appendix 2: Model solution methods

The model is solved in the loglinearised form above using a projection method set out in Minford et al. (1984, 1986); it is of the same type as Fair and Taylor (1983) and has been used constantly in forecasting work, with programme developments designed to ensure that the model solution is not aborted but re-initialised in the face of common traps (such as taking logs of negative numbers); the model is solved by a variety of standard algorithms, and the number of passes or iterations is increased until full convergence is achieved, including expectations equated with forecast values (note that as this model is loglinearised, certainty equivalence holds). Terminal conditions ensure that the transversality conditions on government and households are met- equivalent to setting the current account to zero). The method of solution involves first creating a base run which for convenience is set exactly equal to the actual data over the sample. The structural residuals of each equation are either backed out from the data and the model when no expectations enter as the values necessary for this exact replication of the data; or, in equations where expectations enter, they are estimated using a robust estimator of the entering expectations as proposed by McCallum (1976) and Wickens (1982), using instrumental variables; here we use as instruments the lagged variables in univariate time-series processes for each expectational variable. The resulting structural residuals are treated as the error processes in the model and together with exogenous variable processes, produce the shocks perturbing the model. For each we estimate a low-order ARIMA process to account for its autoregressive behaviour. The resulting innovations are then bootstrapped by time vector to preserve any correlations between them. Two residuals only are treated as non-stochastic and not bootstrapped: the residual in the goods market-clearing equation (the GDP identity) and that in the uncovered interest parity (UIP) condition. In the GDP identity there must be mis-measurement of the component series: we treat these measurememt errors as fixed across shocks to the true variables. In the UIP condition the residual is the risk-premium which under the assumed homoscedasticity of the shocks perturbing the model should be fixed; thus the residuals represent risk-premium variations due to perceived but according to the model non-existent movements in the shock variances. We assume that these misperceptions or mismeasurements of variances by agents are fixed across shocks perturbing the model- since, although these shocks are being generated by the true variances, agents nevertheless ignore this, therefore making these misperceptions orthogonally.

To obtain the bootstraps, shocks are drawn in an overlapping manner by time vector and input into the model base run (including the ARIMA processes for errors and exogenous variables). Thus for period 1, a vector of shocks is drawn and added into the model base run, given its initial lagged values; the model is solved for period 1 (as well as the complete future beyond) and this becomes the lagged variable vector for period 2. Then another vector of shocks is drawn after replacement for period 2 and added into this solution; the model is then solved for period 2 (and beyond) and this in turn becomes the lagged

variable vector for period 3. Then the process is repeated for period 3 and following until a bootstrap simulation is created for a full sample size. Finally to find the bootstrap effect of the shocks the base run is deducted from this simulation. The result is the bootstrap sample created by the model's shocks. We generate some 1500 of such bootstraps.

We add these bootstraps to the Balanced Growth Path implied by the model and the deterministic trend terms in the exogenous variables and error processes. We find this BGP by solving for the effect of a permanent change in each error/exogenous variable at the terminal horizon T; we then multiply this steady-state effect by the deterministic rate of change of this variable. When this BGP is incorporated in every bootstrap we have 1500 full alternative scenarios for the economy over the sample period; these bootstrap samples are then used in estimation of the VECM auxiliary equation.

To generate the model-implied joint and individual distributions of the parameters of the VECM estimated on the data, we carry out exactly the same estimation on each bootstrap sample. This gives us 1500 sample estimates which provide the sampling distribution under the null of the model. The sampling distribution for the Wald test statistic, $[a_T - \alpha_S]'W[a_T - \alpha_S]$, is of principal interest. We represent this as the percentile of the distribution where the actual data-generated parameters jointly lie. We also compute the value of the square root of this, the Mahalanobis distance, which is a one-sided normal variate; we reset this so that it has the 95% value of the variate at the same point as the 95th percentile of the bootstrap distribution (which is not necessarily normal). This 'normalised Mahalanobis Distance' we use as a measure of the distance of the model from the data under the bootstrap distribution. Its advantage is that it is a continuous variable representation of the theoretical distribution underlying the bootstrap distribution- which is made finite by the number of bootstraps.

8 Appendix 3: Model Solution with Complete Markets and Non-stationary Shocks

It turns out that under non-stationary shocks the model solution is the same under complete contingent asset contracts.

Consider contingent assets paying 1 consumption unit in specified states of the world (for example when $y_{t+T} = \widetilde{y}$)? Here we write the price of this asset, P, as

$$P_t = \frac{\beta^T u'_{t+T}(y_{t+T} = \widetilde{y}) prob(y_{t+T} = \widetilde{y})}{u'_t} \tag{A3_1}$$

One, the first, problem here is to define this probability. Since GDP has an infinite variance at T as T tends to infinity, we define the probability for a finite T. Such an asset will not be valued anyway for an 'infinite T' since as T tends to infinity β^T tends to zero. In practice therefore an asset paying off in 'infinite' time is not interesting to a household. For earlier finite periods however β^T is non-zero and the probabilities can be defined so that the asset is valued.

Now introduce a foreign country and allow trading of these contingent assets. We now let y stand for the vector of states in both countries. The foreign country's equivalent asset paying one unit of foreign consumption at T would be

$$P_{Ft} = \frac{\beta^T u'_{Ft+T}(y_{t+T} = \widetilde{y}) prob(y_{t+T} = \widetilde{y})}{u'_{Ft}} \tag{A3_2}$$

Now the price a home resident would pay for this foreign asset would be

$$P_{Ft} = \frac{\beta^T u'_{t+T} Q_{t+T}(y_{t+T} = \widetilde{y}) prob(y_{t+T} = \widetilde{y})}{u'_t Q_t} \tag{A3_3}$$

while the price a foreigner would pay for the home asset would be

$$P_t = \frac{\beta^T u'_{Ft+T}/Q_{t+T}(y_{t+T} = \widetilde{y}) prob(y_{t+T} = \widetilde{y})}{u'_{Ft}/Q_t} \tag{A3_4}$$

By equating these two values paid for each asset by home and foreign residents we obtain the Uncovered Parity contingent asset condition:

$$1 = \frac{u'_{t+T} Q_{t+T}/u'_{Ft+T}(y_{t+T} = \widetilde{y})}{u'_t Q_t/u'_{Ft}} \tag{A3_5}$$

or

$$\ln u'_{t+T} - \ln u'_t = (\ln u'_{Ft+T} - \ln u'_{Ft}) + \ln Q_{t+T} - \ln Q_t \tag{A3_6}$$

for any state of the world at T.

This ties together movements in consumption over time in the two countries with the movement in the real exchange rate. Notice that under stationary shocks the probability of the future state at $t+T$ could be defined independently of what t is, provided T is large enough so that the effects of any shocks originating at t have died away. This allowed Chari et al (2002) to fix t at some arbitrary initial date 0 and rewrite the condition

$$\ln u'_T = \ln u'_{FT} + \ln Q_T + \ln u'_0 - \ln u'_{F0} - \ln Q_0 \tag{A3_7}$$

We may then normalise the initial values at zero for convenience to obtain

$$\ln u'_T = \ln u'_{FT} - \ln Q_T \tag{A3_8}$$

However under non-stationary shocks such detachment of the condition from t is impossible because the state at $t+T$ depends crucially on the state at t: the shocks at t are permanent and therefore alter the state at $t+T$.

We may now note that taking rational expectations at t of the condition we obtain:

$$E_t(\ln u'_{t+T} - \ln u'_t) = E_t(\ln u'_{Ft+T} - \ln u'_{Ft}) + E_t(\ln Q_{t+T} - \ln Q_t) \tag{A3_9}$$

The lhs (by our non-contingent asset first order condition in the text — eqs 18 and 20 there) is simply $T \ln R$,the first term on the rhs is from the foreign equivalent $T \ln R_F$; if r is the net real interest rate then $\ln R \approx r$ so that we obtain UIP:

$$r_t = r_{Ft} + T^{-1}(E_t \ln Q_{t+T} - \ln Q_t) \tag{A3_10}$$

What we discover is that under non-stationary shocks contingent assets do not change the rational expectations equilibrium of the model from that with merely non-contingent assets. The reason is that contingent asset values depend critically on the shocks at t and so do not as with stationary shocks produce a condition binding on the expected levels of variables independent of the date at which the expectation is formed.