

Level- k Reasoning and Incentives

**Larbi Alaoui
Antonio Penta**

July 2012

Barcelona GSE Working Paper Series

Working Paper n° 653

Level- k Reasoning and Incentives ^{*}

Larbi Alaoui[†]

Antonio Penta[‡]

July 24, 2012

Abstract

Level- k theories are agnostic over whether individuals stop the iterated reasoning because of their own cognitive constraints, or because of their beliefs over the cognitive constraints of their opponents. In practice, individual level of play may be a function both of their own constraints and their beliefs over their opponents' reasoning process. Moreover, the rounds of introspection that players perform may depend on their incentives to think more deeply. We develop a theory which explicitly models players' reasoning procedure. The rounds of introspection that individuals perform and their actual level of play both follow endogenously. This model delivers testable implications as payoffs and opponents change, and it allows for comparisons across games. It also disentangles the cognitive bound of players for a given game from their beliefs about the play of their opponents. In conjunction with the framework, we present an experiment designed to test its predictions. We modify the Arad and Rubinstein (2012) '11-20' game to serve this precise purpose, and administer different treatments which vary beliefs over payoffs and opponents. The results of this experiment are consistent with the model, and appear to lend support to our theory. This experiment also confirms the central premise that individuals change their level of play as incentives to think more and beliefs over opponents vary.

Keywords: beliefs – bounded rationality – cognitive cost – higher order beliefs – incentives – level- k reasoning – value of reasoning

JEL Codes: C72; C92; D80; D83.

^{*}We thank the Laboratori d'Economia Experimental (LEEX) at Universitat Pompeu Fabra for providing the financial support and for the use of its facilities for this study, Pablo Lopez-Aguilar for programming the experimental software. We thank Cristina Fuentes-Albero, Jacob Goeree and participants to the Junior Lunch Seminar at UW-Madison for their comments. We are particularly grateful to Nagore Iriberri and Rosemarie Nagel for their thoughtful advice.

[†]Universitat Pompeu Fabra and Barcelona GSE. E-mail: larbi.alaoui@upf.edu

[‡]University of Wisconsin-Madison. E-mail: apenta@ssc.wisc.edu

1 Introduction

A large body of experimental research demonstrates that individuals do not follow the precepts of game theory. Systematic deviations arise even in the case of dominance solvable games. These observed departures have led to theories that appeal to notions of bounded rationality. Specifically, models of level- k reasoning assume that individuals have an exogenous type which corresponds to the number of rounds of iterated reasoning they perform. In these models, a level-0 individual represents a non-strategic type that follows some exogenously specified behavior, while a level-1 individual best responds to level-0, and so forth.¹

These models typically do not take a position on whether individuals stop the iterated reasoning because of their own cognitive constraints, or because of their beliefs over the cognitive constraints of their opponents. In practice, it may be the case that players vary their level of play as they change their beliefs over their opponent. In other words, their own cognitive constraints might not be binding, and their chosen strategy might adjust to the number of rounds they expect their opponents to perform. Moreover, if performing additional rounds of introspection is cognitively costly, individuals may decide that the benefit of an extra round is not worth the cognitive effort. That is, a player might be unwilling to perform an additional round of introspection, even though in principle he could do so if sufficiently motivated. In this case, the level at which he stops does not reveal his ‘absolute’ cognitive limitations. Rather, it reveals his limitations *given* the cognitive effort justified by the incentives that the game provides. Consequently, increasing players’ incentives could increase the number of rounds of iterated reasoning that they perform. Awareness of this fact may also lead them to react to changes in their opponents’ incentives. If these conjectures are empirically well-grounded, then they would suggest that level- k models would need to be extended to allow for this behavior. In this paper, we introduce a tractable framework of procedural rationality which endogenizes players’ chosen actions as a function of their incentives and beliefs about the opponents’ reasoning.

Arguably, an important appeal of level- k models is the notion that level of play derives from a reasoning procedure, which suggests that this process should be endogenized.² A plausible

¹Level- k models were first introduced by Nagel (1995) and Stahl and Wilson (1994, 1995). Camerer, Ho and Chong (2002) propose the closely related ‘cognitive hierarchy’ model, in which level- k types respond to a distribution of lower types. Level- k models have been extended to study communication (Crawford, 2003), incomplete information games (Crawford and Iriberri, 2007), etc. Crawford, Costa-Gomes and Iriberri (2012) provide a thorough survey of this literature. See also Camerer, (2003). For recent theoretical work inspired by these ideas, see Strzalecki (2010) and Kets (2012).

²Level- k models are supported by solid experimental evidence. In addition to the references in footnote 1, see for instance Ho, Camerer and Weigelt (1998), Bosch-Domènec, García-Montalvo, Nagel and Satorra (2002) and Costa-Gomez and Crawford (2006). More recently, Bhatt and Camerer (2005), Coricelli and Nagel (2009), and Bhatt, Lohrenz, Camerer and Montague (2010) study strategic thinking via fMRI, also finding further support for

model of this procedure would be instrumental to a deeper understanding of level- k reasoning. In addition, our framework allows for sharper comparisons of behavior across games. While existing level- k models have revealed a consistent regularity of play in different games, this model makes predictions relating behavior in some games to behavior in others.³ In this sense, it paves the way to a more complete model of procedural rationality.

The fundamental feature of our framework is that players weigh the incremental value of additional rounds of reasoning against an incremental cost of learning more about the game from introspection.⁴ They take into consideration that their opponents follow a similar process of reasoning and use their understanding of the game and their beliefs about the opponents' understanding in choosing an action. Moreover, since a player can face opponents of different skills, he may in principle conceive of an opponent who is as sophisticated, or more, than he is himself. Our model permits this eventuality, and resolves this apparent conceptual difficulty.

We then demonstrate that this model has clear testable implications that hold with minimal structure imposed over the value and cost associated with additional rounds reasoning. These predictions allow for comparisons of play across games. That is, a comparative statics analysis can be conducted as incentives and opponents are changed. The rounds of introspection that a player performs can be disentangled from the rounds he believes his *opponents* perform. The procedure used by the agent in selecting actions is separated from the action chosen itself. This model can therefore serve as a tool for identifying the agent's 'as-if' process in choosing actions.

We provide an experimental test of the predictions of our framework, and show that these predictions are consistent with the results. This experiment also serves the more general purpose of testing the conjecture that players vary the number of rounds of reasoning they perform as their incentives and beliefs over opponents change. Note that, with respect to the importance of beliefs on players' actions, a recent experiment by Agranov et. al. (2012) makes the simple but important point that beliefs do change the average number of rounds performed, in a standard beauty contest.⁵ Our focus is distinct, and testing our framework requires an analysis of the interaction between the agents' beliefs and incentives. Furthermore, as our concern is to explicitly

level- k reasoning.

³For an analysis of the general model with different classes of games, see Alaoui and Penta (2012).

⁴Camerer, Ho and Chong (2004) provide an interpretation of level- k play as a consequence of a cost-benefit analysis, although this is outside of their model. Recently, Choi (2012) and Gabaix (2011) have also proposed models that appeal to a cost-benefit analysis to model agents' bounded reasoning. We discuss the connections with their models in Section 3.4.

⁵Palacios-Huerta and Volij (2009) explore a related point in the dynamic context of the centipede game. They analyze how the number of rounds of backward induction performed by subjects vary when the level of sophistication of the opponent is changed.

identify the way individual level- k 's respond to varying beliefs as well as incentives, we focus on a different experimental design.

As a baseline, we use the following variation of the Arad and Rubinstein (2012) '11-20' game. Each individual chooses a number between 11 and 20, and always receives that amount in tokens, which have a specific monetary value. As in the Arad-Rubinstein (2012) game, if the agent gives a number that is exactly one less than his opponent, then he receives 20 extra tokens. Unlike the Arad-Rubinstein game, however, the agent also receives 10 extra tokens if he and his opponent tie. This addition serves the purpose of avoiding cycles in the iterated reasoning of the level- k model. Choosing 19 is only a best response to 20, choosing 18 is only a best response to 19, and so forth. In this variation, an agent's best response to 11 is still 11, since the payoff of 21 is higher than it would be if he chose any other number. Without the reward for the tie, however, the best response to 11 would be 20, hence the cycle.

Using our modification of the 11-20 game, we consider two different ways of changing the agents' beliefs over their opponents' type. In the first, we separate the subjects into two groups by their degrees of study. In the second, subjects are required to take a test of our design, and are then separated by their score, which can either be 'high' or 'low'. In both classifications, we hypothesize that the subjects believe the given labels to be informative over the game theoretic sophistication of their opponents. We then use these labels to vary agents' beliefs over their opponents, and, to test for higher order beliefs effects, over their opponents' opponents.

To test whether players respond to increased incentives of doing more rounds of reasoning, we increase the additional payoff of choosing a number that is exactly one below the opponent from 20 extra tokens to 80 extra tokens. This change does not affect the pure strategy best responses, hence the unique pure strategy equilibrium, which remains at (11,11). It only rewards players who stop at the 'correct' round of reasoning. If the subjects have the cognitive capability of doing more rounds than they exhibit in the lower payoff game, then an increase in this additional payoff may persuade them to incur the additional cost of introspection. Furthermore, if they believe that their opponents will do more rounds, then they themselves should also follow suit, provided that they are able to do so. In addition to these main treatments, we also consider additional treatments as robustness checks. For instance, subjects are given, near the end of the experiment, a game theory 'tutorial', and they play subjects who have not been given the tutorial. These additional results are deferred to Section 5.

Since players' best responses depend on the precise step at which their opponents stop the iterated reasoning, the added noise that this may generate makes this game an especially demanding

test of this theory. We show, however, that the results are consistent with our model, and that it appears to match the observed behavior very closely.

The paper is structured as follows. Section 2 introduces the experiment, and section 3 presents the theoretical model and its predictions. Section 4 analyzes the empirical results, section 5 focuses on additional treatments, and section 6 concludes.

2 Experimental Design

For the sake of exposition, we first describe the experiment before presenting the theoretical model. In essence, the experiment is designed not only to test whether individuals play differently when their incentives and beliefs about opponents change, but also to analyze the direction in which their actions change. Moreover, we aim to disentangle whether their action is dictated by their cognitive constraints, given their incentives, or by their beliefs over their opponents' cognitive constraints. The baseline game remains the modified 11-20 game throughout:

The subjects are matched in pairs. Each subject enters an (integer) number between 11 and 20, and always receives that amount in tokens. If he chooses *exactly one less* than his opponent, then he receives an extra 20 tokens. If they both choose the same number, then they both receive an extra 10 tokens.

This game is a variation of Arad and Rubinstein's (2012) '11-20' game, the difference being that the original version does not include the extra reward in case of tie. As extensively argued by Arad and Rubinstein, the 11-20 game presents a number of advantages in the study of level- k reasoning, which are inherited by our modified version. We recall here the most relevant to our purposes.

First, using level- k reasoning is natural, as there are no other obvious focal ways of approaching the game. The competing alternative of guessing the unique pure-strategy equilibrium seems far from self-evident, and would be difficult to see without going through at least a few steps of iterated reasoning. Secondly, the level-0 type specification is intuitively appealing and unambiguous, since choosing 20 is a natural anchor for an iterative reasoning process. Moreover, it is the unique best choice for a player who ignores all strategic considerations. Thirdly, there is robustness to level-0 specification, in that the choice of 19 would be the level-1 strategy for a wide range of level-0s. For instance, the best response would still be 19 even if level-0 randomized uniformly over the possible actions. Lastly, best-responding to any level- k is simple: level-1 plays 19, level-2 best responds to

19 by playing 18, and so on. In the modified 11-20 game, the best response to 11 remains 11, and is straightforward as well. Since we do not aim to capture cognitive limitations due to computational complexity, having a simple set of best responses is preferable in this case.

In addition to these points, our modification of the game leads to another useful feature for our objectives. By introducing the extra reward in case of tie, the best response to 11 is 11, and not 20, as in the version of Arad and Rubinstein. Thus, our modification breaks the cycle in the chain of best responses, which enables us to assign one specific level of reasoning to each possible announcement (with the exception of 11, which corresponds to any level higher than 8). Action 19 can only be a level-1 strategy, 18 can only be a level-2 strategy, and so forth for every k up to $k = 8$. In the original 11-20 game, action 19 could have been played by a level 1, but also by a level-11, level 21, or other ‘high’ levels (levels of form $10n+1$). Although levels-11 and above appear to be uncommon, it is crucial that these cycles be avoided here. One of the main hypotheses that we aim to test is whether an increase in players’ incentives would shift the distribution of level- k ’s toward higher k ’s, but this hypothesis could not be falsified in the presence of such cycles. In addition, since we allow for the possibility that the observed level- k behavior does not necessarily coincide with a player’s actual cognitive bound, as it may reflect instead his beliefs over his opponent’s bound, the presence of cycles would further complicate identifiability.

Our experiment consists of different treatments based on this game. Each subject plays multiple times and is anonymously paired with a new opponent after every iteration of the game. To focus on one-shot responses and to avoid learning from taking place, the subjects do not observe their payoffs after their play. They only observe their earnings once the session is over. Moreover, subjects are paid randomly, once every six iterations, and therefore do not have any mechanism for hedging against risk by changing their actions.⁶ As an additional control for order effects, the order of treatments is randomized. Furthermore, subjects play the same treatment more than once during a session, allowing us to compare play for each treatment through equality of distribution tests. Lastly, subjects only observe their own overall earnings at the end, and receive no information concerning their opponents’ results. This serves to avoid that subjects focus on goals other than monetary incentives, such as defeating the opponent or winning for its own sake.

The subjects of the experiment were 120 Spanish undergraduate students at Universitat Pompeu Fabra (UPF), in Barcelona. The instructions of the experiment were given in Spanish.⁷

⁶These devices are standard in the literature that focuses on ‘initial responses’, where the classical equilibrium approach is hard to justify. See, for instance, Stahl and Wilson (1994, 1995), Costa-Gomes, Crawford and Broseta (2001) and Costa-Gomez and Crawford (2006). For an experimental study of equilibrium in a related game, see Capra, Goeree, Gomez and Holt (1999).

⁷The details on the pool of subjects, the earnings and the logistics of the experiment are in Appendix A.

2.1 Changing beliefs about the opponents

We consider two different classifications of subjects, an *exogenous classification* and an *endogenous classification*, each with 3 sessions of 20 subjects. In the exogenous classification, subjects are distinguished by their degree of study. Specifically, in each session of the experiment, 10 students are drawn from the field of humanities (humanities, human resources, and translation), and 10 from math and sciences (math, computer science, electrical engineering, biology and economics). They are aware of their own classification when beginning the experiment, and are labeled as ‘humanities’ or ‘math and sciences’. In the endogenous classification, there is no restriction on the pool of subjects. Before playing the game, however, they are required to take a test of our design. Based on their performance on this test, each student is either labeled as ‘high’ or ‘low’, and is shown his own label before playing the game. We defer the description of this test to Section 2.3.2 (see also Appendix B).

These classifications allow us to change subjects’ beliefs about their opponents. In each treatment, the subjects are given information concerning their opponents. They play the baseline game against someone from their own label (homogeneous treatment [A]) and against someone from the other label (heterogeneous treatment [B]). For instance, for the exogenous classification, a student from math and sciences (resp., humanities), is told in homogeneous treatment [A] that his opponent is a student from math and sciences (humanities) as well. In heterogeneous treatment [B], he is told that the opponent is a student from humanities (math and sciences).

Suppose, for the sake of illustration, that there are two cognitive types of subjects, consisting of those with higher game theoretical sophistication and those with lower game theoretical sophistication. For our purposes, it suffices that players *believe* that there is a meaningful difference between these two types, and that the labels we use in the two classifications are informative about their opponents’ type (see Section 2.3). We hypothesize that subjects in the exogenous classification associate the ‘maths and sciences’ label and ‘humanities’ label with higher and lower game theoretical sophistication, respectively, and that subjects in the endogenous classification associate the ‘high’ and ‘low’ labels with higher and lower game theoretical sophistication, respectively.⁸ Then, when playing homogeneous treatment [A] compared to heterogeneous treatment [B], subjects change from believing they are playing against one type of player to another.

Treatments [A] and [B] are designed to test whether the behavior of the subjects varies with the sophistication of the opponent. The next treatment is designed to test whether the subjects believe

⁸This assumption is not required, as it is revealed by the data. Since our results are consistent with this interpretation, we maintain it in this discussion.

Treatment	Own label	Opponent's label	Own payoffs	Opponent's payoffs	Replacement of opponent's opponent
Homogeneous [A]	I (II)	I (II)	Low	Low	No
Heterogeneous [B]	I (II)	II (I)	Low	Low	No
Replacement [C]	I (II)	II (I)	Low	Low	Yes
Homogeneous-high [A+]	I (II)	I (II)	High	High	No
Heterogeneous-high [B+]	I (II)	II (I)	High	High	No
Replacement-high [C+]	I (II)	II (I)	High	High	Yes

Table 1: Treatment summary: Label I (II) refers to ‘math and sciences’ (‘humanities’) or to ‘high’ (‘low’) subjects.

(or are aware of) that the behavior of their opponents also changes when they face opponents of different levels of sophistication. To do so, we consider replacement treatment [C]. A ‘math and sciences’ subject, for instance, is told that his opponent is “a student from humanities facing another student from humanities.”⁹ The theoretical framework that we introduce in the next section makes predictions on the change in the distribution of actions chosen by players of both groups, for each of the three treatments. These predictions are summarized in Section 3, and a summary of the different treatments can be found in Table 1.

2.2 Changing incentives

We next consider a second dimension that would entail a change in players’ chosen actions, according to our framework. In particular, we aim to test the central premise of our theoretical model, that players may perform more rounds of introspection if they are given more incentives to do so. To do this, we change the baseline game in the following way: rather than winning an extra 20 tokens for choosing the number precisely one below the opponent, subjects win an extra 80 tokens. The rest of the game payoffs remain the same.

We consider three treatments for this ‘high payoff game’: homogeneous treatment [A+], heterogeneous treatment [B+], and replacement treatment [C+]. These treatments are equivalent to treatments [A], [B] and [C], respectively, but with the increased reward for undercutting of 80 tokens. We then compare an agent’s play under lower payoffs to his play under higher payoffs, by comparing [A] to [A+], [B] to [B+] and [C] to [C+]. We also compare treatments [A+], [B+] and [C+] in an analogous way to the comparison between treatments [A], [B] and [C]. This concludes our discussion of the main treatments, summarized in Table 1; a description of the additional treatments is deferred to Section 5.

⁹For a related experiment in a beauty contest environment, see Agranov, Potamites, Schotter and Tergiman (2012), in which they consider how undergraduates play when replacing a graduate student in a pool of eight graduate student subjects.

2.3 Experimental Design: Discussion

2.3.1 The modified 11-20 Game

Our theoretical model is applicable to a range of well-known games, for which it makes testable predictions. As discussed in Alaoui and Penta (2012), these predictions appear consistent with observed behavior in a variety of experiments.¹⁰ But since our precise objective here is to test our model as cleanly as possible and while minimizing the impact of confounding factors, we favor the modified 11-20 game in our experiment. As we have described the features which make this game particularly suitable, we now turn to the reasons for not using these other games instead. We maintain, however, that a number of these games fall within the scope of our model, and that it could therefore serve in better identifying the cognitive procedure associated with observed behavior.

Perhaps the most evident candidate is the *beauty contest* (or *p-guessing game*), which dominates the experimental literature on level- k . In this game, a number of players are asked to report a number between 0 and 100, and the player whose number is closest to a fraction $p \in (0, 1)$ of the average report wins a monetary prize.¹¹ A remarkable finding of the literature is that players' reports exhibit a regular pattern concentrated around specific numbers, and that the position of these spikes shifts down as p decreases. This evidence has been interpreted as suggesting that players approach games of this kind by thinking in steps, which has been a central motivation to the development of level- k theories.

While the *beauty contest* game is ideal to test whether players do reason according to some form of iterated reasoning, the goal of the present discussion is different. Rather, we maintain that players follow an iterated reasoning procedure, and we ask whether their depth of reasoning varies with their beliefs about the sophistication of the opponents, and with incentives to reason more deeply about the game. A number of features of the beauty contest makes it less suitable than the modified 11-20 for this task. For instance, level- k reasoning is not necessarily the only 'focal' form of reasoning, and, for related reasons, the beauty contest does not present an obvious specification of the level-0 action.¹² This last point is particularly problematic because the level-1 action is highly sensitive to the specification of level-0. Furthermore, a natural setting for our

¹⁰For instance, the predictions of our theory in the static games considered in Holt and Goeree (2001) are consistent with their experimental findings. These games include matching pennies, battle of the sexes and the traveller's dilemma (see Alaoui and Penta, 2012).

¹¹The guessing game was first introduced by Nagel (1995) and Stahl and Wilson (1994, 1995). Variations of the game have been studied by Ho, Camerer and Weigelt (1998), Bosch-Domènech, García-Montalvo, Nagel and Satorra (2002) and Costa-Gomez and Crawford (2006).

¹²For a thorough description of these different thought processes, see Bosch-Domènech, García-Montalvo, Nagel and Satorra (2002).

experiment is one with few players, as it allows for greater ease in changing beliefs over opponents, and opponents' opponents. But with a small number of players, the best response function in the p -guessing game is difficult to compute. A player must take into account the impact of his own number on the average, which is not as immediate a calculation as computing the best response in our game.¹³ Notwithstanding these factors, we expect the comparative statics predictions of our theoretical model to hold for the beauty contest as well.

Another game that may seem apt is Basu's (1994) traveler's dilemma. In this two-player game, each agent reports a number between 2 and 100, and both players receive the lowest of the two numbers chosen. In addition, if the players report different numbers, then the one who reports a higher number pays a penalty of 2, and the one with the lowest receives a reward of 2. This game shares appealing features of the modified 11-20 game. The main difference is that it is sufficient in the traveler's dilemma to undercut the opponent to receive the additional reward, rather than to be at exactly the right action. The modified 11-20 therefore leads to a more precise mapping between the agent's action and his level of reasoning. Moreover, it would not be an issue in the modified 11-20 game if agents had social preferences such as altruism or fairness. This is because, independent of agents' preferences over the final outcomes, the optimal choice would still require players to identify their opponents' action.¹⁴

2.3.2 Designing the Group Classification: Demarcation and Focality.

In order to vary subjects' beliefs about the opponents, we divide the pool of subjects into two labeled groups. We then change subjects' beliefs about the opponents by changing the opponent's group in the different treatments. This approach requires two features. The first is *demarcation*; the types in the classifications must be perceived as being sufficiently distinct from each other in terms of their strategic sophistication. The second is *focality*. Since subjects' behavior depends not only on their beliefs but also on their beliefs about their opponents' beliefs, it is important that the two types share sufficient agreement about the way the two types differ. That is, they should 'commonly agree' over their relative sophistication. The two classification we consider have been chosen to guarantee that these properties hold.

The Exogenous Classification. The exogenous classification exploits the intuitive, albeit vague, view that 'math and sciences' students are regarded as more accustomed to numerical reasoning

¹³See Grosskopf and Nagel (2008), in which a large majority of subjects in the two-person beauty contest do not play the dominant strategy.

¹⁴A traveler's dilemma in which payoffs are varied can be found in Holt and Goeree (2001). The results of this game fit well with our theoretical predictions; see Alaoui and Penta (2012) for details.

than ‘humanities’ students. Furthermore, the specific degrees of study used to populate the ‘maths and sciences’ group are commonly viewed as being the most selective degrees at UPF, and require the highest entry marks.¹⁵ We would therefore expect the subjects to believe the ‘maths and sciences’ group to be comparatively more sophisticated in game theoretical reasoning than the ‘humanities’ group. However, the subjects are not primed into shaping specific beliefs about either particular group.

The Endogenous Classification and the Test. Since the exogenous classification is based on labels that are salient for the subjects in the pool, it can be expected to ensure both demarcation and focality. But it does not allow us to fully control the agents’ beliefs about these labels. For this reason, we also introduced the endogenous classification, where students are classified solely based on their performance in a test of our design. The goal of the test is thus twofold. It sorts subjects into two groups, and, by labeling the scores obtained by subjects as ‘high’ or ‘low’, the test itself forms the agents’ beliefs over the content of these labels.

The main objective of the test is to convince subjects that the result of the test is informative about their opponents’ game theoretical sophistication. To do so, we ensure that our test questions appear difficult to solve, and that subjects would be likely to infer that an individual with of higher sophistication would respond better to the questions.¹⁶

The cognitive test takes roughly thirty minutes to complete, and consists of three questions. These are all single-agent questions, in that they are not pitted against each other. Rather, subjects are asked to provide the correct answer. Scores are assigned to the answers to each of these tasks, and a formula then takes a weighted average. Those who have a score above the median are labeled ‘high’, and the others are labeled ‘low’. Subjects do not see their numerical grade, but they are told whether they are labeled ‘high’ or ‘low’. (Details of the test are contained in Appendix B).

The three questions are as follows. In the first question, subjects have nine attempts to play a variation of the board game *mastermind*, in which the aim is to deduce a hidden pattern through a sequence of guesses. This game requires skill at logical inference to complete successfully. In the second question, they are given a typical *centipede game* of seven rounds. In the third game, the agents are given a lesser known *pirates game*. As with the centipede game, the pirates game

¹⁵These views emerged from informal conversations with students. They are confirmed by the admission scores, used to select the students admitted in the various fields. These scores can be found at: <http://www.elpais.com/especial/universidades/titulaciones/universidad/universidad-pompeu-fabra/45/nota-corte/>

¹⁶Another possible consequence of being assigned a ‘high’ or a ‘low’ label may impact an individual’s self-esteem, which may in turn impact his subsequent performance in playing the modified 11-20 game. This concern, however, is tangential to the aim of our experiment. Our objective is not to identify the number of rounds of introspection a subject performs in a single game, but how his actions vary across treatments.

can be solved by backward induction. The solution is more difficult to attain, however, in that it involves additional computation in determining how players' strategy profiles map into outcomes. For the latter two games, subjects are not asked how they would play; rather, they are asked how "infinitely sophisticated and rational agents, who each want to get as much money as possible" would play.

These three questions are challenging to most, and performing well is arguably informative of a subject's sophistication. More importantly, for our purposes, it seems fair to assume that the *subjects* believe the score to be a strong indicator of game theoretical sophistication. Arguably, this would be the case for subjects who recognize that the 11-20 game has a recursive pattern reminiscent of the structure of the problems in the test, which appears plausible for players who would perform at least one round of introspection. Furthermore, while the last two games are close enough to the 11-20 game to require a similar kind of analysis, they are sufficiently different that the feedback provided in the form of the 'high' or 'low' score should limit the learning about how to play the 11-20. It is precisely to avoid information leaking from the test to the proper experiment that we have not included the beauty contest, the traveler's dilemma or other closely related games.

2.3.3 Testing for Effects of Higher Order Beliefs

Suppose that label *I* denotes the group perceived by the subjects as relatively more sophisticated, and that label *II* denotes the group perceived as relatively less sophisticated. We would then expect label *I* subjects to play (weakly) lower numbers in treatment [A] than in treatment [B], and the label *II* subjects to play (weakly) lower numbers in treatment [B] than [A].¹⁷ Whereas these treatments test whether subjects' beliefs affect their choices in the game, it is difficult to establish from these treatments alone whether the subjects are aware that their opponents' beliefs may affect their choices. For instance, subjects may expect, as we do, that a label *II* subject plays differently against another label *II* than against label *I*. This would indicate that they have an understanding that the label *II* subject may not necessarily play according to his cognitive limit, and that his play may vary.

The objective of treatment [C] is precisely to test for the degree to which the subjects have a well-formed model of their opponents' reasoning, and whether it is consistent with our theoretical predictions. The precise wording of treatment [C] is designed to pin down the entire hierarchy of

¹⁷Note that we do not make the comparison across groups, but within groups. That is, we do not compare the true relative sophistication of the groups and whether their beliefs are correct.

beliefs, since any potential ambiguity may lead to a misinterpretation of the results. For instance, the full description that a math and sciences student is given concerning his opponent in treatment [C] is: “two students from humanities play against each other. You play against the number that one of them has picked.” It is therefore clear that he is playing a humanities playing a humanities subject, who himself is playing a humanities subject, and so forth. Any ambiguity that allowed players to believe that one of them believes (at some high level) that one of them is a student from math and sciences could result in a sophisticated subject behaving as a less sophisticated one, invalidating the identification of the types.

3 A Theory of Endogenous Level- k Reasoning

This section introduces our model of endogenous level- k reasoning.¹⁸ We follow existing level- k theories in positing that players approach the game with some form of iterated reasoning. While we take this process itself to be exogenous, we endogenize the number of steps at which they stop their reasoning. In particular, their cognitive abilities and the payoff structure of the game determine the number of steps they take. We also endogenize their chosen actions, which depends both on the number of steps of reasoning each agent takes and on his beliefs about the cognitive abilities of his opponents. We then show that this theory delivers clear observable predictions for the different treatments in the experiment. More precisely, the model implies first-order stochastic dominance relations between the distributions of the actions played, by a given ‘label’, in the different treatments of the experiment.

3.1 Steps of Reasoning

Consider a finite two-players game with complete information, $G = (A_i, u_i)_{i=1,2}$, where A_i is the (finite) set of actions of player i , $A_1 = A_2$, and $u_i : A_1 \times A_2 \rightarrow \mathbb{R}$ is player i ’s payoff function. Let G be such that the (pure strategy) best response correspondence $BR_i : A_j \rightarrow A_i$, defined as

$$BR_i(a_j) = \arg \max_{a_i \in A_i} u_i(a_i, a_j) \text{ for each } a_j \in A_j,$$

is single-valued. Exploiting the symmetry and single-valuedness of BR_i , we assume that players’ reasoning about the game is represented by sequences $\{a_1^k\}_{k \in \mathbb{N}}$, $\{a_2^k\}_{k \in \mathbb{N}}$ such that $a_1^0 = a_2^0$, and $a_i^{k+1} = BR_i(a_j^k)$ for each $k \in \mathbb{N}$. We interpret a_i^0 as the action that player i would play by

¹⁸A more general framework is developed in Alaoui and Penta (2012).

default, without any strategic understanding of the game. As player i performs the first step of reasoning, however, he becomes aware that his opponent could play a_j^0 , and thus considers playing $a_i^1 = BR_i(a_j^0)$. Similarly, as player i advances from step $k - 1$ to step k , he realizes that his opponent may play a_j^{k-1} , in which case the best response would be $a_i^k = BR_i(a_j^{k-1})$.

We interpret the steps of reasoning, k , as ‘rounds of introspection’. As an illustration of this concept, consider the modified 11-20 game used in the experimental design. For any player i , action $a_i^0 = 20$ is a natural action for a level-0 player, as it is the number that a non-sophisticated player would report if he ignored all strategic considerations. If player i exerts some cognitive effort and performs the first step of the reasoning process, then he realizes that the opponent may approach the game in the same way, in which case the best response is $a_i^1 = 19$. Similarly, if player i performs one more round of introspection, he realizes that his opponent may also be able to perform the same reasoning, and play 19. In that case, the best response is $a_i^2 = 18$. If he performs another round, his best response is $a_i^3 = 17$. This reasoning continues until he reaches 11, in which case the best response remains at 11.¹⁹

Thus, in our model, players are not ‘boundedly rational’ in the sense of failing to compute best responses. Rather, players are limited in their ability to conceive that the opponent may perform the same steps of reasoning. Thus, after 3 rounds of introspection, a player has become aware that the opponent may perform two steps of reasoning, but not more. In other words, after 3 rounds of introspection, player i understands that *if* j believes that $a_i = 20$, then j would play 19, and that j understands that *if* i expects j to play 20, then he would play $a_i = 19$, in which case $a_j = 18$ would be the optimal response. This, however, does not necessarily mean that player i would play 17 if he performed 3 steps of reasoning. The action chosen by player i depends on his beliefs about whether player j is actually capable of performing two steps reasoning. That is, it depends on player i ’s beliefs about j ’s cognitive abilities.

This interpretation of level- k is related to that of Crawford and Iriberri (2007) and other models of level- k reasoning. In these frameworks, a player’s k identifies both the action that he plays and his ‘understanding of the game’. Here, we distinguish the two, and separate the highest k that a player has reached from his beliefs over the number of rounds performed by his opponents. The action chosen by a player is a function of both factors, which can be disentangled from one another. Denoting by $\hat{k}_i$ player i ’s *cognitive bound*, he may play action $a_i^{k_i}$ such that $k_i \leq \hat{k}_i$ if, given $\hat{k}_i$, he thinks that $a_i^{k_i}$ is preferable to $a_i^{\hat{k}_i}$. Our model aims to endogenize both $\hat{k}_i$ and k_i ,

¹⁹Formally, $BR_i(a_j)$ is single-valued for any pure strategy a_j , and is equal to $BR_i(a_j) = \max\{a_j - 1, 11\}$ for $a_j \in \{11, \dots, 20\}$. The action associated with any given $k \geq 0$ is $a_i^k = \max\{20 - k, 11\}$.

where the former is a function of player i 's incentives and cognitive abilities, while the latter is also a function of his beliefs about the cognitive abilities of the opponent.²⁰

3.2 Cognitive Costs and Value of Introspection.

The model we propose for endogenizing the steps of reasoning taken by players is based on a cost-benefit analysis. Performing additional rounds of reasoning entails incurring a cognitive cost. While these costs reflect a player's cognitive ability, which we view as exogenous, we assume that the benefits of performing an extra step of reasoning depend on the payoff structure of the game.

We stress that we do not envision this 'cost-benefit' analysis as an optimization problem *actually* solved by the agent, but rather as a device to represent a player's reasoning about the game. We hypothesize that agents' understanding of the game varies systematically with the payoff structure. To the extent that players' understanding of the game exhibits this form of consistency, we posit that it can be modeled *as if* the cognitive bound $\hat{k}_i$ results from a cost-benefit analysis.

3.2.1 Individual Understanding of the Game.

Formally, we assume that the value of doing extra steps of reasoning only depends on the payoff structure of the game. This is represented by a function $v_i : \mathbb{N} \rightarrow \mathbb{R}_+$, where $v_i(k)$ represents i 's value of doing the k -th round of reasoning, given the previous $k - 1$ rounds. The cognitive ability of agent i is represented by a cost function $c_i : \mathbb{N} \rightarrow \mathbb{R}_+$, where $c_i(k)$ denotes i 's incremental cost of performing the k -th round of reasoning. The following assumptions on c_i and v_i are maintained throughout.

Condition 1 *Maintained assumptions on the Cost and Value of Reasoning:*

1. **Cost of Reasoning:** $c_i(0) = 0$ and $c_i(k) \geq 0$ for every $k \in \mathbb{N}$.
2. **Value of Reasoning:** $v_i(k) \geq 0$ for every $k \in \mathbb{N}$.

Condition 1 entails minimal restrictions on the cost and value-of-reasoning functions. In particular, it contains no assumptions about their 'shape' (their monotonicity, convexity, etc.). For the results that follow, we do not need to impose additional structure on these functions.

²⁰In the following we introduce different notions of level k . Without a superscript, k_i refers to player i 's actual k . When both a subscript and a superscript are present, the symbol denotes a belief: the subscript indicates the player that a particular k refers to, and the superscript indicates the player holding the belief. For instance, k_j^i denotes i 's belief about j 's behavioral k . Moreover, the k notation refer to behavior while $\hat{k}$ notation refers to cognitive bounds (e.g. $\hat{k}_i$ is i 's cognitive bound, and $\hat{k}_j^i$ is i 's belief about j 's cognitive bound.)

The purpose of maintaining this level of generality is to capture different kinds of plausible cost functions. For instance, in the modified 11-20 game, a player who understands the inductive structure of the problem would have a non-monotonic cost c_i . The first steps of reasoning may be cognitively costly, but subsequent steps of reasoning would not be. We can allow for this type of thought process with a cost function that is strictly positive for the first levels, and then drops to zero after the agent has figured out the inductive step (Figure 1 represents the cost function of an agent who understands the induction after four steps of reasoning. See example 1).

It is useful to introduce the following mapping, which identifies the intersection between the value of reasoning and the cost function: Let $\mathcal{K} : \mathbb{R}_+^{\mathbb{N}} \times \mathbb{R}_+^{\mathbb{N}} \rightarrow \mathbb{N}$ be such that, for any $(c, v) \in \mathbb{R}_+^{\mathbb{N}} \times \mathbb{R}_+^{\mathbb{N}}$,

$$\mathcal{K}(c, v) = \min \{k \in \mathbb{N} : c(k) \leq v(k) \text{ and } c(k+1) > v(k+1)\}, \quad (1)$$

with the understanding that $\mathcal{K}(c, v) = \infty$ if the set in equation (1) is empty. Player i 's *cognitive bound*, which represents his understanding of the game, is then determined by the value that this function takes at (c_i, v_i) :

Definition 1 *Given cost and value functions (c_i, v_i) , the cognitive bound of player i is defined as:*

$$\hat{k}_i = \mathcal{K}(c_i, v_i). \quad (2)$$

Player i therefore stops the iterative process when the value of performing an additional round of introspection exceeds the cost. The point at which this occurs identifies his cognitive bound $\hat{k}_i$. Note that a player does not compare the benefits and costs at higher k 's. That is, he does not consider stages of reasoning higher than his current point. A player who has performed k rounds of introspection is only aware of the portion uncovered by the k steps, and performs a 'one-step ahead' comparison of the incremental cost and value of reasoning.

This process may appear to translate to a standard optimization problem, in which an agent's marginal cost-marginal benefit analysis can be interpreted as first order conditions of a 'total' value and cost tradeoff. This need not always be the case, however. If the functions c_i and v_i cross at most once, then the two procedures are indeed the same. In other words, if the player were able to look ahead to higher k 's, it would not change his decision to stop the process, since the cost c_i would remain higher than the benefit v_i for those k 's. But suppose that the cost and benefit functions were to cross more than once. Then, with a standard optimization problem, the first crossing would not necessarily be the optimal one. The myopic procedure followed by

the players in this model would then not lead to the global optimizer's solution. Thus, player i does not 'optimize' by taking into account the entire curves; rather, he optimizes locally. The 'myopic' (one-step-ahead) behavior that we assume captures the idea, inherent to the very notion of bounded rationality, that the agents do not know (or are not aware of) whatever they have not (yet) thought about. Formalizing this notion is often perceived to be a fundamental difficulty in developing a theory of bounded rationality; in this model it emerges naturally.²¹

Notice that, in our framework, the cognitive bound $\hat{k}_i$ is monotonic in the level of the cost and of the value functions, irrespective of the shape of these functions: $\hat{k}_i$ (weakly) decreases as the cognitive costs increase, and it (weakly) increases as the value of reasoning increases.

Remark 1 *Under the maintained assumptions of Condition 1:*

1. *For any c_i , the cognitive bound $\hat{k}_i$ (weakly) increases as v_i is replaced by v'_i such that $v'_i(k) \geq v_i(k)$ for all k .*
2. *For any v_i , the cognitive bound $\hat{k}_i$ (weakly) decreases as c_i is replaced by c'_i such that $c'_i(k) \geq c_i(k)$ for all k .*

This result is immediate, as shown in the following example of players' reasoning process.

Example 1 *Figure 1 represents a non-monotonic cost function, c_i . The value of reasoning, v_i , is drawn as a constant for simplicity. Player i 's cognitive bound, $\hat{k}_i$, is determined by the first intersection of the two functions, c_i and v_i . In the graph on the left, $\hat{k}_i = 2$, meaning that player i has 'become aware' of one round of reasoning of the opponent. The grey area represents the 'unawareness region' of player i about the opponents' steps of reasoning of level higher than 1. As the value v_i is continuously increased, $\hat{k}_i$ remains constant at first, but then increases to $\hat{k}'_i = 3$ when level v'_i is reached. Correspondingly, the grey area shifts to the right, uncovering one more round of reasoning of the opponent. If v_i is further increased, $\hat{k}_i$ stays constant at first, and it increases to $\hat{k}_i = 4$ once v^* is reached. As soon as $v_i > v^*$, however, $\hat{k}_i$ jumps to ∞ . The non-monotonic cost function c_i thus captures the situation of a player that suddenly understands 'everything' after having performed a few rounds of reasoning (four in this case). From the figure it is also easy to see that if we lower c_i , holding v_i constant, then $\hat{k}_i$ (weakly) increases (cf. point 2 in Remark 1).*

²¹The results we present here, however, would not change if $\mathcal{K}(\cdot)$ were defined as the optimal solution resulting from a cost-benefit analysis. We maintain the 'myopic' formulation because we find it more natural in the context of level- k reasoning.

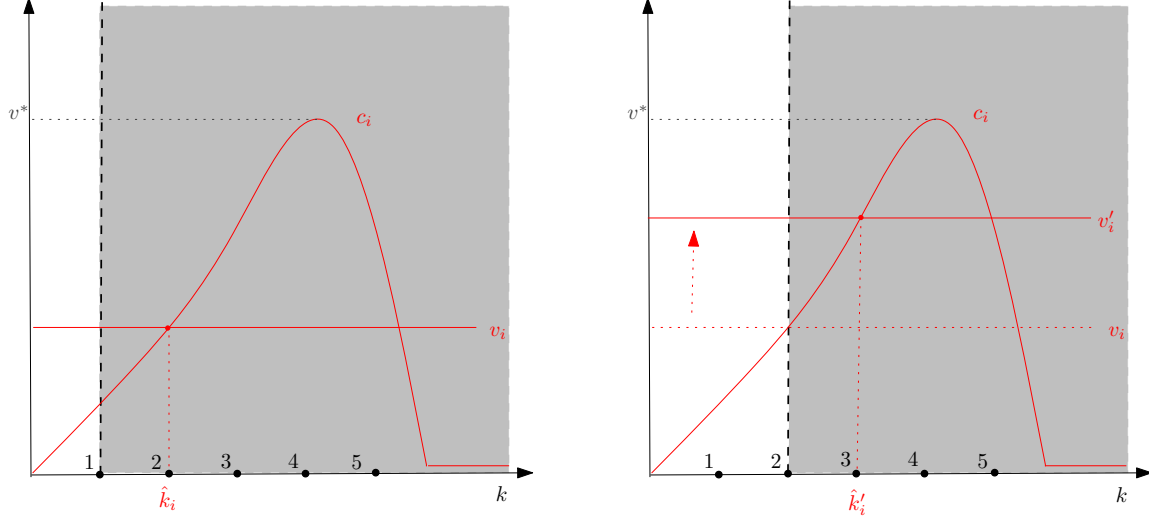


Figure 1: As v_i increases, $\hat{k}_i$ (weakly) increases.

Note that the cost-benefit analysis conducted by the agent and the ensuing bound $\hat{k}_i$ are independent of the player's opponent. In this respect, the bound $\hat{k}_i$ can be seen to be determined before the game is played and before the beliefs about the opponent's sophistication are formed. But while we view player i 's understanding of the game as distinct from his beliefs over others, his choice of action does depend on these beliefs. We now characterize the relationship between players' beliefs over their opponents' understanding of the game and their chosen action.

3.2.2 Reasoning about the opponents.

Before choosing an action, players take into account the sophistication of their opponents. Players' beliefs about their opponents' sophistication, however, need not be correct, as we do not seek for an equilibrium concept and correctness of beliefs is not guaranteed by introspection alone.

A key conceptual difficulty in level- k models is the notion that a player may be playing someone who he envisions as more sophisticated than him. Generally, level- k models assume that this cannot be the case. Our model overcomes this limitation and gives meaning to the idea of playing a more sophisticated opponent.

Since a player's reasoning ability in this model is captured by the cost function, we use the same tool to model players' beliefs over others' sophistication. We specifically define sophistication in the following manner:

Definition 2 Consider two cost functions, c' and c'' . We say that cost function c' corresponds to a 'more sophisticated' player than c'' , if $c'(k) \leq c''(k)$ for every k .

The reason we do not define sophistication directly in terms of the cognitive bounds $\hat{k}_i$ and $\hat{k}_j$ is that these bounds are a consequence of the cost-benefit analysis, which is determined endogenously by the game. When payoffs are different for the two players, it may be that $\hat{k}_i < \hat{k}_j$, even if i is more sophisticated than j , in the sense of definition 2. For instance, this may be the case if, despite his higher cognitive abilities, player i has lower incentives than player j .

Based on definition 2, we can separate the space of cost functions as follows. For any $c_i \in \mathbb{R}_+^N$, let

$$C^+(c_i) = \{c \in \mathbb{R}_+^N : c_i(k) \geq c(k) \text{ for every } k\} \text{ and}$$

$$C^-(c_i) = \{c \in \mathbb{R}_+^N : c_i(k) \leq c(k) \text{ for every } k\}.$$

Thus, $C^+(c_i)$ and $C^-(c_i)$ are comprised of the cost functions that are respectively ‘more’ and ‘less’ sophisticated than c_i .

Definition 3 *Let c_j^i denote player i ’s beliefs over j ’s cost function. Then, player i ‘believes that his opponent is more (resp., less) sophisticated than himself’, if and only if $c_j^i \in C^+(c_i)$ (resp., $c_j^i \in C^-(c_i)$).*

When choosing his action, player i considers the cost-benefit analysis of his opponent. Appealing to equation (1), we first define $\bar{k}_j^i$ in the following manner:

$$\bar{k}_j^i = \mathcal{K}(c_j^i, v_j). \quad (3)$$

Given player i ’s beliefs c_j^i and given v_i and v_j , level $\bar{k}_j^i$ would represent player i ’s view of his opponent’s understanding of the game, *if* player i had performed at least $\bar{k}_j^i + 1$ rounds of reasoning (i.e., $\hat{k}_i \geq \bar{k}_j^i + 1$). But if $\hat{k}_i \leq \bar{k}_j^i$, then player i ’s cognitive bound cannot accommodate the full depth of player j ’s reasoning. In that case, player i would believe that the opponent’s cognitive bound is $\hat{k}_i - 1$, which is the maximum that player i can ‘conceive’, given $\hat{k}_i$. Hence, we define player i ’s belief about player j ’s bound, k_j^i , as

$$\hat{k}_j^i = \min \left\{ \hat{k}_i - 1, \bar{k}_j^i \right\}. \quad (4)$$

Equations (3) and (4) capture the idea that player i considers the cost-benefit analysis of player j using c_j^i and v_j , but only in the region uncovered by his own reasoning, that is up to level $\hat{k}_i - 1$. Thus, i ’s beliefs c_j^i actually represent the *path* of i ’s beliefs about the opponent’s reasoning, as i

uncovers more and more steps of reasoning.²²

We are now in position to specify the behavior of player i . Given his beliefs about his opponent's bound, $\hat{k}_j^i$, we assume that player i believes that the opponent plays according to level $k_j^i = \hat{k}_j^i$. Hence, player i plays the action $a_i^{k_i}$ associated with the 'behavioral level', k_i , defined as

$$k_i = k_j^i + 1. \quad (5)$$

Hence, whenever $\hat{k}_i \leq \bar{k}_j^i$, player i plays according to his own cognitive bound: $k_i = \hat{k}_i$. If instead player i believes that the opponent has performed *less* rounds of introspection ($\hat{k}_i > \bar{k}_j^i$), then player i assumes that the opponent would play according to *his* maximal bound (as perceived by player i), $k_j^i = \hat{k}_j^i$, so that i best responds by playing action $a_i^{k_j^i+1}$.

In brief, there are four k 's that are determined endogenously. Player i 's *cognitive bound*, $\hat{k}_i$, arises from his cost-benefit analysis, and represents his understanding of the game. Player i 's *beliefs about player j 's understanding of the game*, $\hat{k}_j^i$, derive from i 's beliefs about the opponent's cost function (c_j^i) and incentives (v_j), given i 's understanding of the game ($\hat{k}_i$). Beliefs $\hat{k}_j^i$ in turn determine i 's beliefs about j 's behavior, k_j^i . Finally, the '*behavioral level*' of player i , k_i , follows from k_j^i and determines the action chosen in the game, $a_i^{k_i}$.

Example 2 *To illustrate the workings of our model, consider the example in Figure 2. This figure represents an agent with cost function c_i which, given v_i , induces a cognitive bound of $\hat{k}_i = 3$. Player i has thus uncovered only two levels of reasoning about the opponent. In Figure 2.a, player i reasons about an opponent that he regards as less sophisticated. Since the intersection between c_j^i and v_j (which determines $\bar{k}_j^i = 1$) falls in the region already uncovered by i 's cognitive bound, then i 's belief about i 's cognitive bound is in fact $\hat{k}_j^i = 1$, which also represents i 's belief about j 's behavior, k_j^i . Hence, player i best responds playing the action associated with level $k_i = k_j^i + 1 = 2$. Notice that, in this case, the cognitive bound $\hat{k}_i$ is not binding: $k_i < \hat{k}_i$. Figure 2.b instead represents the same player reasoning about someone he regards as more sophisticated. In this case, the intersection between c_j^i and v_j (i.e. $\bar{k}_j^i$) falls in the 'unawareness area' of player i . Since player i 's has only uncovered two rounds of reasoning of his opponent, his 'perceived' cognitive bound for player j is not $\bar{k}_j^i$, but $\hat{k}_j^i = \hat{k}_i - 1$. Hence, player i best responds playing according to level $k_i = \hat{k}_j^i + 1$, that is according to his own cognitive bound $\hat{k}_i$.*

For later reference, we summarize some implications of our model, which will translate to

²²Hence, rather than 'actual' beliefs, functions c_j^i incorporate potentially counterfactual beliefs, conditional on subjunctive propositions about i 's own depth of reasoning.

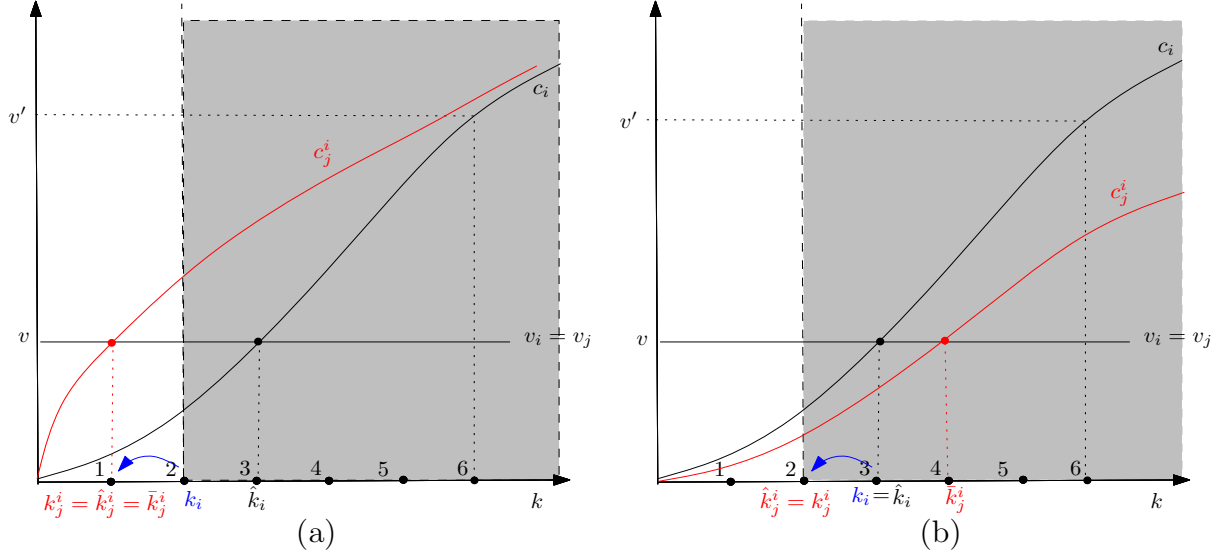


Figure 2: Reasoning about the opponents: on the left, $c_j^i \in C^-(c_i)$; on the right, $c_j^i \in C^+(c_i)$.

predictions in the experiment conducted.

Remark 2

1. If $v_i = v_j$ and if $c_j^i \in C^+(c_i)$, then $k_i = \hat{k}_i$: with symmetric incremental benefits, the cognitive bound $\hat{k}_i$ is always binding for a player who believes he is playing against a more sophisticated opponent. The cognitive bound $\hat{k}_i$ thus coincides with the behavioral k_i .
2. If $v_i = v_j$ and if $c_j^i \in C^-(c_i)$, then $k_i \leq \hat{k}_i$. Together with the previous point, this implies that in the modified 11-20 game, players play (weakly) higher actions against opponents that they regard as less sophisticated ($c_j^i \in C^-(c_i)$) than against ones they regard as more sophisticated ($c_j^i \in C^+(c_i)$).
3. If $v_i = v_j$ and the functions are increased (preserving the symmetry), both k_i and $\hat{k}_i$ (weakly) increase. Hence, in the modified 11-20 game, players play lower actions.

3.2.3 Higher Order Reasoning

The analysis above is based on the implicit assumption that players believe their opponents have correct beliefs over their own cost function. This assumption is used to conclude that, if player i believes that the opponent performed less rounds of introspection, than he would play according to his own cognitive bound. The model however can be extended to accommodate higher order uncertainty, allowing players to view their opponents' beliefs to be incorrect. This extension is

relevant for the analysis of treatments $[C]$ and $[C+]$, relative to $[B]$ and $[B+]$, in the experiment. Specifically, let c_i^{ij} denote i 's beliefs about j 's beliefs about i 's cost function. The model discussed so far implicitly assumes that $c_i^{ij} = c_i$. This, however, need not be the case if player i 's opponent knows that he does not play against player i , but against someone else. In treatment $[C]$ for instance, a player faces the action of an opponent playing someone else. In that case, we need to model i 's beliefs about j 's opponent explicitly.

Let the triple (c_i, c_j^i, c_i^{ij}) represent player i 's cost c_i , his beliefs over his opponents cost c_j^i , and his beliefs over his opponent's beliefs about his own cost, c_i^{ij} . We then extend the model by having player i take into account that player j may play at a lower k than his cognitive bound. In particular, while $\hat{k}_j^i$, as defined in equation (4), still defines player i 's perception of player j 's cognitive bound, it does not define his perception of player j 's actual play. Instead, player i considers player j 's reasoning about i using the pair (c_j^i, c_i^{ij}) . That is, he puts himself in player j 's 'shoes' and calculates j 's opponent's bound from his standpoint.²³ Formally, player i views player j 's perception of player i 's limit to be

$$\hat{k}_i^{ij} = \min\{\bar{k}_i^{ij}, \hat{k}_j^i - 1\}, \quad (6)$$

where $\bar{k}_i^{ij} = \mathcal{K}(c_i^{ij}, v_i)$. If player i is capable of conceiving such a level, that is, if $\hat{k}_i^{ij} + 1 \leq \hat{k}_i - 1$, then he expects j to play according to level $\hat{k}_i^{ij} + 1$. Hence, we define i 's perception of j 's play as

$$k_j^i = \min\{\hat{k}_i^{ij} + 1, \hat{k}_i - 1\}. \quad (7)$$

Player i then best responds playing action $a_i^{k_i}$, where $k_i = k_j^i + 1$.²⁴

Example 3 Figure 3 represents a player with cost function c_i reasoning about an opponent that he regards as less sophisticated. On the left, we represent a situation in which i believes that j thinks that i is even less sophisticated (that is, $c_i^{ij} \in C^-(c_j^i)$). The dark grey area represents the 'unawareness region' of player i , whose cognitive bound is $\hat{k}_i = 5$. Given beliefs c_j^i , i 's perception of j 's cognitive bound is $\hat{k}_j^i = 3$ (which is equal to $\bar{k}_j^i$ because the latter falls in the area already uncovered by i). This bound, however, does not coincide necessarily with k_j^i . To identify that, player

²³More precisely, he uses the triple $(c_j^i, c_i^{ij}, c_j^{iji})$ for his opponent, where $c_j^{iji} = c_j^i$. Hence, the implicit assumption is that beliefs of order higher than 2 correspond to common certainty of c_j^i .

²⁴Note that setting $c_i^{ij} = c_i$, we obtain the case of the previous subsection, where $k_j^i = \hat{k}_j^i$. This is so because, in that case, $\bar{k}_i^{ij} = \hat{k}_i$, hence (6) delivers $\hat{k}_i^{ij} = \hat{k}_j^i - 1$. By definition of $\hat{k}_j^i$ (4), $\hat{k}_j^i - 1 < \hat{k}_i - 1$, hence by (7), $k_j^i = \hat{k}_i^{ij} + 1 = \hat{k}_j^i$.

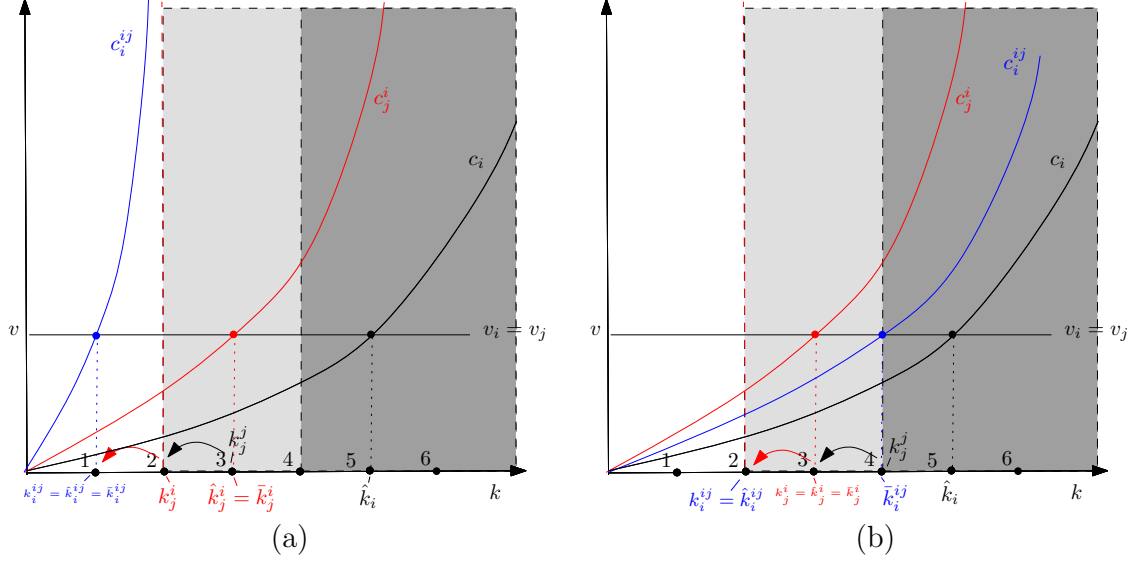


Figure 3: Higher Order Reasoning: $c_j^i \in C^-(c_i)$, with $c_i^{ij} \in C^-(c_j^i)$ on the left; on the right $c_i^{ij} \in C^+(c_j^i)$.

i must consider (his beliefs about) j 's reasoning about i , which is determined by the intersection between c_i^{ij} and v_i , that is $\bar{k}_i^{ij} = 1$. Since this falls in the region already uncovered by the light grey area (which represents the unawareness region of j , as perceived by i), then $\hat{k}_i^{ij} = k_i^{ij} = 1$, hence j 's best response is to play according to $k_j^i = 2$, which is less than $\hat{k}_j^i$. Given this, i 's best response requires playing according to $k_i = 3$. In figure 3.b, $c_i^{ij} \in C^+(c_j^i)$, and therefore $\bar{k}_i^{ij}$ falls in the unawareness region of player j (the light grey area). It follows that $\hat{k}_i^{ij} = \hat{k}_j^i - 1$. In this case, player i expects j to play at his maximum bound, $k_j^i = \hat{k}_j^i = 3$.²⁵ The best response is thus to play according to $k_i = 4$.

3.3 Theoretical Predictions for the Experiment

Recall that we use the terminology 'label I ' (resp., 'label II ') to refer indiscriminately to the 'high score' ('low score') subjects in the endogenous classification, or to the 'math and sciences' ('humanities') subjects for the exogenous. Accordingly, we introduce notation $l_i = \{I, II\}$ to refer to player i 's label. We describe next how we apply our theory to the treatments of the experiment.

We assume that the value of reasoning, v_i , is determined by player i 's payoffs, and does not depend on the players' labels. Hence, it remains constant throughout treatments $[A], [B], [C]$, as well as in treatments $[A+], [B+], [C+]$. We assume, however, that the value of reasoning increases

²⁵This example shows that the result from the previous subsection that $k_j^i = \hat{k}_j^i$ if $c_j^i \in C^-(c_i)$, requires only that $c_i^{ij} \in C^+(c_j^i)$; it is not necessary that $c_i^{ij} = c_i$.

when the extra reward for choosing a number exactly one below the opponent's is increased, and therefore v_i does vary when comparing treatments $[X]$ to $[X+]$ for $X \in \{A, B, C\}$:

Condition 2 For each $X = A, B, C$, $v_i^{[X+]}(k) \geq v_i^{[X]}(k)$ for all k .²⁶

We also assume that an agent's cognitive ability, c_i , remains constant throughout all treatments. His beliefs, c_j^i and c_i^{ij} , however, change with his opponent's label, although not with the game payoffs. Hence, for $X = A, B, C$, $c_j^{i,[X]} = c_j^{i,[X+]}$ and $c_i^{ij,[X]} = c_i^{ij,[X+]}$.

To apply our model to the experiment we posit that whenever a subject plays against a label I opponent (resp., label II), then he believes that the opponent is more (less) sophisticated than himself. For higher order beliefs, we assume that $c_i^{ij,[X]} = c_i$ for $X = A, B$. In treatment $[C]$, we apply to j 's beliefs the same logic we applied to i 's first order beliefs: if player j 's opponent has label I (resp., II), then $c_j^{ij} \in C^+(c_j^i)$ (resp., $c_j^{ij} \in C^-(c_j^i)$). Hence, $c_i^{ij,[C]} \in C^-(c_j^i)$ if $l_i = I$ and $c_i^{ij,[C]} \in C^+(c_j^i)$ if $l_i = II$. Define first a player's *type* to be:

$$t_i = (c_i, c_j^{i,I}, c_j^{i,II}, c_i^{ij,[C]}),$$

where c_i is his cost function; $c_j^{i,I}$ are his beliefs when he plays against a label I subject; $c_j^{i,II}$ his beliefs when he plays against a label II subject; $c_i^{ij,[C]}$ are his higher order beliefs in treatments $[C]$ and $[C+]$.

Define the sets T^* , T_I^* and T_{II}^* as follows:

$$T^* = \left\{ t_i \in (\mathbb{R}_+^N)^4 : c_j^{i,I} \in C^+(c_i) \text{ and } c_j^{i,II} \in C^-(c_i) \right\},$$

$$T_I^* = \left\{ t_i \in T^* : c_i^{ij,[C]} \in C^-(c_j^{i,II}) \right\}, \quad (8)$$

$$\text{and } T_{II}^* = \left\{ t_i \in T^* : c_i^{ij,[C]} \in C^+(c_j^{i,I}) \right\}. \quad (9)$$

The assumptions discussed above imply the following conditions:

Condition 3 If i is a subject with label l_i , then $t_i \in T_{l_i}^*$, where $l_i \in \{I, II\}$.

The next proposition summarizes the implications of the model under these assumptions.

Proposition 1 Under Conditions 1, 2 and 3, the following holds: (1) For any player i and for any $X = A, B, C$, $a_i^{[X+]} \leq a_i^{[X]}$; (2) If $l_i = I$, then $a_i^{[A]} \leq a_i^{[B]} \leq a_i^{[C]}$ and $a_i^{[A+]} \leq a_i^{[B+]} \leq a_i^{[C+]}$; if $l_i = II$, then $a_i^{[A]} \geq a_i^{[B]} \geq a_i^{[C]}$ and $a_i^{[A+]} \geq a_i^{[B+]} = a_i^{[C+]}$.

²⁶This assumption is consistent with the axiomatization proposed in Alaoui and Penta (2012).

Labels	Changing beliefs (low payoffs)	Changing beliefs (high payoffs)	Changing payoffs
$l_i = I$	$F_C \succsim F_B \succsim F_A$	$F_{C+} \succsim F_{B+} \succsim F_{A+}$	$F_X \succsim F_{X+}$ for $X = A, B, C$
$l_i = II$	$F_B \succsim F_A; F_B \approx F_C$	$F_{A+} \succsim F_{B+}; F_{B+} \approx F_{C+}$	$F_X \succsim F_{X+}$ for $X = A, B, C$

Table 2: Summary of distribution results across treatments

Applying proposition 1 leads to unambiguous first order stochastic relations between the distributions of actions in the different treatments.²⁷ Specifically, suppose that the subjects in our experiment of label I are independently drawn from a distribution G_I over T_I^* , and subjects from label II are independently drawn from a distribution G_{II} over T_{II}^* . Let F_X^l denote the cumulative distribution of actions $a \in \{11, \dots, 20\}$ in treatment X for label $l \in \{I, II\}$. Then, letting $\succsim$ denote the first order stochastic dominance relation, it is immediate from Proposition 1 that $F_C^I \succsim F_B^I \succsim F_A^I$, and that $F_A^{II} \succsim F_B^{II}$ and $F_A^{II} \succsim F_C^{II}$. Distributions F_B^{II} and F_C^{II} , however, should coincide. Intuitively, a label I subject plays according to a (weakly) lower action (higher k) in treatment $[B]$ than in $[A]$, as he believes that he is playing a less sophisticated player. He plays an even lower action against a less sophisticated opponent who himself believes he is playing a less sophisticated player. Similarly, a label II subject plays according to a higher action in treatment $[B]$ than $[A]$. The same results hold when comparing treatments $[A+]$, $[B+]$ and $[C+]$. Lastly, for all $X = A, B, C$ and $l = I, II$, we have that $F_X^l \succsim F_{X+}^l$: as payoffs are increased, players perform more rounds of reasoning (and they expect their opponents to do so), hence they play lower actions.

These results, summarized in Table 2, remain true under several variations of the assumptions entailed by the definition of the sets T_I^* and T_{II}^* . For instance, the results hold if we allow subjects to be ‘misabeled’, in the sense that label I (II) subjects may believe they are less (more) sophisticated than label II (I) subjects, provided that they kept the belief that label I subjects are known to be more sophisticated than label II subjects. The framework can also be extended to allow for a stochastic component to the reasoning process, which would introduce an element of noise to the stochastic dominance properties described above.

3.4 Related Models

The main feature of our framework consists of modeling the cognitive bound as a consequence of a tradeoff between value of reasoning and costs of cognition. The general notion that players follow

²⁷Given two cumulative distributions $F(x)$ and $G(x)$, we say that F (weakly) first order stochastically dominates G if $F(x) \leq G(x)$ for every x .

a cost-benefit analysis is present in the language of Camerer, Ho and Chong (2002), but not in the cognitive hierarchy (CH) model itself, as players’ cognitive types remain exogenous. A recent paper by Choi (2012) extends the CH model by letting cognitive types result from an optimal choice. This optimization serves to provide identification restrictions to estimate the distribution of types across different environments, and is motivated through an evolutionary argument. In contrast, our goal is to provide an explicit model of reasoning, in which players think about both the game and about the reasoning process of their opponents. The objectives and modeling choices are therefore distant.

Gabaix (2012) also proposes a framework in which the accuracy of players’ beliefs about the opponents’ behavior is determined by a cost-benefit analysis. At a conceptual level, the main goal of Gabaix (2012) is to provide an equilibrium concept that allows players to have both incorrect beliefs as well as to respond non-optimally. Our model focuses on agents’ cognitive limitations in reasoning about the opponents, and ignores the orthogonal issue of limitations in computing best responses. Furthermore, unlike Gabaix (2012) and CH models of Camerer, Ho and Chong (2002) and Choi (2012), our framework does not impose equilibrium-like restrictions that relate agents’ beliefs to the actual distribution of actions or types. Since our model is purely one of introspection, we avoid conditions that would not be a consequence of players’ reasoning process alone. This feature of our model is shared by the level- k models of Nagel (1995) and Crawford and Iriberri (2007).²⁸

4 Experimental results

We present, for brevity, only the experimental results for the grouped exogenous and endogenous classifications.²⁹ We group the label *I* subjects (‘math and sciences’ for exogenous treatments and ‘high’ for endogenous treatments), and we group the label *II* subjects (‘humanities’ for exogenous treatments and ‘low’ for endogenous treatments). Recall that we do not take these labels to be an indication of actual game theoretical sophistication, but of *perceived* sophistication by the subjects. Moreover, we present the results by grouping together the treatments when they are repeated. For these repetitions, our groupings appear justified by tests for equality of distribution.³⁰ We analyze

²⁸The ‘noisy introspection’ model of Goeree and Holt (1999) introduces noisy responses in a non-equilibrium solution concept.

²⁹The figures for the separate classifications are consistent with the results for the grouped classifications. They are available upon request, as are other supplementary materials.

³⁰These equality of distribution tests also provide a strong indication that ‘feedback-free’ learning is not an important factor in this experiment (see Appendix C, Table 3). As an additional control, the order of treatments was also randomized (see Appendix A.2 for details).

first the results when subjects' payoffs are changed, followed by the results when their beliefs over opponents are varied. Lastly, while the stochastic dominance results present a sharper test of our theoretical predictions, the sign of the coefficients for the relevant regressions may also serve as an additional test of the model. We therefore present results of these regressions as well, and note that these statistics appear to confirm the theory, as most of the relevant ones are significant with the expected sign of the coefficient.³¹ This lends support to the theory, and to the general claim that beliefs and incentives affect the individuals' depth of reasoning and their behavior.

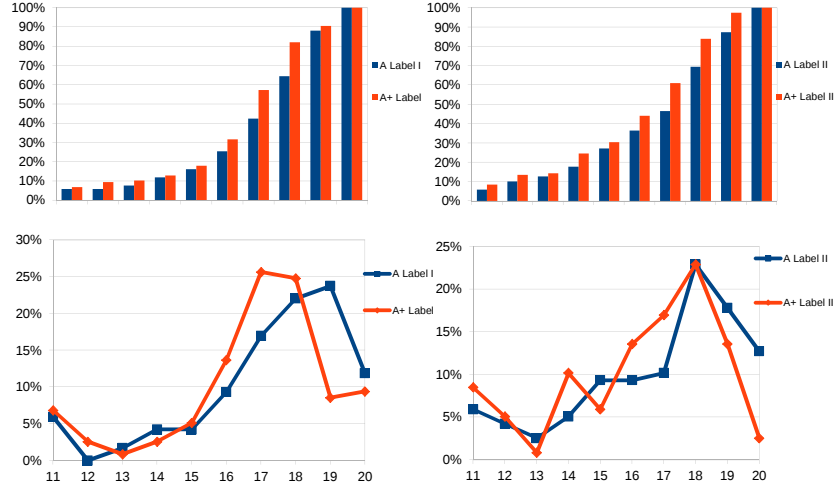
In what follows, the first order stochastic dominance tests use the method introduced by Davidson and Duclos (2000) (henceforth, DD). For the (random-effects) ordinary least squares estimations (OLS), we regress, for each label, the outcome on a dummy for the treatments, and another for the classification (endogenous or exogenous). The latter is never significant.

To control for 'feedback-free' learning, we exploit two factors. First, we use randomization of treatments, particularly within [A], [B] and [C], and within [A+], [B+] and [C+]. Second, we exploit the repetition of treatments to do equality of distribution tests for the same treatment. For instance, each agent plays treatment [A] twice with other treatments in between, and we test for equality of distribution between the first and the second time treatment [A] occurs. The results of Mann-Whitney-Wilcoxon (rank sum) tests and Kolmogorov-Smirnov tests are highly suggestive that the distributions are equal. As an added robustness check in comparing low to high payoffs in the OLS estimations, we also control for the round at which they are played, and find that it is never significant.

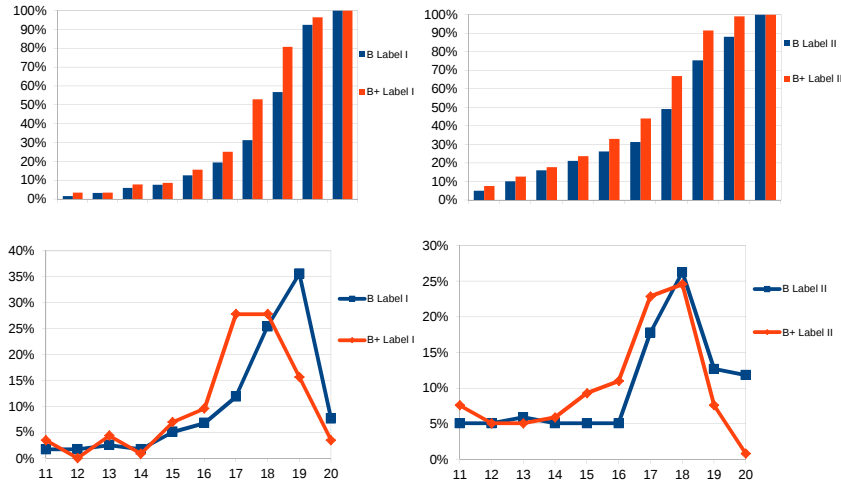
4.1 Changing incentives

As the value of introspection increases for players and their opponents, the model predicts that they would choose actions associated with higher k 's. Specifically, comparing treatments across different marginal values of payoffs, $F_A \succeq F_{A+}$, $F_B \succeq F_{B+}$ and $F_C \succeq F_{C+}$. These implications hold for both label *I* and label *II* subjects. Beginning with label *I*, it is clear from Figure 4.a (left) that the empirical distribution [A] stochastically dominates [A+] everywhere. Distribution [B] stochastically dominates [B+] everywhere, and [C] clearly stochastically dominates [C+] everywhere (Figures 4.b and 4.c). Using a DD test for each comparison, we find that these results are indeed consistent with first order stochastic dominance. These results therefore seem to follow the predictions of the model. Conducting an OLS regression, we find that the coefficients are highly significant ($< 1\%$) for distributions [A] compared to [A+], [B] to [B+] and [C] to [C+], and of the correct sign.

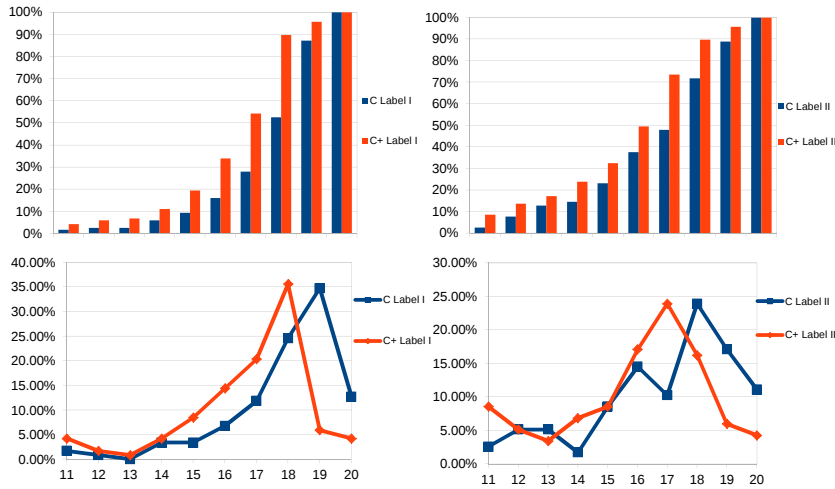
³¹See Table 4 in Appendix C. All regressions and statistical tests are in Appendix C.



(a) : A and A+



(b) : B and B+



(c) : C and C+

Figure 4: Changing Payoffs, label *I* (left) and label *II* (right)

The Mann-Whitney-Wilcoxon tests and Kolmogorov-Smirnov tests are both significant ($< 5\%$) or highly significant ($< 1\%$) for all of these comparisons of distribution as well.

These findings are consistent with the model, and with the view that agents perform more rounds of reasoning if the incentives are increased. These results also indicate that changing from an extra 20 tokens to an extra 80 tokens determines a large enough shift in the value function that it leads agents to increase their level of reasoning.³² The graphs in Figure 4 are suggestive of the idea that there is a shift in the distribution. We note as well that level-1 and level-2 play is modal for treatments [A], [B] and [C], while level-2 and level-3 play is modal for [A+], [B+] and [C+]. The means of the distributions change from 17.3, 17.7 and 18.0 for treatments [A], [B] and [C], respectively, to 16.8, 17.1 and 16.9 for treatments [A+], [B+] and [C+], respectively.

For label *II*, the stochastic dominance relationships hold everywhere for all three comparisons, [A] to [A+], [B] to [B+] and [C] to [C+], as shown in Figure 4 (right). The DD tests also confirm that these results are consistent with stochastic dominance, and the coefficients from the OLS regression are significant ($< 5\%$) for distributions [A] compared to [A+] and highly significant ($< 1\%$) for distributions [B] compared to [B+] and [C] to [C+]. These coefficients are of the correct sign. The Mann-Whitney-Wilcoxon test is significant ($< 5\%$) or highly significant ($< 1\%$) for all of these comparisons, and the Kolmogorov-Smirnov test is significant ($< 5\%$) for [B] compared to [B+] and highly significant ($< 1\%$) for [C] compared to [C+]. As with label *I*, the figures for label *II* are suggestive of a shift in the distribution. Here as well level-1 and level-2 play is modal for treatments [A] and [C] (level-2 and level-3 play is modal for [B]), while level-2 and level-3 play is modal for [A+], [B+] and [C+]. The means of the distributions change from 16.9, 16.8 and 16.9 for treatments [A], [B] and [C], respectively, to 15.6, 15.6 and 15.5 for treatments [A+], [B+] and [C+], respectively.

4.2 Changing beliefs about the opponents

Consider the comparison between homogeneous treatment [A], heterogeneous treatment [B] and replacement treatment [C]. According to the theoretical model, $F_C \succsim F_B \succsim F_A$ for label *I* players.

From Figure 5, this result seems consistent with the data. Distribution [C] clearly stochastically dominates [B] everywhere, and [B] stochastically dominates [A] nearly everywhere.³³ We also note

³²See the discussion in Section 5.2 on disentangling $\hat{k}_i$ from k_j^i .

³³The only exception is at action 19, which is consistent with the well-known observation that stochastic dominance relations are often violated near the endpoints, even when the true distributions are stochastically-dominance ranked.

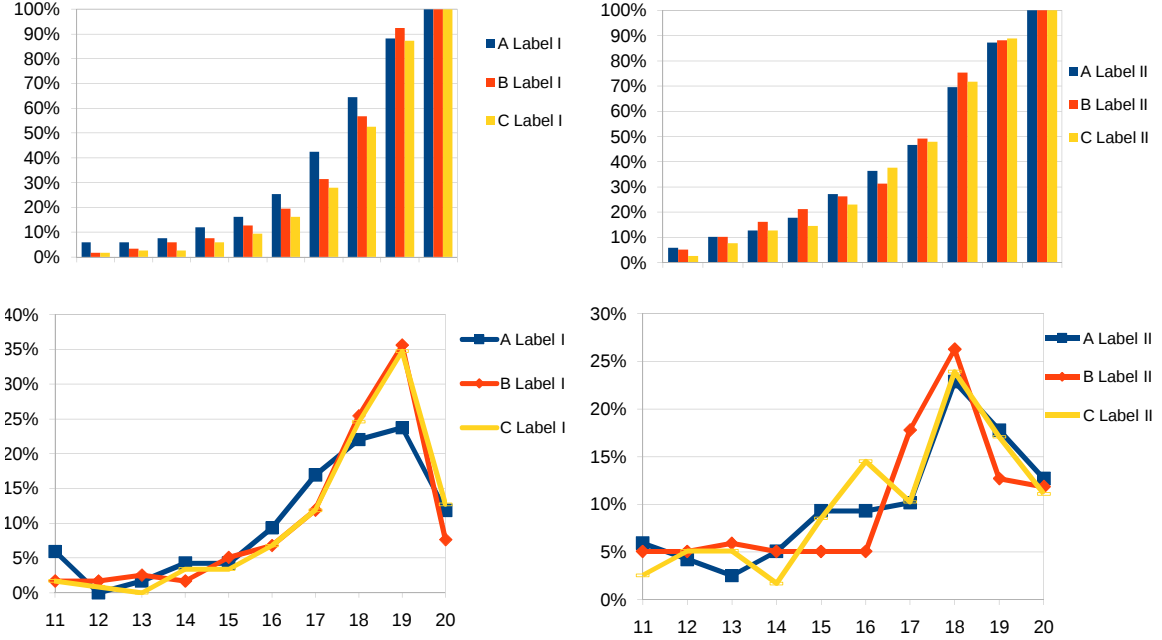


Figure 5: Treatments [A], [B] and [C] for label *I* (left) and *II* (right)

that [C] clearly stochastically dominates [A] everywhere. Comparing F_C to F_B , F_B to F_C and F_A to F_C using the DD test, we find that our results are indeed consistent with stochastic dominance.

Note that the theory model does not predict *strict* stochastic dominance relations $F_C \succ F_B$ or $F_B \succ F_A$. That is, it allows for individuals' beliefs over their relative sophistication to be such that they would play the same against lower or higher levels of sophistication. The distinct pattern that emerges from Figure 5 indicates that label *I* individuals view the cost function associated with label *II* as sufficiently far from their own to induce a difference in their chosen action. We also note that this pattern could not be explained by existing models of level- k reasoning, as they do not endogenize that level of play may vary even if the structure of the game remains constant.

The OLS estimates comparing [A] to [B] are significant at $< 10\%$, and the estimates comparing [A] to [C] are highly significant ($< 1\%$). The estimates comparing [B] to [C], however, are not significant. Figure 5 reveals that distributions [B] and [C] remain very close to each other, and so the lack of significance is not surprising.

Turning next to label *II* players, the model predicts $F_A \lesssim F_B \approx F_C$. Here, no clear difference emerges from Figure 5 between the three cumulative distributions. Conducting Mann-Whitney-Wilcoxon and Kolmogorov-Smirnov equality of distribution tests confirms the visual intuition, and the OLS estimates are not significant for any of the comparisons of [A] to [B], [B] to [C] or [A] to

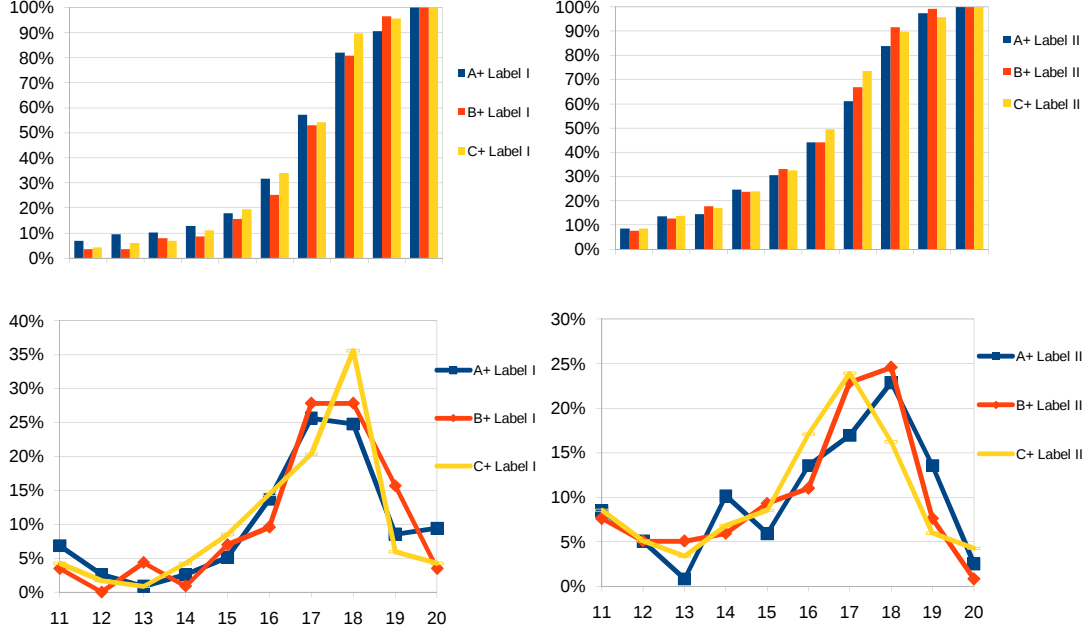


Figure 6: Treatments [A+], [B+] and [C+] for label *I* (left) and *II* (right)

[C]. While $F_B \approx F_C$ is the exact prediction of the theory model, the result that $F_A \approx F_B$ indicates that label *II* subjects do not view the sophistication of other label *II* subjects as significantly lower than their own, and therefore do not adjust their level of play in a measurable way. This result can therefore serve as a first step towards identifying subjects' beliefs over their opponents.

We now compare the high payoff treatments [A+], [B+] and [C+], in which an individual whose action is exactly one below his opponent's receives an additional 80 tokens rather than 20. The model makes analogous predictions in these cases as it does in treatments [A], [B] and [C], namely that $F_{C+} \succsim F_{B+} \succsim F_{A+}$ for label *I* and $F_{C+} \succsim F_{B+} \approx F_{A+}$ for label *II*. From Figure 6, no discernible pattern emerges either for label *I* or for label *II*, and we note that the (frequency) distributions are close to each other. Both Mann-Whitney-Wilcoxon and Kolmogorov-Smirnov tests for equality of distribution are consistent with this reading. None of the OLS estimates for the comparison of [A+] and [B+] or [A+] and [C+] are significant, for either label *I* or label *II*. The OLS estimates of [B+] and [C+] for label *II* are not significant either, although we note that they are significant from [B+] to [C+] ($< 5\%$) for label *I* in the other direction. Given the closeness of the distribution as confirmed by the equality of distribution tests, this significance does not seem to be of first order.

The last results comparing treatments [A+], [B+] and [C+], viewed together with the results

for treatments [A], [B] and [C], are indicative of label I subjects' beliefs. Specifically, these results suggest that label I subjects believe that the cost functions associated with label II subjects are higher than their own at low levels of k , but become closer to their own cost function at higher k 's. In other words, label I subjects believe that, when sufficiently motivated, label II subjects are essentially the same as label I . An example of cost functions that satisfy this property is provided in Figure 2.a in Section 3. While the present analysis does not allow for precise identification of subjects' cost function, an extension of this approach could potentially be used for this purpose.

5 Additional Treatments

We now describe additional treatments conducted on a subset of the subjects, with the aim of exploring less immediate predictions of our theory as well as possible directions for future research. These designs are more complex and cognitively demanding for the subjects, and some rely on our theory more than they directly test it. Overall, the results are encouraging, especially in light of these factors. The smaller sample sizes of these treatments, however, suggest caution in interpreting these findings as conclusive.

5.1 Identifying Beliefs

The theoretical model of Section 3 offers a clear distinction between a player's cognitive bound, $\hat{k}_i$, and his behavioral k_i , which is determined by his beliefs about the opponent, k_j^i . Recall that we have made the simplifying assumption that subjects view label I opponents to be more sophisticated than they are themselves. As previously mentioned, this condition is not necessary for our theoretical predictions. Here, we present a test of whether players overall do indeed play according to their bound $\hat{k}$ when playing label I opponents. As we demonstrate below, the evidence seems consistent with this assumption.

Specifically, we aim to test whether in treatments [A] and [B] subjects play according to their own cognitive bound or whether they are responding to their beliefs about the opponents' cognitive bound. In the context of our model, we could achieve this if we had a mechanism for setting the players' own costs to zero while holding their beliefs about their opponents constant. For instance, in the example of Figure 2 (p. 21), if we change the cost function of player i so that $c_i(k) = 0$ for every k , then $k_j^i = \bar{k}_j^i$, whereas $k_j^i < \bar{k}_j^i$ if the cognitive bound is binding. Our treatments [D], [E] and [F] are designed precisely to operationalize this thought experiment.

After having administered the main treatments, we expose subjects from four of the six sessions

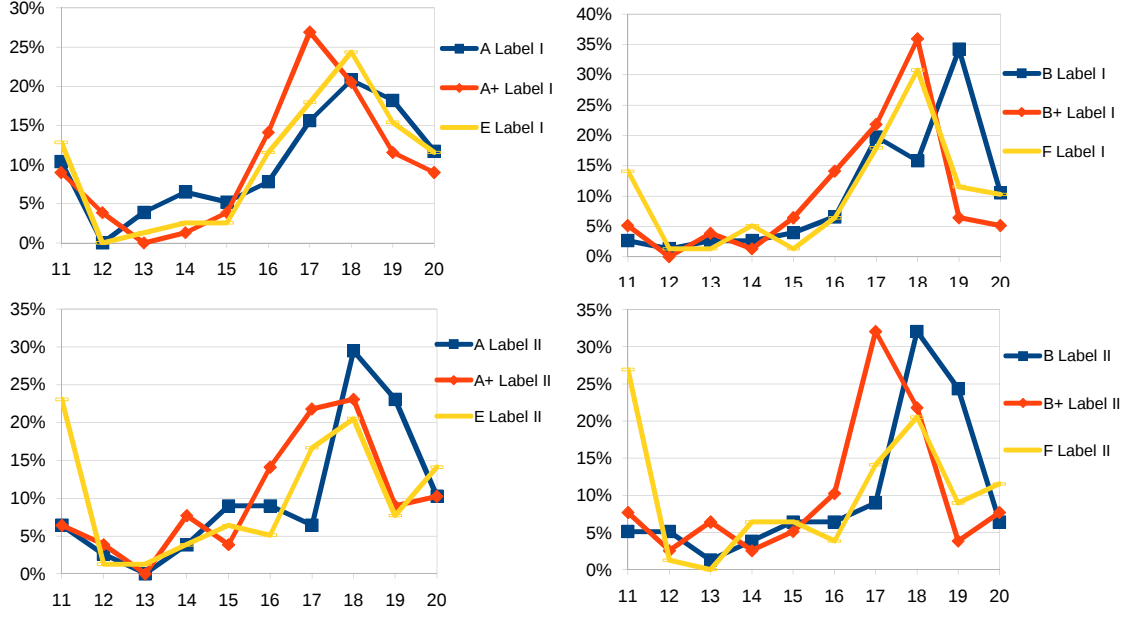


Figure 7: Treatments [E] and [F]

(two for the endogenous and two for the exogenous classifications) to a ‘game theory tutorial’, in which we explain how, through the chain of best replies, ‘infinitely sophisticated and rational players’ would play (11, 11) (see Appendix A.1.4 for details). We interpret offering this explanation as setting the subjects’ cognitive costs to zero. We then proceed with the new treatments. In treatment [D], we instruct each subject to play the baseline game (with low payoffs) against another subject who has also been given the same tutorial. Not surprisingly, a high fraction of the subjects (48% of label *I* and 55% of label *II*) announce 11. In treatment [E], we instruct the subjects who had previously been given the tutorial to play the baseline game against a player of the same label who had *not* been given the tutorial (that is, as in the homogenous treatment [A]). Analogously, treatment [F] contains the same instructions but with the heterogenous treatment (as in [B]). Hence, in treatments [E] and [A] (and in [F] and [B]), subjects face essentially the same opponents, but in treatments [E] and [F] their cost function has been ‘turned off’. If their cognitive bound are *not* binding in [A] and [B], then the distributions of actions in [E] and [F] should be the same as in [A] and [B], respectively. Figure 7 displays the results for the two labels. Comparing [A] to [E] and [B] to [F], we observe that (with different degrees across labels and treatments) the distributions of actions shift to the left. Through the lens of our model, the data suggest that the cognitive bound for label *I* in treatment [A] and for label *II* in treatment [B] are binding for at

least some of the agents.

We would like to emphasize, however, that this interpretation should be taken with a grain of salt, in addition to the caution due to the reasons mentioned above. It is not obvious that a highly manipulative intervention such as providing the ‘game theory tutorial’ would change players’ cost function without affecting their beliefs. Nonetheless, given the complexity of the design and the instructions, we find these results to be encouraging overall.

Figure 7 also shows the distributions for the treatments with high payoffs, [A+] and [B+]. This figure reveals how, as predicted by our model, increasing the incentives (from [A] to [A+] and from [B] to [B+]) produces effects analogous to reducing the cost function (from [A] to [E] and from [B] to [F]). Unlike treatments [E] and [F], however, subjects in treatments [A+] and [B+] play against opponents who also have higher incentives than in [A] and [B]. Thus, treatments [A+] and [B+] cannot be directly compared to [E] and [F].

5.2 Reasoning about opponents’ incentives

In the design of treatments [A+], [B+] and [C+], relative to [A], [B], [C], we increase the payoff for undercutting the opponent for both players in the game. Thus, the shifts in the distributions towards lower numbers observed in Section 4.1 may conflate two distinct effects. The first effect is the (possible) increase in the cognitive bound of player i , and the second is the change in i ’s beliefs about j ’s cognitive bound due to the change in j ’s incentives. Both effects would determine an increase in the behavioral k_i , hence a shift of the distribution towards lower actions.

To illustrate these effects using our model, consider the example in Figure 2.b, p.21. When $v_i = v_j = v$, the cognitive bound and behavioral level of play are $\hat{k}_i = k_i = 3$. Now, suppose that v_i is increased up to v^* , whereas $v_j = v$. In this case, $\hat{k}_i = 6$, $\hat{k}_j^i = \bar{k}_j^i = k_j^i = 4$, and i ’s response is to play according to $k_i = 5$, which is higher than the original level of 3. Suppose next that v_j is also increased to $v_j = v^*$. In that case, player i ’s cognitive bound becomes binding, and k_i increases to $k_i = \hat{k}_i = 6$. The movement from 3 to 5 is thus due to the increase in $\hat{k}_i$ alone, induced by an increase in v_i ; the further change from 5 to 6 instead is determined by i ’s reasoning about the change in his opponent’s incentives.³⁴

The following treatments are aimed at testing whether subjects in our experiment reason about

³⁴In general, our model implies that $\hat{k}_i$ (weakly) increases whenever v_i is increased. Furthermore, if v_j is held constant, $\hat{k}_j^i$ increases only if $\bar{k}_j^i$ (the intersection between c_j^i and v_j) had been larger than $(\hat{k}_i - 1)$ in the first place. In Figure 2.a, for instance, increasing v_i without changing v_j does not affect $\hat{k}_j^i$, hence k_i . The opposite is true in Figure 2.b, where $\hat{k}_j^i$ increases as v_i is increased, until $\hat{k}_j^i = \bar{k}_j^i$ (that is, when v_i is sufficiently high that $\hat{k}_i \geq 5$.)

their opponents' incentives independently of their own. In a sense, the exercise is of a similar spirit to treatment [C], in which subjects play against the number chosen by an opponent in treatment [B]. Similarly, in treatments [K] and [L] agents play the high- payoff game against the number chosen by an opponent in treatments [A] and [C], respectively. This treatment adds a further layer of complexity, as the individual is told in treatment [K] (resp., [L]) that he is playing the high-payoff game against the number chosen by an opponent of the same (other) label himself playing the low payoff game against opponent of the same (other) label (see Appendix A.1.3 for the precise instructions).

By comparing treatments [K] and [L] with treatments [A] and [C] and with treatments [A+] and [C+], we can, in principle, disentangle the two effects mentioned above. The shift from [A] to [K] (and from [C] to [L]), due solely to the increase of each subject's own payoffs and not his opponent's, may be imputed to the increase of subjects' own cognitive bound. It should thus be observed only if the cognitive bound in treatments [A] and [C] was binding (see footnote 34); the further shift from [K] to [A+] (and from [L] to [C+]) instead can be imputed to the increase in subjects' beliefs about their opponents' behavior due to the increase of their payoffs.

The complexity of treatments [K] and [L] is due to the players' higher order beliefs being manipulated along two dimensions: the first, as in [C], is that the label of a player i 's opponent need not coincide with i 's own label (this is the case in treatment [L]); the second is that in [K] and [L] both the payoffs of i 's opponent and of that opponent's opponent are 'low', whereas i 's payoffs are 'high'.

Figures 8 and 9 show the results of these treatments for labels I and II , respectively. The results are roughly in line with the predictions of the theory. The empirical distribution of [A] first order stochastically dominates [K] everywhere for label I other than at 14, and in most of the curve for label II . Distribution [K] first order stochastically dominates [A+] in the majority of the curve, although this appears more tentative. The results for [L], however, are surprisingly clean: the distribution of [L] lies 'in between' the distributions of [C] and [C+] nearly everywhere for label I and in most regions for label II . Furthermore, for label I , the theory predicts that the increase in k_i from [C] to [L] should be at most one, which seems roughly confirmed by the small shift in distribution from [C] to [L]. In this case, and consistently with the theory, the movement from [C] to [C+] for label I is mainly due to the increase in the opponents' payoffs, and not solely to the agent's own incentives.

In light of the complexity of these treatments, we find these results to be surprisingly good. As with treatments [D], [E] and [F], however, we note that treatments [K] and [L] were conducted on a

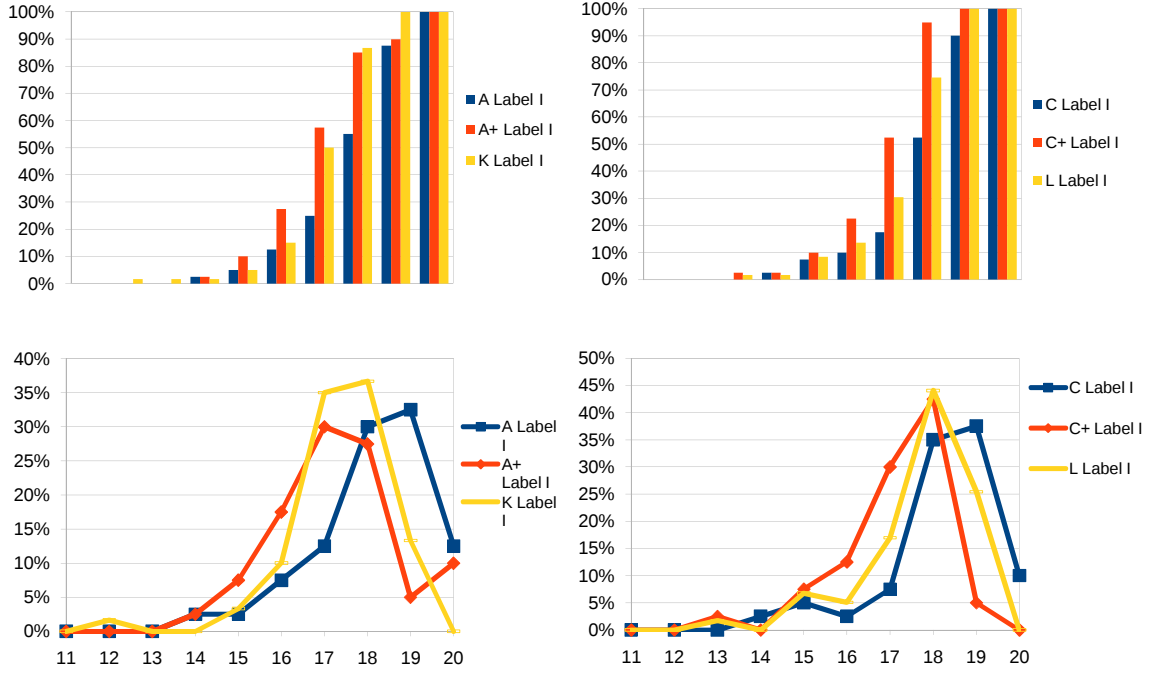


Figure 8: Treatments [K] and [L] for label *I*

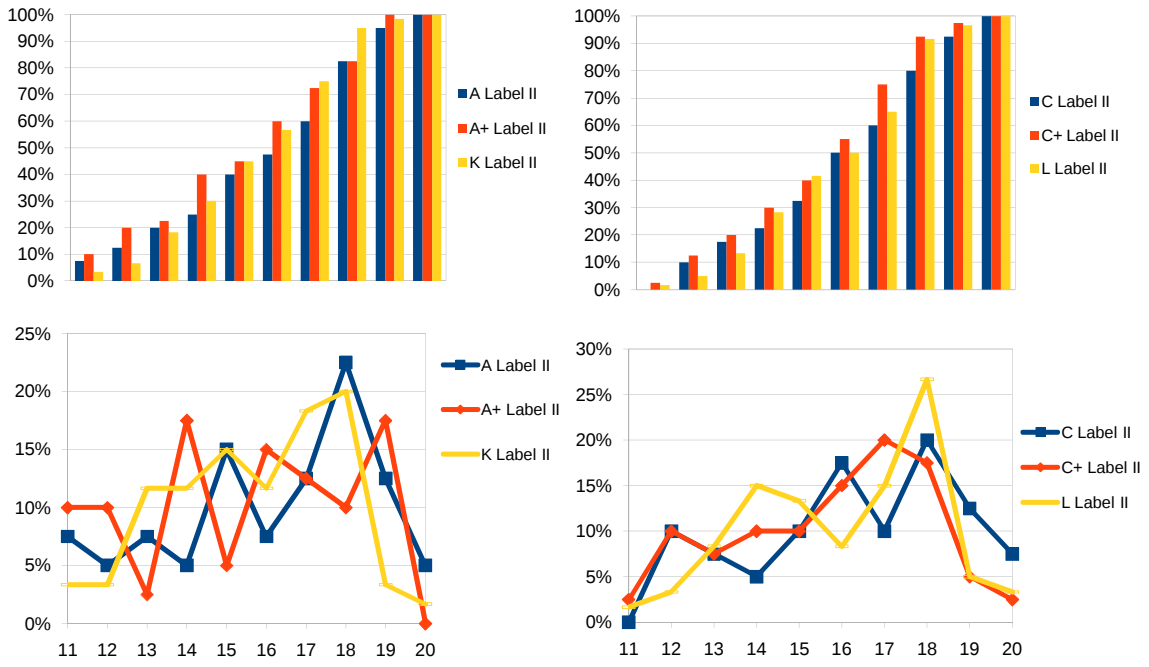


Figure 9: Treatments [K] and [L] for label *II*

restricted subset of subjects. In particular, they were administered in only two sessions (one in the endogenous and one in the exogenous classification). The size of the sample and the complexity of the instructions therefore suggest some caution in interpreting these results as conclusive.

6 Concluding Remarks

The theoretical model introduced here endogenizes the reasoning process of individuals in strategic environments in a tractable way, and allows for a differentiation between players' cognitive bounds and their perception of the cognitive bound of others. The modeling of the cost-benefit analysis aims to capture the bounded rationality feature of this reasoning process and the importance of beliefs over the opponent in a strategic environment, while remaining within the economics approach of tradeoffs. The findings of the experiment designed to test the theoretical predictions appear to support our framework.

We view this model as leading towards a more complete model of procedural rationality, and of strategic behavior under non-equilibrium play. A natural next step is a characterization of the benefit function (see Alaoui and Penta (2012)), and a better understanding of the cost curves. A rigorous identification of the properties of the cost functions would lead to added predictions, particularly on the magnitudes of the change in play across games. These last empirical studies are outside the scope of this paper, and remain open for further research.

Appendix

A Logistics of the Experiment

The experiment was conducted at the Laboratori d'Economia Experimental (LEEX) at Universitat Pompeu Fabra (UPF), Barcelona. Subjects were students of UPF, recruited using the LEEX system. No subject took part in more than one session. Subjects were paid 3 euros for showing up (students coming from a campus that was farther away received 4 euros instead). Subjects' earnings ranged from 10 to 40 euros, with an average of 15.8.

Each subject went through a sequence of 18 games. Payoffs are expressed in 'tokens', each worth 5 cents. Subjects were paid randomly, once every six iterations. The order of treatments is randomized (see below). Finally, subjects only observed their own overall earnings at the end, and received no information concerning their opponents' results.

Our subjects were divided in 6 sessions of 20 subjects, for a total of 120 subjects. Three sessions were based on the exogenous classification, and each contained 10 students from the field of humanities (humanities, human resources, and translation), and 10 from math and sciences (math, computer science, electrical engineering, biology and economics). Three sessions were based on the endogenous classification, and students were labeled based on their performance on a test of our design. (See Appendix B). In these sessions, half students were labeled as 'high' and half as 'low'.

A.1 Instructions of the Experiment

We describe next the instructions as worded for a student from math and sciences. The instructions for students from humanities would be obtained replacing these labels everywhere. Similarly, labels high and low would be used for the endogenous classification.

A.1.1 Baseline Game and Treatments [A], [B] and [C]

Pick a number between 11 and 20. You will always receive the amount that you announce, in tokens.

In addition:

- if you give the same number as your opponent, you receive an extra 10 tokens.
- if you give a number that's exactly one less than your opponent, you receive an extra 20 tokens.

Example:

- If you say 17 and your opponent says 19, then you receive 17 and he receives 19.
- If you say 12 and your opponent says 13, then you receive 32 and he receives 13.
- If you say 16 and your opponent says 16, then you receive 26 and he receives 26.

Treatments [A] and [B]:

Your opponent is:

- a student from maths and sciences (treatment [A]) / humanities (treatment [B])
- he is given the same rules as you.

Treatment [C]:

In this case, the number you play against is chosen by:

- a student from humanities facing another student from humanities. In other words, two students from maths and sciences play against each other. You play against the number that one of them has picked.

A.1.2 Changing Payoffs: Treatments [A+], [B+] and [C+]

You are now playing a high payoff game.

- if you give the same number as your opponent, you receive an extra 10 .

Example:

- If you say 17 and your opponent says 19, then you receive 17 and he receives 19.
- If you say 12 and your opponent says 13, then you receive 92 and he receives 13.
- If you say 16 and your opponent says 16, then you receive 26 and he receives 26.

Treatments [A+] and [B+]

Your opponent is:

- a student from maths and sciences playing the high payoff game (treatment [A]) / humanities (treatment [B])
- he is given the same rules as you.

Treatment [C+]

In this case, the number you play against is chosen by:

- a student from humanities playing the high payoff game with another student from humanities.

In other words, two students from humanities play the high payoff game with each other (extra 10 if they tie, 80 if exactly one less than opponent). You play against the number that one of them has picked.

A.1.3 Treatments [K] and [L]

Treatment [K]

In this case, the number you play against is chosen by:

- a student from maths and sciences playing the low payoff game with another student from maths and sciences. In other words, two students from maths and sciences play the low payoff game with each other (extra 10 if they tie, 20 if exactly one less than opponent). You play against the number that one of them has picked.

Treatment [L]

In this case, the number you play against is chosen by:

- a student from humanities playing the low payoff game with another student from humanities. In other words, two students from humanities play the low payoff game with each other (extra 10 if they tie, 20 if exactly one less than opponent). You play against the number that one of them has picked.

A.1.4 Treatments [D], [E] and [F]

Before playing treatments [D], [E] and F, the subjects were given the following ‘tutorial’:

Game Theory Tutorial: According to game theory, if the players are infinitely rational, then the game should be played in the following way. Both players should say 11.

Explanation: Suppose the two players are named Ana and Beatriz. If Ana thinks Beatriz plays 20, then Ana would play 19. But then Beatriz knows that Ana would play 19, so she would play 18. Ana realizes this, and so she would play 17.... they both follow this reasoning until both would play 11. Notice that if Beatriz says 11, then the best thing for Ana is to also say 11.

Treatment [D]

Your opponent is:

- a student who has also been given the game theory tutorial.

Treatment [E]

Your opponent is:

- a student from maths and sciences.
- he has not been given the game theory tutorial.

Treatment [F]

Your opponent is:

- a student from humanities.

- he has not been given the game theory tutorial.

A.2 Sequences

Our 6 groups (3 for the endogenous and 3 for the exogenous classification) went through four different sequences of treatments. Two of the groups in the exogenous treatment followed Sequence 1, and one followed Sequence 2. The three groups of the endogenous classification each took a different sequence: respectively sequence 1, 3 and 4. All the sequences contain our main treatments, [A], [B], [C], [A+], [B+], [C+]. In addition, sequences 2 and 4 contain the [K] and [L] treatments, whereas sequences 1 and 3 conclude with the additional treatments [D], [E] and [F]. The order of the main treatments is different in each sequence, both in terms of changing the beliefs and the payoffs.

- **Sequence 1:** $A, B, C, B, A, C, A^+, B^+, C^+, B^+, A^+, C^+, D, E, F, D, E, F$
- **Sequence 2:** $A, B, B, A, C, C, K, L, K, L, K, L, A^+, B^+, B^+, A^+, C^+, C^+$
- **Sequence 3:** $A^+, B^+, C^+, B^+, A^+, C^+, A, B, C, B, A, C, D, E, F, D, E, F$
- **Sequence 4:** $B, A, C, B, A, C, K, L, K, L, K, L, B^+, A^+, C^+, B^+, A^+, C^+$

B The Test for the Endogenous Classification

The cognitive test takes roughly thirty minutes to complete, and consists of three questions. In the first, subjects are asked to play a variation of the board game Mastermind. In the second question, the subjects are given a typical centipede game of seven rounds, and are asked what an infinitely sophisticated and rational agent would do. In the third game, the subjects are given a lesser known ‘pirates game’, which is a four player game that can be solved by backward induction. Subjects are asked what the outcome of this game would be, if players were ‘infinitely sophisticated and rational’. Each question was given a score, and then a weighted average was taken. Subjects whose score was higher (lower) than the median score were labeled as ‘high’ (‘low’). We report next the instructions of the test, as administered to the students (see the online appendix for the original version in Spanish).

Instructions of the Test. This test consists of three questions. You must answer all three, within the time limit stated.

Question 1:

In this question, you have to guess four numbers in the correct order. Each number is between 1 and 7. No two numbers are the same. You have nine attempts to guess the four numbers. After each attempt, you will be told the number of correct answers in the correct place, and the number of correct numbers in the wrong place.

Example: Suppose that the correct number is: 1 4 6 2.

If you guess : 3 5 4 6, then you will be told that you have 0 correct answers in the correct place and 2 in the wrong place.

If you guess : 3 5 6 4, then you will be told that you have 1 correct answer in the correct place and 1 in the wrong place.

If you guess : 3 4 7 2, then you will be told that you have 2 correct answers in the correct place and 0 in the wrong place.

If you guess : 1 4 6 2, then you will be told that you have 4 correct answers, and you have reached the objective.

Notice that the correct number could not be (for instance) 1 4 4 2, as 4 is repeated twice. You are, however, allowed to guess 1 4 4 2, in any round.

You have 15 minutes in total for this question.

Question 2:

Consider the following game. Two people, Antonio and Beatriz, are moving sequentially. The game starts with 1 euro on the table. There at most 6 rounds in this game:

Round 1) Antonio is given the choice whether to take this 1 euro, or pass, in which case the game has another round. If he takes the euro, the game ends. He gets 1 euro, Beatriz gets 0 euros. If Antonio passes, they move to round 2.

Round 2) 1 more euro is put on the table. Beatriz now decides whether to take 2 euros, or pass. If

she takes the 2 euros, the game ends. She receives 2 euros, and Antonio receives 0 euros. If Beatriz passes, they move to round 3.

Round 3) 1 more euro is put on the table. Antonio is asked again: he can either take 3 euros and leave 0 to Beatriz, or pass. If Antonio passes, they move to round 4.

Round 4) 1 more euro is put on the table. Beatriz can either take 3 euros and leave 1 euro to Antonio, or pass. If Beatriz passes, they move to round 5.

Round 5) 1 more euro is put on the table. Antonio can either take 3 euros and leave 2 to Beatriz, or pass. If Antonio passes, they move to round 6.

Round 6) Beatriz can either take 4 euros and leaves 2 to Antonio, or she passes, and they both get 3.

Assume Antonio and Beatriz are infinitely sophisticated and rational, and they each want to get as much money as possible. What will be the outcome of the game?

- a) Game stops at Round 1, with payoffs: (Antonio: 1 euro Beatriz: 0 euros)
- b) Game stops at Round 2, with payoffs: (Antonio: 0 euro Beatriz: 2 euros)
- c) Game stops at Round 3, with payoffs: (Antonio: 2 euros Beatriz: 1 euro)
- d) Game stops at Round 4, with payoffs: (Antonio: 1 euro Beatriz: 3 euros)
- e) Game stops at Round 5, with payoffs: (Antonio: 3 euros Beatriz: 2 euros)
- f) Game stops at Round 6, with payoffs: (Antonio: 2 euros Beatriz: 4 euros)
- g) Game stops at Round 6, with payoffs: (Antonio: 3 euros Beatriz: 3 euros)

You have 8 minutes in total for this question.

Question 3)

Four pirates (Antonio, Beatriz, Carla and David) have obtained 10 gold doblones and have to divide up the loot. Antonio proposes a distribution of the loot. All pirates vote on the proposal. If half the crew or more agree, the loot is divided as proposed by Antonio.

If Antonio fails to obtain support of at least half his crew (including himself), then he will be killed. The pirates start over again with Beatriz as the proposer. If she gets half the crew (including herself) to agree, then the loot is divided as proposed. If not, then she is killed, and Carla then makes the proposal. Finally, if her proposal is not agreed on by half the people left, including herself, then she is killed, and David takes everything.

In other words:

Antonio needs 2 people (including himself) to agree on his proposal, and if not he is killed.

If Antonio is killed, Beatriz needs 2 people (including herself) to agree on her proposal, if not she is killed.

If Beatriz is killed, Carla needs 1 person to agree (including herself) to agree on her proposal, and if not she is killed.

If Carla is killed, David takes everything.

The pirates are infinitely sophisticated and rational, and they each want to get as much money as possible. What is the maximum number of coins Antonio can keep without being killed?

Notice that *the proposer* can also vote, and that exactly half the votes is enough for the proposal to pass.

You have 8 minutes in total for this question.

Scoring. In the *mastermind* question, subjects were given 100 points if correct, otherwise they received 15 points for each correct answer in the correct place and 5 for each correct answer in the wrong place in their last answer. In the *centipede* game, subjects were given 100 points if they answered that the game would end at round 1, otherwise points were equal to $\min\{0, (6 - \text{round}) \cdot 15\}$. In the *pirates* game, subjects obtain 100 if they answer 100, 60 if they answer 10, $\min\{0, (80 - x) \cdot 10\}$. The overall score was given by the average of the three.

C Statistical Tests and Regressions

	Two-Sample KS D-stat (exact p-value)	Mann-Whitney-Wilcoxon p-values
A1 vs A2 Label <i>I</i>	0.06 (0.999)	0.55
B1 vs B2 Label <i>I</i>	0.03 (1.000)	0.90
C1 vs C2 Label <i>I</i>	0.05 (1.000)	0.76
A+1 vs A+2 Label <i>I</i>	0.05 (1.000)	0.65
B+1 vs B+2 Label <i>I</i>	0.11 (0.803)	0.35
C+1 vs C+2 Label <i>I</i>	0.06 (0.999)	0.62
A1 vs A2 Label <i>II</i>	0.10 (0.920)	0.71
B1 vs B2 Label <i>II</i>	0.05 (1.000)	0.96
C1 vs C2 Label <i>II</i>	0.15 (0.412)	0.47
A+1 vs A+2 Label <i>II</i>	0.16 (0.365)	0.13
B+1 vs B+2 Label <i>II</i>	0.084 (0.985)	0.43
C+1 vs C+2 Label <i>II</i>	0.06 (1.000)	0.69

Table 3: Equality of Distributions Tests: Rounds

	Relevant dummy	Classification dummy	Constant	Number of obs.
From A to A+ Label <i>I</i>	-0.50*** (0.17)	0.22 (0.52)	17.21	235
From B to B+ Label <i>I</i>	-0.62*** (0.19)	0.36 (0.38)	17.50	233
From C to C+ Label <i>I</i>	-1.15*** (0.18)	0.34 (0.38)	17.76	236
From A to A+ Label <i>II</i>	-0.64** (0.27)	-0.10 (0.45)	16.91	236
From B to B+ Label <i>II</i>	-0.74*** (0.25)	0.38 (0.47)	16.57	236
From C to C+ Label <i>II</i>	-0.97*** (0.25)	-0.07 (0.45)	16.97	234
From B to A Label <i>I</i>	-0.36* (0.20)	0.34 (0.43)	17.50	236
From B to C Label <i>I</i>	0.25 (0.25)	0.44 (0.37)	17.46	236
From A to C Label <i>I</i>	0.62*** (0.18)	0.36 (0.42)	17.13	236
From A to B Label <i>II</i>	-0.09 (0.26)	0.50 (0.48)	16.6	236
From B to C Label <i>II</i>	0.16 (0.25)	0.36 (0.46)	16.59	235
From A to C Label <i>II</i>	0.07 (0.27)	0.30 (0.44)	16.71	235
From B+ to A+ Label <i>I</i>	-0.26 (0.17)	0.25 (0.46)	16.96	232
From B+ to C+ Label <i>I</i>	-0.29** (0.13)	0.27 (0.45)	16.93	233
From A+ to C+ Label <i>I</i>	0.03 (0.16)	0.21 (0.48)	16.71	236
From A+ to B+ Label <i>II</i>	-0.18 (0.21)	-0.22 (0.52)	16.33	236
From B+ to C+ Label <i>II</i>	-0.07 (0.25)	-0.04 (0.50)	16.05	235
From A+ to C+ Label <i>II</i>	-0.25 (0.21)	-0.47 (0.51)	16.46	235

Table 4: Regressions for Labels *I* and *II*.

	ABC (Label <i>I</i>)	ABC (Label <i>II</i>)	A+B+C+ (Label <i>I</i>)	A+B+C+ (Label <i>II</i>)
opp	-0.36** (0.19)	-0.09 (0.26)	-0.26 (0.16)	-0.18 (0.21)
R	0.25 (0.19)	0.17 (0.26)	-0.29* (0.16)	0.06 (0.21)
Classification dummy	0.38 (0.39)	0.38 (0.44)	0.25 (0.46)	0.25 (0.50)
Constant	17.49	16.66	16.96	16.34
Number of obs.	354	353	350	353

Table 5: Changing Opponents (II): regressions for *ABC* and *A+B+C+*, Labels *I* and *II*.

	Two-Sample KS D-stat (exact p-value)	Mann-Whitney-Wilcoxon p-values
A vs A+ Label <i>I</i>	0.18 (0.040) **	0.02 **
B vs B+ Label <i>I</i>	0.24 (0.002) ***	0.0004 ***
C vs C+ Label <i>I</i>	0.37 (0.000) ***	0.0000 ***
A vs A+ Label <i>II</i>	0.14 (0.128)	0.019 **
B vs B+ Label <i>II</i>	0.17 (0.048) **	0.0039 ***
C vs C+ Label <i>II</i>	0.25 (0.001) ***	0.0007 ***

Table 6: Equality of Distributions Tests: Changing Payoffs

	Two-Sample KS D-stat (exact p-value)	Mann-Whitney-Wilcoxon p-values
A vs B Label <i>I</i>	0.11 (0.463)	0.21
B vs C Label <i>I</i>	0.05 (0.998)	0.31
A vs C Label <i>I</i>	0.14 (0.163)	0.03 **
A+ vs B+ Label <i>I</i>	0.06 (0.952)	0.47
B+ vs C+ Label <i>I</i>	0.08 (0.738)	0.24
A+ vs C+ Label <i>I</i>	0.07 (0.869)	0.72
A vs B Label <i>II</i>	0.05 (0.986)	0.68
B vs C Label <i>II</i>	0.06 (0.957)	0.76
A vs C Label <i>II</i>	0.04 (1.000)	0.96
A+ vs B+ Label <i>II</i>	0.07 (0.793)	0.42
B+ vs C+ Label <i>II</i>	0.06 (0.938)	0.57
A+ vs C+ Label <i>II</i>	0.12 (0.277)	0.22

Table 7: Equality of Distributions Tests: Changing Opponents

References

1. Agranov, Marina, Elizabeth Potamites, Andrew Schotter and Chloe Tergiman. 2012. "Beliefs and Endogenous Cognitive Levels: An Experimental Study" *Games and Economic Behavior*, forthcoming.
2. Alaoui, Larbi and Antonio Penta. 2012 "A Model of Procedural Rationality", *mimeo*.
3. Arad, Ayala and Ariel Rubinstein. 2012. "The 11-20 Money Request Game: A Level- k Reasoning Study", *American Economic Review*, forthcoming
4. Basu, Kaushik, "The Traveler's Dilemma: Paradoxes of Rationality in Game Theory." *American Economic Review Papers and Proceedings*, Vol. 84, No. 2: 391-395
5. Bhatt, Meghana A., and Colin F. Camerer. 2005. "Self-referential Thinking and Equilibrium as States of Mind in Games: fMRI Evidence." *Games and Economic Behavior*, 52(2): 424-459.
6. Bhatt, Meghana A., Terry Lohrenz, Colin F. Camerer, and P. Read Montague. 2010. "Neural Signatures of Strategic Types in a Two-Person Bargaining Game." *Proceedings of the National Academy of Sciences*, 107(46): 19720-19725.
7. Bosch-Domènech, Antoni, Jose García-Montalvo, Rosemarie Nagel, and Albert Satorra. 2002. "One, Two, (Three), Infinity...: Newspaper and Lab Beauty-Contest Experiments." *American Economic Review*, 92(5), 1687-1701.
8. Camerer, Colin F. 2003. *Behavioral Game Theory: Experiments in Strategic Interaction*. Princeton, N.J.: Princeton University Press.
9. Camerer, Colin F., Teck-Hua Ho, and Juin Kuan Chong. 2004. "A Cognitive Hierarchy Model of Games." *Quarterly Journal of Economics*, 119(3): 861-898.
10. Capra, C. Monica, Jacob K. Goeree, Rosario Gomez, Charles A. Holt. 1999. "Anomalous Behavior in a Traveler's Dilemma?" *American Economic Review*, Vol. 89 (3): 678-690
11. Coricelli, Giorgio, and Rosemarie Nagel. 2009. "Neural Correlates of Depth of Strategic Reasoning in Medial Prefrontal Cortex." *Proceedings of the National Academy of Sciences*, 106(23): 9163-9168.

12. Costa-Gomes, Miguel A., and Vincent P. Crawford. 2006. "Cognition and Behavior in Two-Person Guessing Games: An Experimental Study." *American Economic Review*, 96(5): 1737-1768.
13. Costa-Gomes, Miguel A., Vincent P. Crawford, and Bruno Broseta. 2001. "Cognition and Behavior in Normal-Form Games: An Experimental Study." *Econometrica*, 69(5): 1193-1235.
14. Choi, Syungoo. 2012. "A Cognitive Hierarchy Model of Learning in Networks, February 2012." *Review of Economic Design*, forthcoming.
15. Crawford, Vincent P. 2003. "Lying for Strategic Advantage: Rational and Boundedly Rational Misrepresentation of Intentions." *American Economic Review*, 93(1): 133-149.
16. Crawford, Vincent P., Miguel A. Costa-Gomes, and Nagore Iriberri. 2012. "Structural Models of Nonequilibrium Strategic Thinking: Theory, Evidence, and Applications." *Journal of Economic Literature*, forthcoming.
17. Crawford, Vincent P., and Nagore Iriberri. 2007. "Level- k Auctions: Can a Nonequilibrium Model of Strategic Thinking Explain the Winner's Curse and Overbidding in Private-Value Auctions?" *Econometrica*, 75(6): 1721-1770.
18. Davidson, R. and Duclos, J.-Y. 2000. "Statistical inference for stochastic dominance and for the measurement of poverty and inequality." *Econometrica* 68: 1435-1464.
19. Gabaix, Xavier. "Game Theory with Sparsity-Based Bounded Rationality." *mimeo*
20. Goeree, Jacob K., and Charles A. Holt. 2001. "Ten Little Treasures of Game Theory and Ten Intuitive Contradictions." *American Economic Review*, 91(5): 1402-1422.
21. Goeree, Jacob K., and Charles A. Holt. 2004. "A Model of Noisy Introspection." *Games and Economic Behavior*, 46(2): 365-382.
22. Grosskopf, Brit, and Rosemarie Nagel. 2008. "The Two-Person beauty contest." *Games and Economic Behavior*, 62: 93-99.
23. Kets, Willemien. 2012. "Bounded Reasoning and Higher-Order Uncertainty." *mimeo*, Northwestern Univ.
24. Ho, Teck-Hua, Colin Camerer, and Keith Weigelt. 1998. "Iterated Dominance and Iterated Best Response in Experimental 'p-Beauty Contests'." *American Economic Review*, 88(4): 947-969.

25. Nagel, Rosemarie. 1995. "Unraveling in Guessing Games: An Experimental Study." *American Economic Review*, 85(5): 1313-1326.
26. Palacios-Huerta, Ignacio and Oscar Volij. 2009. "Field Centipedes." *American Economic Review*, vol. 99(4): 1619-1635.
27. Stahl, Dale O., and Paul R. Wilson. 1994. "Experimental Evidence on Players' Models of Other Players." *Journal of Economic Behavior and Organization*, 25 (3): 309-327.
28. Stahl, Dale O., and Paul R. Wilson. 1995. "On Players' Models of Other Players: Theory and Experimental Evidence." *Games and Economic Behavior*, 10(1): 218-254.
29. Strzalecki, Tomasz. 2010. "Depth of Reasoning and Higher-Order Beliefs." Harvard Institute of Economic Research Discussion Paper Number 2184.