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Testing quantum-like models of judgment for question order effects

Thomas Boyer-Kassem∗, Sébastien Duchêne†, Eric Guerci†

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Abstract

Lately, so-called “quantum” models, based on parts of the mathematics of quantum mechanics, have been developed in decision theory and cognitive sciences to account for seemingly irrational or paradoxical human judgments. In this paper, we limit ourselves to such quantum-like models that address order effects. It has been argued that such models are able to account for existing and new empirical data, and meet some a priori predictions. From the quantum law of reciprocity, we derive new empirical predictions that we call the Grand Reciprocity equations, that must be satisfied by quantum-like models on the condition that they are non-degenerate. We show that existing non-degenerate quantum-like models for order effects fail this test on several existing data sets. We take it to suggest that degenerate quantum-like models should be the focus of forthcoming research in the area.

"Research is needed which designs tests of quantum properties such as the law of reciprocity" (Busemeyer and Bruza 2012, 342)

1 Introduction

In decision theory and in cognitive sciences, classical cognitive models of judgment rely on Bayesian probabilities and suppose that agents’ decisions or choices are guided by preferences or attitudes that are determined at any time. Yet, various empirical results have threatened the predictive and explanatory power of these classical models: human judgments display order effects — the answers given to two questions depend on the order of presentation of these questions — (Schuman and Presser 1981, Tourangeau, Ribs and Rasinski 2000, Moore 2002), conjunction fallacies — someone is judged less likely to have characteristic C than characteristics C and D — (Tversky and Kahneman 1982 and 1983, Gavanski and Roskos-Ewoldsen 1991, Stolarz-Fantino et al. 2003), violate the sure-thing principle — stating that preferring X to Y given any possible state of the world should imply preferring X to Y when the exact state of the world is not known — (Allais 1953, Ellsberg 1961, Shafir and Tversky 1992), or asymmetries in similarity — X is judged more similar to Y than Y to X — (Tversky 1977, Krumhansl 1978, Ashby and Perrin 1988). In the classical cognitive framework, these behaviors are usually dubbed as irrational or paradoxical.

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Recently, so-called “quantum” or “quantum-like” models have been proposed to account for such behaviors. The qualifier “quantum” is used to indicate that these models exploit the mathematics of a contemporary physical theory, quantum mechanics. Note that only some mathematical tools of quantum mechanics are employed, and that the claim is not that these models are justified by an application of quantum physics to the brain. For that reason, we shall prefer to call them “quantum-like” models. Such models put into question two classical characteristics recalled above: they abandon Bayesian probabilities for others which are similar to probabilities in quantum mechanics, and they allow for preferences to be undetermined. Quantum-like models have received much interest from psychologists, physicists, economists, cognitive scientists and philosophers.

In particular, quantum-like models have been proposed to account for the seemingly paradoxical behaviors recalled above (for a review, see Pothos and Busemeyer 2013). First, as the mathematics involved in quantum mechanics are well-known for their non-commutative features, one of their natural application is the account of order effects. For instance, Wang and Busemeyer (2013) and Wang et al. (2014) offer a general quantum-like model for attitude questions that are asked in polls. Conte et al. (2009) present a model for mental states of visual perception, and Altmanspacher and Römer (2012) discuss non-commutativity. Quantum-like models have also been offered to explain the conjunction fallacy. For instance, Franco (2009) argues that it can be recovered from interference effects, which are central features in quantum mechanics and at the origin of the violation of classical probabilities. Busemeyer et al. (2011) present a quantum-like model that could explain conjunction fallacy from some order effects. The violation of the sure thing principle has also been investigated by means of quantum-like models. Busemeyer, Wang and Townsend (2006), and Busemeyer and Wang (2007) use quantum formalism to explain this violation by introducing probabilistic interference and superposition of states. Khrennikov and Haven (2009) also explain Ellsberg’s paradox, and Aerts and al. (2011) model the Hawaii problem. Dynamical models, that rely on a time evolution of the mental state, have also been proposed (Pothos and Busemeyer 2009, Busemeyer et al. 2009, and Trueblood and Busemeyer 2011). Several other empirical features, such as asymmetry judgments in similarity have also been offered a quantum-like model (Pothos and Busemeyer, 2011).

The question that motivates and underlies the present paper is: to what extent have these quantum-like models been already empirically tested? And can the existing tests be considered as sufficient? As the task is huge, we shall concentrate here on one class of quantum-like models, namely quantum-like models for order effect. But a natural continuation of the present work is to turn to quantum-like models for conjunction fallacy or for the violation of the sure-thing principle, for instance.

Actually, when quantum-like models for order effects were proposed, it was argued that they were able to account either for existing empirical data (e.g. Wang and Busemeyer 2013 for the data from Moore 2002, Wang et al. 2014 for data from around 70 national surveys) or for data from new experiments (e.g. Conte et al. 2009, Wang and Busemeyer 2013). This is not all; a supplementary a priori constraint has been derived (the “QQ equality”) that all quantum-like models should verify, and it has been successfully verified on the above-mentioned data (Wang and Busemeyer 2013, Wang et al. 2014). So, it seems that quantum-like models for order effects are well verified and can be considered as adequate, at least for a vast set of experiments.

In this paper, we argue that some quantum-like models for order effect can actually be empirically tested in another unstudied — although simple — way. After reconstructing a general quantum-like model for order effect (Section 2), we derive from the quantum law of reciprocity a set of constraints that we call the Grand Reciprocity (GR) equations
(Section 3). They apply for a kind of quantum-like models that has been considered by almost all the papers in the quantum-like literature on order effects, and that is often represented in figures, namely those for which answers are represented by one-dimensional subspaces, and that are called non-degenerate models. We study the relation of these GR equations to double-stochasticity and to the QQ equality. Then, we discuss to which of the existing non-degenerate quantum-like models of the literature the GR equations apply (Section 4). We argue that it is the case for Conte et al. (2009), for Busemeyer and Bruza (2012), Pothos and Busemeyer (2013), Wang and Busemeyer (2013) and Wang et al. (2014). All the non-degenerate models in these papers have to obey the GR equations. For the cases in which empirical data is available, we put the GR equations to the test (Section 5), and it turns out that a vast majority of empirical cases fail to satisfy the GR equations. So, non-degenerate models are not possible for these data sets. In other words, our paper shows that non-degenerate quantum-like models, which are popular in the literature because they are the most simple ones and can be easily represented, are actually not empirically adequate in most cases. Note that this conclusion is reached without carrying out new experiments, but just using existing available data. Finally (Section 6), we discuss the consequences of these results for quantum-like models for order effect at large. The positive conclusion of our results, we suggest, is that research on quantum-like models should be directed towards degenerate models — this is the place where exciting results should be expected.

2 A general quantum-like model

Several recent papers present or discuss quantum-like models which can account for order effects (Aerts 2009, Aerts, Gabora and Sozzo 2013, Busemeyer et al. 2011, Conte et al. 2009, Pothos and Busemeyer 2013, Wang and Busemeyer 2013, Wang et al. 2014). Instead of presenting and discussing all these models in turn, we choose for simplicity to present one simple and general model, which sets the notations, on which the discussions will be made. Then, it will be easy for the reader to replace it with such or such model from the literature, and have the corresponding discussion.

The model is about the beliefs of a person, about which dichotomous yes-no questions can be asked, for instance “is Clinton honest?”. A vector space on the complex numbers is introduced to represent the beliefs of the agent, and the answers to the questions. In the model, two dichotomous questions $A$ and $B$ are posed successively to an agent, in the order $A$-then-$B$ or $B$-then-$A$. Answer “yes” (respectively “no”) to $A$ is represented by the vector $|a_0\rangle$ (respectively $|a_1\rangle$), and similarly for $B$, with vectors $|b_0\rangle$ (respectively $|b_1\rangle$. It is important to note that an answer is supposed to be represented by a vector (or more exactly, see below, by the ray defined by this vector), and not by a plan or by any subspace of dimension greater than 1. Since there are no other possible answers to question $A$ (respectively $B$) than 0 and 1, the set $(|a_0\rangle, |a_1\rangle)$ (respectively $(|b_0\rangle, |b_1\rangle)$) forms a basis of the vector space of possible answers, and the vector space is thus of dimension 2. Note that it is the same vector space that is used to represents answers to questions $A$ and $B$; this space just has two different bases.\footnote{In the literature, questions that can be represented in this way are known as “incompatible”. “Compatible” questions have a common basis, but the model is then equivalent to a classical one, and there is nothing quantum about it. Cf. for instance Busemeyer and Bruza (2012, 32–34).}

The vector space is supposed to be equipped with a scalar product, thus becoming a Hilbert space: for two vectors $|x\rangle$ and $|y\rangle$, the scalar product $\langle x|y \rangle$ is a complex number; its complex conjugate, $\langle x|y \rangle^*$, is just $\langle y|x \rangle$. The Hilbert space is on the complex numbers,
and vectors can be multiplied by any complex number. The basis \((|a_0\rangle, |a_1\rangle)\) is supposed to be orthonormal, i.e. \(\langle a_0|a_1\rangle = 0\) and \(\langle a_0|a_0\rangle = \langle a_1|a_1\rangle = 1\); similarly for \(B\). Note that there exists a correspondence between the two bases (cf. figure 1 left):

\[
|b_0\rangle = \langle a_0|b_0\rangle|a_0\rangle + \langle a_1|b_0\rangle|a_1\rangle, \tag{1}
\]

\[
|b_1\rangle = \langle a_0|b_1\rangle|a_0\rangle + \langle a_1|b_1\rangle|a_1\rangle, \tag{2}
\]

and similarly for \(|a_0\rangle\) and \(|a_1\rangle\) expressed as a function of \(|b_0\rangle\) and \(|b_1\rangle\).

Figure 1: [Left:] The basis vectors \(|b_0\rangle\) and \(|b_1\rangle\) can be decomposed on the other basis vectors \(|a_0\rangle\) and \(|a_1\rangle\), so as to be expressed as in eq. 1 and 2. The scalar products are either equal to \(\cos \delta\) or to \(\sin \delta\). [Right:] The state vector \(|\psi\rangle\) can be expressed in the two orthonormal bases \((|a_0\rangle, |a_1\rangle)\) and \((|b_0\rangle, |b_1\rangle)\). These figures assume the special case of a Hilbert space on real numbers.

An agent’s beliefs about the subject matter of the questions \(A\) and \(B\) are represented by a normalized belief state \(|\psi\rangle\) from this vector space \((||\psi||^2 = 1)\). It is supposed to gather all the relevant information to predict her behaviour in the situation (like in orthodox quantum mechanics). \(|\psi\rangle\) can be expressed in the basis \((|a_0\rangle, |a_1\rangle)\) as

\[
|\psi\rangle = \alpha_0|a_0\rangle + \alpha_1|a_1\rangle, \tag{3}
\]

with \((\alpha_0, \alpha_1) \in \mathbb{C}^2\) (cf. fig. 1 right). It can also be expressed in the basis \((|b_0\rangle, |b_1\rangle)\) as

\[
|\psi\rangle = \beta_0|b_0\rangle + \beta_1|b_1\rangle, \tag{4}
\]

with \((\beta_0, \beta_1) \in \mathbb{C}^2\), and with an appropriate correspondence between the coefficients.

The belief state \(|\psi\rangle\) determines the answer in a probabilistic way, and changes only when a question is answered, according to the following rules:

- Born’s rule: the probability for the agent to answer \(x_i\) \((i = 0, 1)\) to question \(X\) \((X = A, B)\) is given by the squared modulus of the scalar product between \(|\psi\rangle\) and \(|x_i\rangle\):  

\[
\Pr(x_i) = |\langle x_i|\psi\rangle|^2 \tag{5}
\]

\footnote{We consider here this general case. In the literature, some quantum-like models consider a real Hilbert space, in which scalar products are real numbers, and vectors can be multiplied by real numbers only.}
projection postulate: the agent’s belief state just after the answer \( x_i \) is the normalized projection of her belief state prior to the question onto the vector \( |x_i\rangle \) corresponding to her answer:

\[
|\psi\rangle \quad \rightarrow \quad \frac{\langle x_i|\psi\rangle}{|\langle x_i|\psi\rangle|} |x_i\rangle.
\]  (6)

For instance, if an agent is described by the state vector \( |\psi\rangle = \alpha_0|a_0\rangle + \alpha_1|a_1\rangle \), the probability that she answers \( i \) to question \( A \) is given by \( |\alpha_i|^2 \), in which case the state after the answer is \( \frac{\alpha_i}{|\alpha_i|} |a_i\rangle \). On fig. 1, this probability can be obtained by first orthogonally projecting \( |\psi\rangle \) on the basis vector corresponding to the answer, and then taking the square of this length. A consequence of the projection postulate in this model is that, just after an answer \( i \) to question \( X \) has been given, the state is of the form \( \lambda|x_i\rangle \) with \( \lambda \in \mathbb{C} \) and \( |\lambda| = 1 \). The fact that the state after the answer is equal to \( |x_i\rangle \) “up to a phase factor”, as one says, is true whatever the state prior to the question. On fig. 1, in the case of a real Hilbert space, the projection postulates can be interpreted as follow: project \( |\psi\rangle \) on the basis vector corresponding to the answer, then normalize it (i.e. expand it so that it gets a length 1); the result is \( \pm|x_i\rangle \), according to the relative orientation of \( |\psi\rangle \) and \( |x_i\rangle \). In the general case, answering a question modifies the agent’s state of belief, but there are exceptions: if an agent’s state of belief is \( \lambda|x_i\rangle \) (with \( |\lambda| = 1 \)), Born’s rule states that she will answer \( i \) (“yes” or “no”) with probability 1, and her state of belief is thus unchanged. Such vectors from which an answer can be given with certainty are called “eigenvectors”, and their set is the “eigenspace”, for the “eigenvalue” \( \lambda \). In this model, all eigenspaces are of dimension 1, and are called “rays” (equivalently, one can say that the eigenvalue is not degenerate), since it has been supposed that an answer is represented by a vector, and not by several independent vectors; this will be of decisive importance in the next section. Another consequence of the projection postulate is that, once an agent has answered \( i \) to question \( A \), she will answer \( i \) with probability 1 to the same question \( A \) if it is posed again just afterwards.  

Such a quantum-like model displays order features. Compare for instance \( p(a_0, b_0) \), the probability to answer 0 to question \( A \) and then 0 to question \( B \), and \( p(b_0, a_0) \), the probability to give the same answers but in the reverse order. To compute \( p(a_0, b_0) \), one can project the initial state on \( |a_0\rangle \) without normalizing the result, then project the result on \( |b_0\rangle \), still without normalizing, and take the squared modulus of the final result.  

In other words, to compare the two probabilities, one can just compare the length of successive projections of \( |\psi\rangle \), first on \( |a_0\rangle \) and then \( |b_0\rangle \), or in the reverse order. Figure 2 shows that they are not necessary equal. Because quantum-like models display these order features, it has been naturally suggested that they can account for experimentally documented order effects; Section 4 discusses several such models.

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3 Of course, this is not true if another question, say \( B \), is posed in-between.

4 Proof. Note \( p(y_j|x_i) \) the probability to answer \( j \) to question \( Y \) given that question \( X \) has been answered with \( i \). After the answer \( x_i \), the state is \( \lambda|x_i\rangle \) with \( |\lambda| = 1 \), so \( p(y_j|x_i) = |\langle y_j|x_i\rangle|^2 \). Note \( p(x_i) \) the probability to answer \( i \) to question \( X \) when this question is asked first. \( p(a_0, b_0) = p(a_0) \cdot p(b_0|a_0) = |\langle a_0|\psi\rangle|^2 \cdot |\langle b_0|a_0\rangle|^2 = |\langle a_0|\psi\rangle \cdot \langle b_0|a_0\rangle|^2 = |\langle b_0|a_0\rangle|^2 |\langle a_0|\psi\rangle|^2 \), where \( |\alpha'_0\rangle = \langle a_0|\psi\rangle |a_0\rangle \). Since \( |\alpha'_0\rangle \) is \( |\psi\rangle \) projected onto \( |a_0\rangle \) and not normalized, computing \( |\langle b_0|a_0\rangle|^2 \) means that \( |a_0\rangle \) is projected onto \( |b_0\rangle \) and not normalized, before the squared modulus is computed. QED.
3 Constraints on the quantum-like model: the Grand Reciprocity equations

From the model presented in the previous section, we derive now some general empirical predictions.

3.1 Derivation of the Grand Reciprocity equations

Within the model presented in the previous section, be \(x_i\) and \(y_j\) the answers an agent gives to two successive questions \(X\) and \(Y\), in one order or in the other (\(X\) may be equal to \(Y\), and \(i\) to \(j\)). Because of the projection postulate, the states just after the answers \(x_i\) and \(y_j\) are respectively \(\lambda |x_i\rangle\) and \(\lambda' |y_j\rangle\) with \((\lambda, \lambda') \in \mathbb{C}^2\) and \(|\lambda| = |\lambda'| = 1\). On the one hand,

\[
p(y_j|x_i) = |\langle y_j|\lambda|x_i\rangle|^2 = |\langle y_j|x_i\rangle|^2. \quad (7)
\]

On the other hand,

\[
p(x_i|y_j) = |\langle x_i|\lambda'|y_j\rangle|^2 = |\langle x_i|y_j\rangle|^2. \quad (8)
\]

Because a mathematical property of the scalar product is that \(\langle y_j|x_i\rangle = \langle x_i|y_j\rangle^*\), as a consequence, \(|\langle y_j|x_i\rangle|^2 = |\langle x_i|y_j\rangle|^2\), and so

\[
p(y_j|x_i) = p(x_i|y_j). \quad (9)
\]

This equation is well-known in quantum mechanics, and is called the law of reciprocity (cf. Peres 1993, p. 35–36 and 56). The only condition for this law is that the eigenvalues are not degenerate\(^5\). It is a quantum law: it is not verified in general by a classical model, in

\(^5\)If the eigenvalues are degenerate, i.e. if the eigenspaces are of dimension greater than 1, then the projection postulate is generalized in the following way: the agent’s belief state just after the answer \(x_i\) is the normalized projection of her belief state prior to the question onto the eigenspace corresponding to her answer. This eigenspace is the ray spanned by \(|x_i\rangle\) if the eigenvalue is not degenerate, but more generally it can be a plan or a hyperspace. Here is why the hypothesis of non-degeneracy is necessary to the reciprocity law. Suppose for simplicity that the Hilbert space is on the real numbers. Suppose that the eigenspace for the eigenvalue 0 for question \(A\) is a ray, while the eigenspace for the eigenvalue 0 for \(B\) is a plan, with an angle \(\pi/4\) between them. Once the answer 0 is obtained from question \(A\), the state vector is on the ray, and if \(|a_0\rangle\) is projected onto the plan, then there is by hypothesis an angle of \(\pi/4\), so \(p(b_0|a_0) = \cos^2(\pi/4)\). Now, once the answer 0 is obtained from question \(B\), the state vector can be anywhere in the plan, for instance with a right angle to the ray, so \(p(a_0|b_0)\) can be null. The reciprocity law does not hold anymore.
which \( P(b_j|a_i) = P(a_i|b_j) \times P(b_j)/P(a_i) \), so \( P(b_j|a_i) \neq P(a_i|b_j) \) as soon as \( P(b_j) \neq P(a_i) \).

For quantum-like models of judgment, the law of reciprocity is somehow well-known\(^6\), but it has not been fully investigated. In the case of our model, it can be instanciated in the following ways:

\[
\begin{align*}
    p(b_0|a_0) &= p(a_0|b_0), \\
    p(b_1|a_0) &= p(a_0|b_1), \\
    p(b_0|a_1) &= p(a_1|b_0), \\
    p(b_1|a_1) &= p(a_1|b_1).
\end{align*}
\]

For each of these equations, the left member is about questions posed in one sense (\(A\)-then-\(B\)), while the right member is about the other sense (\(B\)-then-\(A\)). This set of equations is actually equivalent to another set, in which each equation is about one order of questions:

\[
\begin{align*}
    p(b_0|a_0) &= p(b_1|a_1), \\
    p(a_0|b_0) &= p(a_1|b_1), \\
    p(b_1|a_0) &= p(b_0|a_1), \\
    p(a_1|b_1) &= p(a_0|b_0).
\end{align*}
\]

Here’s a proof of the first equation (others are similar): \( p(b_0|a_0) = 1 - p(b_1|a_0) \) because there are only two possible answers (\(b_0\) and \(b_1\)) to question \(B\); then \(1 - p(b_1|a_0) = 1 - p(a_0|b_1)\) because of eq. 11; then \(1 - p(a_0|b_1) = p(a_1|b_1)\) because there are only two possible answers to question \(A\); then \(p(a_1|b_1) = p(b_1|a_1)\) because of eq. 13.

This set of equations 14 to 17 shows that there exist some equations that a quantum-like model must satisfy even if the questions are not reversed, although the reciprocity law (eq. 9) is originally about questions in reverse orders. The reciprocity law actually gives some constraints on one unchanged experimental setup too; quantum-like models do not only put constraints when the order is change. The equivalence of the two sets of equations (reversing-the-order or not-reversing-the-order) suggests that there is nothing special about equations which compare the reverse order.

An equality exists between the eq. 14 and 15, because of eq. 10 or 13; similarly for eq. 16 and 17. So, a new set of equations, that we call the Grand Reciprocity equations, or just the GR equations, can be written:

\[
\begin{align*}
    \begin{cases}
        p(b_0|a_0) = p(a_0|b_0) = p(b_1|a_1) = p(a_1|b_1), \\
        p(b_1|a_0) = p(a_0|b_1) = p(b_0|a_1) = p(a_1|b_0)
    \end{cases}
\end{align*}
\]

This set of equations is equivalent to either of the two previous sets, but its form is more explicit, so we shall prefer it. It is also equivalent to the reciprocity law itself, because it states it for all possible cases in the model. Further, the two equations 18 and 19 are equivalent to one another, because \( p(y_0|x_i) + p(y_1|x_i) = 1 \) by definition. So, each of these equations is actually equivalent to the other sets and to the reciprocity law itself.

To the best of our knowledge, these equations have not yet been fully written in the literature on quantum-like models. Many papers note the law of reciprocity and the

consequences for conditional probabilities (cf. footnote 6), but always for one order of the questions only.

The GR equations set the value for all possible conditional probabilities: among the eight quantities that can be experimentally measured, there is just one free real parameter. In the case of a real Hilbert space, the origin of this constraint is to be found in the \( \delta \) angle between the two bases (cf. fig. 3).

\[ \langle a_0 | b_0 \rangle, \langle a_0 | b_1 \rangle, \langle a_1 | b_0 \rangle, \langle a_1 | b_1 \rangle \]

\[ \langle b_0 | a_0 \rangle, \langle b_0 | a_1 \rangle, \langle b_1 | a_0 \rangle, \langle b_1 | a_1 \rangle \]

Figure 3: A graphical illustration of the Grand Reciprocity equations, in the special case of a real Hilbert space. A \( \delta \) angle is enough to define the two orthonormal bases relatively from one another. Then, conditional probabilities, of the form \( |\langle x_i | y_j \rangle|^2 \), are either equal to \( \cos^2(\delta) \) [Left] or to \( \sin^2(\delta) \) [Right], with a sum to 1.

### 3.2 Generalizations

Let us now consider some generalizations of the GR equations to cases which go beyond the quantum-like model presented in Section 2.

First, the GR equations have been shown for a single agent; by averaging, they obviously still hold for any population of \( N \) agents, with various initial state vectors \( |\psi_k\rangle \) \((k \in \{1, \ldots, N\}\). Another possible generalization concerns the basis vectors: what if they depend on the agent \( k \), with \( |a_{0,k}\rangle, |a_{1,k}\rangle, |b_{0,k}\rangle, |b_{1,k}\rangle \)?7 In this case, the above GR equations can be established for each agent \( k \) in the same way as previously, except that the terms are now indexed for each agent \( k \), in the form \( p_k(x|y) = p_k(z|t) \) (or respectively \( = 1 - p_k(z|t) \)), with \( (x,y,z,t) \in \{a_{0,k}, a_{1,k}, b_{0,k}, b_{1,k}\}^4 \). For the whole population, \( p(x|y) \) is now defined by the average of the statistical probabilities \( p_k \) on all \( N \) agents:

\[ p(x|y) = \frac{1}{N} \Sigma_k p_k(x|y), \quad (20) \]

and similarly for \( p(z|t) \). But because for any \( k \), \( p_k(x|y) = p_k(z|t) \), then

\[ \frac{1}{N} \Sigma_k p_k(x|y) = \frac{1}{N} \Sigma_k p_k(z|t). \quad (21) \]

So finally \( p(x|y) = p(z|t) \) (or respectively \( = 1 - p(z|t) \)), and the GR equations hold for the whole population in case the basis vectors depend on the agent. This is just because it holds for any agent, and that the average on the agents is a linear operation that enables to keep the form of the equations.

7This situation is considered for instance by Busemeyer et al. 2011 p. 212.
Also, quantum systems can be described with more general mixed states, instead of pure states. Here is the difference: $|\psi\rangle$ has been supposed to be equal to a vector from the Hilbert space, i.e. a pure state, and this supposes that we know to which vector the state is equal. But there are cases in which we don’t know exactly to which vector the state is equal (for instance, we only know that there are 50% chances that the state is $|a_0\rangle$, and 50% chances that it is $|a_1\rangle$). To express this, the belief state is considered as a statistical ensemble of pure states, and it is represented by a density matrix (for instance, we write $\rho = 0.5|a_0\rangle\langle a_0| + 0.5|a_1\rangle\langle a_1|$ for the above case). The question is then: are the GR equations valid for pure states only, or also for mixed states? For both, because no particular hypothesis has been made on the state before the first question is asked. What matters for the demonstration is the state after the first question, and it is doomed to be the corresponding eigenvector (up to a phase factor), whether the belief state was described as a statistical mixture or not before the question was asked. So, the GR equations hold even if mixed states are assumed.

The hypothesis that eigenvalues are non-degenerate, i.e. that eigenspaces are of dimension 1, can be relaxed in some cases. Suppose a model $M$ with eigenspaces of dimensions greater than 1 is empirically equivalent to a model $M'$ with eigenspaces of dimension 1. So to speak, the supplementary dimensions of $M$ are theoretically useless, and empirically meaningless. As the GR equations hold for $M'$, it will also hold for $M$, to which it is equivalent. So, more generally, the GR equations hold for models which have eigenspaces of dimension 1, or which are reducible (i.e. equivalent) to it.

The GR equations concern a model with 2 questions with 2 possible answers each, but both these parameters can be easily extended. Consider a model with $q$ dichotomous questions $A, B, C, \ldots$ that are asked in a row, in various orders, with non-degenerate eigenvalues, in a Hilbert space of dimension 2. For the questions considered 2 by 2, the reciprocity law (eq. 9) holds, and so the GR equations hold too. This gives a great set of GR equations (actually, $q(q-1)/2$ equations like eq. 18 or 19) that this model must satisfy. Consider now a model with 2 questions, each with $r$ possible answers, with non-degenerate eigenvalues, and so in a complex Hilbert space of dimension $r$. The reciprocity law still holds in this case, and GR equations hold for each couple of indexes $(i, j) \in \{1, \ldots, r\}^2$. The conclusion is that the GR equations can be generalized to apply for models with $q$ questions and $r$ possible answers, as soon as eigenvalues are non-degenerate.

These results may have a larger impact still. Suppose that there exists a set of elementary questions, the answers of which would be represented by non-degenerate eigenspaces. These elementary rays would define the fundamental basis belief states, on which any belief could be decomposed. The number of such elementary rays would define the dimensionality of the Hilbert space representing human beliefs. A similar idea is expressed, for emotions, by Pothos and Busemeyer:

"one-dimensional sub-spaces (called rays) in the vector space would correspond to the most elementary emotions possible. The number of unique elementary emotions and their relation to each other determine the overall dimensionality of the vector space. Also, more general emotions, such as happiness, would be represented by subspaces of higher dimensionality." (Pothos and Busemeyer 2013, 258).

Generalized GR equations would hold for the elementary questions. Then, all models with degenerate eigenspaces would just be coarse-grained models, which could be refined with more elementary questions and rays. For instance, a degenerate answer represented by a plan would combine two elementary dimensions because it failed to distinguish between them. As a consequence, the meaning of a degenerate answer could be specified through the
more specific elementary answers, and the degenerate model could actually be tested, with
generalized GR equations, on these fundamental questions. In other words, if elementary
questions existed, any model could be tested with generalized GR equations.

3.3 Link with double stochasticity

Parts of the GR equations are actually known in the literature as the requirement of
“double stochasticity”. Let us analyse the links between them.

Define the change of basis matrix $\mu^{a,b}$ between the two bases $(|a_0\rangle, |a_1\rangle)$ and $(|b_0\rangle, |b_1\rangle)$, as:

$$\mu^{a,b} = \begin{pmatrix} \langle a_0 | b_0 \rangle & \langle a_0 | b_1 \rangle \\ \langle a_1 | b_0 \rangle & \langle a_1 | b_1 \rangle \end{pmatrix}. \quad (22)$$

As the two bases are orthonormal by hypothesis, this matrix is unitary. From it, a transi-
tion matrix $T^{a,b}$ can be defined by $T^{a,b}_{ij} = |\mu^{a,b}_{ij}|^2$:

$$T^{a,b} = \begin{pmatrix} |\langle a_0 | b_0 \rangle|^2 & |\langle a_0 | b_1 \rangle|^2 \\ |\langle a_1 | b_0 \rangle|^2 & |\langle a_1 | b_1 \rangle|^2 \end{pmatrix} = \begin{pmatrix} p(a_0 | b_0) & p(a_0 | b_1) \\ p(a_1 | b_0) & p(a_1 | b_1) \end{pmatrix}. \quad (23)$$

$T^{a,b}$ contains the probabilities for the answers to question A given the previous answer to
question B ($T^{a,b}$ could be also called $T^{A|B}$).

Transition matrices are left stochastic: they are square matrices of non-negative real
numbers, of which each column sums to 1. This expresses the fact that, once an answer
has been given to the first question, there is a probability 1 that one of the answers is
given to the second question. A matrix is said to be doubly stochastic in case all columns
and all rows sum to 1. Saying that $T^{a,b}$ is doubly stochastic amounts to the following
equations:

$$\begin{cases} p(a_0 | b_0) + p(a_1 | b_0) = 1 & \quad (24) \\ p(a_0 | b_1) + p(a_1 | b_1) = 1 & \quad (25) \\ p(a_0 | b_0) + p(a_0 | b_1) = 1 & \quad (26) \\ p(a_1 | b_0) + p(a_1 | b_1) = 1 & \quad (27) \end{cases}$$

By substracting these equations one by one, one gets eq. 15 and 17 — but nothing more⁸.
So, one gets one half of the reciprocity relations. This should be no surprise: the double
stochasticity constraint bears on only one experiment, when the order of the questions
is the same, and cannot be informative about the reverse order. It is easy to see that
if the double stochasticity of the reverse transition matrix $(T^{b,a})$ is assumed, then one
gets the other half of our reciprocity relations (eq. 14 and 16), and then also the GR
equations. Conversely, our GR equations imply the double stochasticity of both transition
matrices. So, the GR equations are equivalent to the requirement of double stochasticity for
transition matrices in both senses. Yet, an advantage of the GR equations is to present in a
single equation all the relations that hold, instead of having to write anew the consequences
of double stochasticity for two matrices. Also, the link between the GR equations and a
fundamental law of quantum mechanics, namely the law of reciprocity, is straightforward,
whereas double stochasticity of transition matrices is usually no presented as a fundamental
law of quantum mechanics.

⁸eq. 14 and 16 cannot be derived from eq. 24–27. Indeed, it is easy to imagine a case in which the
former are false, whereas the latter are true, because they bear on questions asked in different orders.
In the literature, double stochasticity is a well-known property of transition matrices for quantum-like models\footnote{Cf. for instance Busemeyer and Bruza (2012, 53–54), Busemeyer, Wang and Lambert-Mogiliansky (2009), Pothos and Busemeyer (2013, 269).}, but it has generally been required for only one transition matrix (i.e. for questions posed in one order). An exception is Khrennikov (2010, 24 and 36), who studies double stochasticity for transition matrices in both senses, and shows that it must be verified by quantum-like probability models with non-degenerate eigenvalues, i.e. like in our general model. However, he doesn’t insist on testing experimentally this property in a systematic way, as we shall do here, with important consequences for existing models.

3.4 Link with the QQ equality

Wang and Busemeyer (2013) and Wang, Solloway, Shiffrin and Busemeyer (2014) defend the test of a relation called “the QQ equality”, that they present as a consequence of the law of reciprocity. It can be derived from our GR equations in the case of our quantum-like model of Section 2. With the notations of our model (and adding the notation $p_{XY}(z_i)$ for the probability to answer $i$ to question $Z$, when posed in the order $X$-then-$Y$), the QQ equality (Wang and Busemeyer 2013, 698) can be written as

$$p(a_0, b_1) + p(a_1, b_0) = p(b_0, a_1) + p(b_1, a_0),$$

that is,

$$p(b_1|a_0)p_{AB}(a_0) + p(b_0|a_1)p_{AB}(a_1) = p(a_1|b_0)p_{BA}(b_0) + p(a_0|b_1)p_{BA}(b_1).$$

Because $p_{XY}(z_0) + p_{XY}(z_1) = 1$, it is equivalent to:

$$p(b_1|a_0)p_{AB}(a_0) + p(b_0|a_1)[1 - p_{AB}(a_0)] = p(a_1|b_0)p_{BA}(b_0) + p(a_0|b_1)[1 - p_{BA}(b_0)],$$

which can be rewritten as:

$$[p(b_1|a_0) - p(b_0|a_1)]p_{AB}(a_0) + p(b_0|a_1) = [p(a_1|b_0) - p(a_0|b_1)]p_{BA}(b_0) + p(a_0|b_1).$$

The square brackets are null because of eq. 16 and 17, and the two remaining terms are equal because of eq. 19. So, the QQ equality is demonstrated from the GR equations. Note however that the QQ equality remains true in the case the eigenspaces are not of dimension 1, cf. Wang and Busemeyer (2013, 18). Below, Section 5 gives an example of data which satisfy the QQ equality, but not our GR equations, and this shows that the two tests are not equivalent.

3.5 Testing the Grand Reciprocity equations

We suggest that (one of) the GR equations be directly tested experimentally for any quantum-like model to which it applies. By “directly tested”, we mean that what should be measured are the conditional probabilities — or data that enable their computation. Consequences that are not equivalent to the GR equations (like the QQ equality) are not suitable. Also, we urge to submit all possible models to the test. The GR equations apply to the quantum-like model presented in Section 2, but also to other quantum-like models that incorporate its hypotheses (cf. subsequent sections for examples).

Why testing these GR equations? First, they are a plain experimental prediction of the model — shouldn’t any model be empirically adequate? — and the needed data are basic ones, easy to get experimentally. Second, the GR equations are each equivalent to
a fundamental property of quantum-like models, namely the law of reciprocity. It is not a remote consequence of it (like the QQ equality is), but it is just the expression of this law for all possible cases in the model. Testing the GR equations is exactly testing the reciprocity law. The benefit is that testing the GR equations is more economical: they state that only 4 conditional probabilities need to be compared to test the law in general. Another reason to test the GR equations is that a classical model does not verify them; so they can be seen as a test of the quantum-like character of the data\textsuperscript{10}.

Note that, to be tested, the GR equations require that the experiment be conducted in both question order (A-then-B and B-then-A). If only one question order is studied, then the test can be done, but it is only partial. So, whenever a quantum-like model of the kind depicted here is proposed to account for the results of a succession of two questions A-then-B, the model should also be tested for the reverse order B-then-A.

4 Applying the GR equations to existing quantum-like models for order effect

As indicated in the introduction, empirically testing quantum-like models is a huge task, and we shall be content here with testing only one kind of models. As non-commutativity is a well-known feature in quantum mechanics, order effect models are perhaps the most straightforward application for quantum mathematics — as noted in Sect. 2, our general quantum-like model displays such order effects. In this section, we review the quantum-like models for order effects to which the GR equations apply. For that, we only need to show that they entail the general quantum-like model from Section 2, and in particular that the answers are represented with subspaces of dimension one. In this section, we are only concerned with theoretical aspects of the models — the empirical test of these models with the GR equations, when some data is available, is considered in the next section. Our general model is entailed by several papers in the literature.

Consider first Conte \textit{et al.} (2009), who propose a quantum-like model to account for order effects in mental states during visual perception of ambiguous figures. Their model can be cast into the lines of our general model: the two dichotomous questions or tests are also called A and B, and the probabilities of the answers are noted \( p(A = +) \), \( p(A = -) \), \( p(B = +) \) and \( p(B = +) \), corresponding to our \( p(a_0), p(a_1), p(b_0) \) and \( p(b_1) \). The questions concern visual perception of ambiguous figures, and a vector state, noted here \( \phi \) (instead of our \( \psi \)) represents the state of consciousness about the perception. It belongs to a complex Hilbert space, and an answer is represented by a one-dimensional subspace (this is implicit in their formula (3)). The usual Born rule makes the link with probabilities. The model of Conte \textit{et al.} also involves a projection postulate: during perception, the “potential” state of consciousness is collapsed onto “an actual or manifest state of consciousness” (p. 6 and 7). Note however a slight difference: for Conte \textit{et al.}, the projection of the state arises during perception of the figure, not during the answering to a question. But as both are quickly followed by one another, and as an agent answers a question only once her perception has stabilized, this does not make any difference in practice. Overall, since Conte \textit{et al.} (2009)’s model matches our general model, it has to obey the GR equations.

Consider now the model proposed by Wang and Busemeyer (2013), called the QQ model, which is to account for several types of order effects. The clear presentation of the

\textsuperscript{10}Indeed, the GR equations are equivalent to the reciprocity law, which is not true in general for classical models (cf. Section 3.1). Note that classical models which satisfy the GR equations are uninteresting: all \( P(x_i) \) and all \( P(x_i | y_j) \) must be equal to 0.5.
model in their Section 2, together with their geometric approach which provides figures similar to ours (in the case of a real Hilbert space), easily enables to see that it matches our own general model. There is one difference only: they explicitly allow an answer to be represented either with a subspace of dimension 1, or with a subspace of dimension larger than 1. So, the GR equations apply only to the former case. For instance, they apply to figures 1 and 2 (p. 4 and 5), which represent “a two-dimensional example of the quantum-like model of question order effects”.

Wang and Busemeyer’s model is also at the basis of two other presentations: in Busemeyer and Bruza’s 2012 book (p. 99–116), and in the review article of Pothis and Busemeyer (2013). As these presentations do not change the model, the GR equations apply there with the same conditions.

Atmanspacher and Römer (2012, p. 277) make the suggestion that Wang and Busemeyer’s model could be extended to mixed states, instead of just pure states. Although this is an interesting theoretical possibility, it must be clear that it cannot be used to avoid the GR equations, since they are valid for pure as for mixed states (cf. Sect. 3.2). In other words, if some data sets do not respect the GR equations, then Wang and Busemeyer’s non-degenerate pure state model is ruled out, and so would be a mixed state version of it.

Building on the paper of Wang and Busemeyer (2013), Wang et al. (2014) take up the same quantum-like model for order effect, in order to apply it to a much larger set of experiments. Here again, answers can be represented with subspaces of any dimension. So, the GR equations apply only to the special case of models with non-degenerate answers, i.e. represented with rays.

5 An empirical glimpse

In this section, we address the issue of statistically testing the GR equations for the models that should respect them, and that have been discussed in the previous section. Actually, the papers themselves provide some needed empirical data, from either laboratory or field experiments, that enable us to carry out this statistical investigation. To study these empirical data, we divide them in two sets: first, in Sections 5.1 to 5.3, the data concern experiments about judgment on social or political questions, which enable to test the models proposed or considered by Busemeyer and Bruza (2012), Pothis and Busemeyer (2013), Wang and Busemeyer (2013) and Wang et al. (2014); second, in Section 5.4, the data concern an experiment about visual perception, and it enables us to test the model proposed by Conte et al. (2009) — another reason to test separately this last model is that the experiment is run on a low number of respondents.

5.1 The data set

Our first data set gathers 72 experiments on order effects, with 70 national surveys made in the USA plus two laboratory experiments:

- three surveys from Moore (2002) concerning Gallup public opinion polls — they are considered in Wang and Busemeyer (2013) and in Wang et al. (2014). The questions of these surveys are noted in Table 1 as experiments 1 to 3;
- one further Gallup survey reported by Schuman et al. (1981), also considered in Wang and Busemeyer (2013) (experiment 4);
- sixty-six national surveys from the Pew Research Center run between 2001 and 2011 about politics, religion, economic policy and so on, considered in Wang et al. (2014)
two laboratory experiments conducted by Wang and Busemeyer (2013). In the first experiment, the authors replicate in a laboratory setting the experiment proposed by Moore (2002) about racial hostility, whereas, in the second one, they replicate the survey of Wilson et al. (2008) (experiments 71 and 72).

Table 1: Questions of the experiments included in the first data set.

1. A. Do you generally think Bill Clinton is honest and trustworthy?
   B. Do you generally think Al Gore is honest and trustworthy?

2. A. Do you think Newt Gingrich is honest and trustworthy?
   B. Do you think Bob Dole is honest and trustworthy?

3. A. Do you think that only a few or many white people dislike black people?
   B. Do you think that only a few or many black people dislike white people?

4. A. Do you think it should be possible for a pregnant woman to obtain a legal abortion if she is married and does not want any more children?
   B. Do you think it should be possible for a pregnant woman to obtain a legal abortion if there is a strong chance of serious defect in the baby?

5-40 (26 experiments).

A. Do you approve or disapprove of the way Bush/Obama is handling the job as President?
B. All in all, are you satisfied or dissatisfied with the way things are going in this country today?

41-55 (15 experiments).

A. Do you approve or disapprove of the job the Republican leaders in Congress are doing?
B. Do you approve or disapprove of the job the Democratic leaders in Congress are doing?

56-70 (25 experiments).

These experiments include diverse questions covering topics, from religious beliefs to support to economic policy.

71. A. Do you think that only a few or many white people dislike black people?
   B. Do you think that only a few or many black people dislike white people?

72. A. Do you generally favor or oppose affirmative action (AA) programs for racial minorities?
   B. Do you generally favor or oppose affirmative action (AA) programs for women?

Figure 4 reports the box plot of the distribution of the total number of subjects per experiment. The median experiment (741 subjects) is thus supposed to involve half of the samples (about 370 subjects) in the A-then-B treatment and the remaining half in the B-then-A treatment. The circle on the left part of the box-plot, outside the left whisker, pinpoints the two laboratory experiments 71 and 72, which involve the smallest sample,
that is, 228 subjects.

Each experiment can be described according to the following common model and notations: two response categorical variables,

- the bernoulli random variable $A \in \{a_0, a_1\}$ that represents the possible replies (respectively “yes” or “no”) to the dichotomous question $A$,

- the bernoulli random variable $B \in \{b_0, b_1\}$ that represents the possible replies (respectively “yes” or “no”) to the dichotomous question $B$,

and one explanatory categorical variables,

- the variable $O \in \{0, 1\}$ that represents the order of appearance of the questions — 0 for the $A$-then-$B$ order and 1 for the $B$-then-$A$ order.

Empirically, what is observed is not conditional probabilities like $p(a_i|b_j)$, but joint frequencies $n(b_j, a_i)$, that is, the outcome of the counting process of the people responding $j$ to the first question $B$ and then $i$ to the second question $A$ (the order matters in this notation: $n(a_i, b_j)$ refers to the $A$-then-$B$ experiment). This information can be reported as a contingency table (Table 2).

Table 2: The contingency table of a generic experiment.

<table>
<thead>
<tr>
<th></th>
<th>$O = 0$</th>
<th>$O = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B = b_0$</td>
<td>$B = b_1$</td>
</tr>
<tr>
<td>$A = a_0$</td>
<td>$n(a_0, b_0)$</td>
<td>$n(a_0, b_1)$</td>
</tr>
<tr>
<td>$A = a_1$</td>
<td>$n(a_1, b_0)$</td>
<td>$n(a_1, b_1)$</td>
</tr>
</tbody>
</table>

Validity conditions for hypothesis testing are sensitive to the number of counts — it is commonly required at least 5 counts for the joint frequencies. To this end, Table 3 reports some statistical information about the distributions of the joint frequencies over the 72 experiments. In particular, Table 3 focuses on the region of the distributions corresponding to small values: the rows report the first decile of each distribution (D1 - 10%), the smallest number of subjects occurred in one experiment (min) and the number of experiments presenting a joint frequency value lower than 5. Summarizing, more than 90% of the experiments have joint frequency values greater or equal to 5, 5 out of 8 joint frequencies never present values less than 5, and only 6 experiments present for only one joint frequency a value smaller than 5.
We suppose here that (eq. 33), we can perform the following test: frequencies (cf. Appendix A). For instance, instead of computing the first statistical test

\[ \log(OR) = \log \left( \frac{f(a_0|b_0) f(b_1|a_1)}{f(a_1|b_0) f(b_0|a_1)} \right) = 0. \]

A continuity correction is also applied, because the normal approximation to the binomial is used, which is effective in particular for small values of \( n(a_i, b_j) \) or \( n(b_j, a_i) \). The above test becomes:

\[ \log(OR) = \log \left( \frac{(n(a_0, b_0) + 0.5)(n(b_0, a_1) + 0.5)}{(n(a_1, b_0) + 0.5)(n(b_0, a_0) + 0.5)} \right) = 0. \]

We suppose here that

\[ \frac{\log(OR)}{\text{SE}_{OR}} \sim N(0,1), \]
where $SE_{OR}$ is the standard error of the log odds ratio. It is estimated as the square root of the sum of the inverse of all the joint frequencies that are considered in the estimation of the OR:

$$
SE_{OR} = \sqrt{\frac{1}{n(a_0, b_0)} + \frac{1}{n(b_0, a_1)} + \frac{1}{n(a_0, b_1)} + \frac{1}{n(b_0, a_0)}}.
$$

Finally, a Bonferroni correction of the type I error is adopted because of the multiple comparisons (six). Note that this statistical correction is a very conservative one, which means that it makes false positive rejections much less liable to occur. Concretely, the two-tailed test implies the null hypothesis of equality between the two conditional frequencies at the $K\%$ significance level is rejected if:

$$
p-value = 1 - CDF_{stdNorm}\left(\frac{|\log(OR)|}{SE_{OR}}\right) \leq \frac{K}{2 \cdot 100 \cdot 6}.
$$

where $CDF_{stdNorm}$ is the cumulative distribution function of the standard normal distribution (mean = 0 and standard deviation = 1).

5.3 Results

Recall that only one rejection of the six tests T1-T6 is sufficient for a quantum-like model to be considered as empirically inadequate. In this sense, satisfying 5 tests over 6 is not more desirable than satisfying 1 — this is an all-or-nothing problem, or a “zero-or-non-zero” rejection matter. However, for the sake of the presentation of the results, the experiments are listed according to a ranking based on the number of test rejections per experiment at the 5% significance level: Table 6 in Appendix B reports the $p$-values of the tests (columns) for every experiment (rows), that is, 6 columns and 72 lines in total.

First, take a general viewpoint on the rejection question. Some summary statistics are provided in Table 4 relative to the distribution of number of test rejections per experiment (left part) or per test (right part). The rejection rate per experiment is high, above 3. The distribution of the test rejections is highly skewed with respect to a uniform distribution. More than 75% of the experiments (Q1 - first quartile) reject half or more than half of the 6 tests and both the median and the mode are equal to 4. The key point is that the first decile (D1) is equal to 1, that is, more than 90% of the experiments exhibit at least one test rejection, thus confirming the hypothesis that quantum-like modeling in which one dimensional subspaces (rays) represent answers to questions is not empirically adequate, at least for the experiments considered in our data set.

<table>
<thead>
<tr>
<th>Number of rejections per experiment</th>
<th>Number of rejections per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean 3.63</td>
<td>T1: $f(b_0</td>
</tr>
<tr>
<td>mode 4</td>
<td>T2: $f(b_0</td>
</tr>
<tr>
<td>Q3 (75%) 5</td>
<td>T3: $f(b_0</td>
</tr>
<tr>
<td>median 4</td>
<td>T4: $f(a_0</td>
</tr>
<tr>
<td>Q1 (25%) 3</td>
<td>T5: $f(a_0</td>
</tr>
<tr>
<td>D1 (10%) 1</td>
<td>T6: $f(b_1</td>
</tr>
</tbody>
</table>
Second, it is possible to have an individual analysis of the experiments: as the last column of Table 6 reports the experiment identifier (exp ID) adopted in Table 1, it is possible to track each question in the ranked-based list of experiments. For each experimental protocol (laboratory or field experiments, Gallup or Pew Research Center surveys and different authors), at least one experiment rejects at least one test, which shows that the result of non-satisfaction is robust also against experimental settings and questions. For instance, the two laboratory experiments (71 and 72) of the data set exhibit one test rejection — as they involve the smallest subject pools, one can conjecture that a even higher number of test rejections would occur with more subjects involved.

Now, out of the 72 experiments, focus on the 7 ones which exhibit zero rejections. They all belong to the set of 66 national surveys from the Pew Research Center. From a statistical viewpoint, they all satisfy the validity condition of the test in terms of joint frequency values, that is more than 5 counts. These experiments have $p$-values for all joint frequencies significantly greater than the 5% significance level with the Bonferroni correction (see Equation 43) that is, with respect to the reference value of 0.00417. Further investigation would be required to understand the rationale of the occurrence of no test rejection in these experiments, in order to exclude spurious statistical arguments. As a first step, Figure 5 plots the number of test rejections against the size of the subject pool among the experiments. The plot suggests that low subject pools (say, below 500) tend to reject less tests. It also suggests that it would be interesting to replicate these 7 experiments with a larger subject pool (say, 1000 or 1500), to check whether they still verify all the tests.

![Figure 5: Scatterplot of the number of test rejections versus the size of the subject pool. The red triangles pinpoint the 7 experiments with 0 test rejections, the green plus (+) the two overlapping laboratory experiments.](image)

To sum up, the 6 experiments considered in Wang and Busemeyer (2013) fail at least one test, and thus we can safely say that they cannot be accounted for with a non-

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At present, the notation 5-70 is adopted for the Pew Research Center data because we currently lack the information of how questions and experiments are linked. This point is under investigation and should soon be solved.
degenerate quantum-like model. Among the 66 extra experiments considered in Wang et al. (2014), which all come from the PEW Research Center, a very large majority (59) fail at least one test, and cannot be accounted for with a non-degenerate quantum-like model either. 7 experiments remain that, in front of the available data, can be accounted for with such a model. It would be interesting to replicate these experiments with a larger number of subjects, to check whether this still holds.

Finally, remark that 65 experiments over 72 do not satisfy the GR equations, whereas they do satisfy the QQ equality (cf. Wang et al. 2014). Given that the GR equations are equivalent to the law of reciprocity (for non-degenerate models), this shows that there exist (many) cases in which the QQ equality is true and the law of reciprocity is not. This contradicts the claim that the test of the QQ equality would amount to the test of the law of reciprocity\(^{12}\). A discussion of these results is led in Section 6.

5.4 Testing Conte et al. 2009

This section considers the experiments conducted by Conte et al. (2009) about visual perception. The validity conditions of the statistical test are not fully met because the number of subjects per experiment is around 60, posing the issue of statistical significance for such a small sample. In particular, the statistical requirement of at least 5 counts for each joint frequency is not always met in this data set. However, for the sake of completeness, the statistical tests are reported here.

4 distinct experiments are performed, each comprising two treatments involving a population of 19 to 22 year-old students. The first treatment of each experiment involves a group of subjects exposed only to test \(A\), whereas the second one to test \(B\) and “soon after” to test \(A\). Only the second (52 subjects) and the third (64 subjects) experiments implement the same visual task (subjects have to look at ambiguous figures of animal), but exchanging tests \(A\) and \(B\). Therefore, by considering only the treatment/group 2 of these two experiments, the full GR equations can be tested.

Table 5 reports the \(p\)-values for the 6 tests. As the third test is rejected at the 5% significance level\(^{13}\), here again we can conclude that a quantum-like non-degenerate model does not adequately fit the experimental data. Here, too, the small size of the sample is associated with a small number of test rejections. A more rigorous empirical test of the GR equations would require a larger subject pool.

Table 5: Experiments of Conte et al. (2012), \(p\)-values for the 6 tests. The rejection is highlighted at the 5% significance level with a *.

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\(^{12}\)This point is claimed by Busemeyer and Bruza (2012, 105), Wang and Busemeyer (2013, 10), Pothos and Busemeyer (2013, 269).

\(^{13}\)As the GR equations are not satisfied, neither is double-stochasticity in both senses (cf. Section 3.3). Actually, the fact that the data of Conte et al. (2009) fail to follow double-stochasticity is noted by Khrennikov (2010, 107 and 114).
6 Discussion: the future of quantum-like models for order effects

Beyond the negative rejection of the quantum-like models discussed in the previous section, can we offer a more constructive suggestion? And how should we consider the future of research in the modelling of order effects, after these results? Several lines of possible research can be distinguished.

First, it is true that there are other possible lines of research than standard quantum-like models. For instance, Khrennikov has proposed to consider hyperbolic Hilbert space models, instead of complex ones like in this paper. Hyperbolic models enable probabilities that are not constrained in the same way, and can account for some data that are out of reach for complex quantum-like models (see Khrennikov 2010 for a synthesis). Also, one could try to abandon or modify other axioms of standard quantum-like models, like the projection postulate or the Born rule.

But all quantum-like models have not been ruled out here: those to which the GR equations do not apply, i.e. which consider degenerate eigenvalues, have not been tested. So, a natural line of research is to turn to degenerate quantum-like models, where answers are represented by sub-spaces of dimension 2 or larger. For instance, this is the solution adopted by Busemeyer, Wang and Lambert-Mogiliansky (2009): after showing that a 2D quantum-like model is not doubly stochastic (which implies that it doesn’t respect our GR equations), they switch to a 4D model, in which answers are represented by plans. In the same way, our results do not indicate the end of quantum-like models in general. They can be seen as suggesting instead more research on quantum-like models, albeit on degenerate ones.

A strong suggestion in favor of this line of research comes from an intriguing fact. While the non-degenerate quantum-like model fails to respect the GR equations on a vast majority of experimental data sets, it remains true that the QQ equality is experimentally verified in all these cases. This might be considered as surprising, because the QQ equality has been derived from the quantum-like model itself. But actually, it has been derived from quantum-like models in general, whether they are degenerate or not (cf. Section 3.4) — and not only from a non-degenerate model that turns out to be empirically inadequate. Obviously, if degenerate models were true for these experiments, then the QQ equality would hold. So, the fact that the QQ equality is valid for data sets for which the GR equations are not, might be interpreted as a sign that degenerate quantum-like models of higher dimensions are actually true.

However, some difficulties with switching to degenerate models have to be acknowledged. First is the question whether the replacing degenerate model is empirically equivalent to, and thus can be reduced to, a non-degenerate model (cf. Section 3.2). If this is the case, then the GR equations do apply. So, once a non-degenerate model has been ruled out, a degenerate model is not an escape route unless it is shown not to be reducible to a non-degenerate one. Second, degenerate models are not a priori freed from any constraint. Some other equations than the GR equations might hold — some research is urgently needed here. So, we suggest that, when research efforts are made to have degenerate models fit some data sets, similar efforts be also directed towards the testing of these models, in all possible forms, so as to prevent later “bad surprises”. Thirdly, as discussed in the end of Section 3.2, there is the possibility that degenerate eigenspaces actually hide non-degenerate and more fundamental eigenspaces. If so, then degenerate models can be tested against generalized GR equations for these more fundamental questions (such GR equations might help to look for the constraints that degenerate models must satisfy in
general, as suggested just above). Another difficulty with developing degenerate models is that introducing supplementary dimensions of degeneracy should be somehow justified, so as not to be accused of being just *ad hoc*. In quantum physics for instance, degeneration is usually theoretically justified with a symmetry in the Hamiltonian, and the degeneration can be experimentally removed by introducing some factor that was not present (e.g., the spin degeneration is removed by introducing a magnetic field). For quantum-like judgment models as well, one could try to provide a similar theoretical justification and an experimental way of removing the degeneration. In this respect, the idea that there exists fundamental questions and rays could be helpful; the work is then to find out which fundamental dimensions are hidden behind the degenerate eigenspace.

In spite of the possible difficulties that we have just discussed, we remain convinced that quantum-like models are a promising research line. They have brought to discussion many provoking and seminal ideas, such as the hypotheses that preferences might be underdetermined instead of only unknown, or that non-classical probabilities could be considered. At the same time, our contribution has clarified some important aspects of quantum-like probabilistic models applied to human cognition. In the past, these aspects have been neglected by the quantum literature. We believe that our results pose the appropriate challenge to the scientific community to guide them on where to make future investigations in the field. Indeed, our results suggest that non-degenerate quantum-like models should be considered more as toy models than as empirically adequate models, and that future investigations should focus on degenerate models.

**Acknowledgments**

We have greatly benefited from comments and suggestions from Corrado Lagazio for the statistical part, and from Ariane Lambert-Mogiliansky on the whole structure of the paper. Thanks also to Camille Aron and Gabriel Lemarié for theoretical helps. Of course, we are solely responsible for our analysis and conclusions. We further thank Zheng J. Wang, Tyler Solloway, Richard M. Shiffrin, and Jerome R. Busemeyer who kindly gave us the opportunity to use their data set to test our theoretical conditions.

**References**


of Mathematical Psychology 53: 423–433.


Khrennikov, Andrei (2010), Ubiquitous Quantum Structure. From Psychology to Finance, Heildelberg: Springer.


A  Testing the equality of two conditional relative frequencies

The statistical test is to compare two conditional relative frequencies $y$ and $x$, with the null hypothesis that they are equal. The test is therefore

$$y = x,$$  \hspace{1cm} (44)

where both $y$ and $x$ are observed conditional relative frequencies.

Testing equation 44 is equivalent to test

$$\log \left( \frac{y}{1-y} \right) = \log \left( \frac{x}{1-x} \right),$$

given that $y$ and $x$ are not equal to zero.

Alternatively, we can formulate the test in terms of the log odds ratio (OR)

$$\log(\text{OR}) = \log \left( \frac{y}{1-y} \right) = 0.$$  \hspace{1cm} (45)

Now let’s suppose that we want to test

$$f(a_0|b_0) = f(b_0|a_0).$$

We can thus test the following condition:

$$\log \left( \frac{f(a_0|b_0)}{1 - f(a_0|b_0)} \right) = \log \left( \frac{f(b_0|a_0)}{1 - f(b_0|a_0)} \right),$$

or

$$\log \left( \frac{f(a_0|b_0)}{f(a_1|b_0)} \right) = \log \left( \frac{f(b_0|a_0)}{f(b_1|a_0)} \right).$$

By expressing the conditional relative frequencies in terms of joint frequencies, that is,

$$f(a_0|b_0) = \frac{n(b_0,a_0)}{n(b_0,\cdot)}, \ f(b_0|a_0) = \frac{n(a_0,b_0)}{n(a_0,\cdot)}, \ldots,$$

being $n(b_0,\cdot)$ and $n(a_0,\cdot)$ the marginal frequencies of $B$ and $A$, respectively, we obtain

$$\log \left( \frac{n(b_0,a_0)}{n(b_0,\cdot)} \cdot \frac{n(b_0,\cdot)}{n(b_0,a_1)} \right) = \log \left( \frac{n(a_0,b_0)}{n(a_0,\cdot)} \cdot \frac{n(a_0,\cdot)}{n(a_0,b_1)} \right),$$

or simplifying

$$\log \left( \frac{n(b_0,a_0)n(a_0,b_1)}{n(b_0,a_1)n(a_0,b_0)} \right) = 0.$$  \hspace{1cm} (46)

We can thus test indifferently eq. 45 or 46.
## Results of the tests for the first data set

Table 6: *p*-values of the 6 tests for the 72 experiments of the first data set. The rejection is highlighted at the 5% (1%) significance level with a * (**) . The 7th column reports the \#R at the 5% significance level, and the 8th one reports the experiment ID (cf. Table 1).

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