E PLURIBUS UNUM: MACROECONOMIC MODELLING FOR MULTI-AGENT ECONOMIES

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E Pluribus Unum: Macroeconomic Modelling for Multi-agent Economies*

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Abstract

From the macroeconomist’s viewpoint, agent based modelling has an obvious drawback: It makes impossible to think in aggregate terms. The modeller, in fact, can reconstruct aggregate variables only "from the bottom up" by summing the levels of a myriad of individual variables. We propose a modelling strategy which reduces the dimensionality of an agent based framework by replacing the actual distribution with the first and second moments of the distribution itself. We put this strategy at work in a Macroeconomic and Agent Based Model (M&ABM) model of the financial accelerator in which firms’ heterogeneous degree of financial robustness affect investment in a bankruptcy risk context à la Greenwald-Stiglitz.

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1 Introduction

It is almost a commonplace that the Representative Agent (RA) assumption is inadequate to deal with multi-agent economies characterized by persistent and relevant heterogeneity. Heterogeneity is persistent when agents’ differences – in the present paper heterogeneity of financial conditions across firms – is not bound to disappear "in the long run". Heterogeneity, even if persistent, can be irrelevant if the second and higher moments of the distribution of agents’ characteristics play an insignificant role in explaining aggregate behaviour. In this case, in fact, the dynamics of the average agent captures almost all of the dynamics of the aggregate.¹

If heterogeneity is both persistent and relevant, the RA assumption should be dismissed and the analysis should identify and track the evolution of the entire distribution of agents’ characteristics over time. Starting from heterogeneous behavioural rules at the micro level, the aggregate (macroeconomic) variable – such as GDP – can be determined from the bottom up, i.e. by adding up the levels of a myriad of individual variables. The increasing availability of computational power has allowed the implementation of this procedure in multi-agent (or agent based) models.²

Multi-agent modelling therefore is the most straightforward way of tackling the heterogeneity issue. From the point of view of the macroeconomist, however, it has a destructive consequence. The dimension of the model explodes. In an agent-based framework macroeconomic thinking – i.e. thinking in terms of aggregate or macro-variables – is prima facie impossible.

The main message of the present paper is that the difficulty of thinking in macroeconomic terms when dealing with multi-agent economies can be circumvented by means of an appropriate aggregation procedure – which we label the Modified-Representative Agent (MRA) – that essentially approximates the evolution over time of the entire distribution of agents’ characteristics by means of the dynamics of a finite set of moments of the distribution. In the following we adopt the minimal version of the procedure which employs only

¹In Krusell and Smith (1998), for example, aggregate behaviour is explained almost entirely by the dynamics of the first moment – i.e. the mean – of the distribution of wealth. If heterogeneity is persistent but "does not matter", the modeller is entitled to ignore higher moments of the distribution and rely comfortably again on the Representative (average) Agent as a reasonable approximation to reality.

²In the following we will use the expressions agent-based model or multi-agent model as synonymus. See Tesfatsion (2006) for a thorough introduction to agent based modelling.
the first and second moments. The bottom line is that *the mean and the variance of the distribution play the role of macroeconomic variables*. In this way one can resume macroeconomic thinking in a multi-agent framework.

As a benchmark for the application of this methodology, we build a *Macroeconomic and Agent Based Model (M&ABM)* as follows. First we build a *macroeconomic* framework of the Greenwald-Stiglitz type starting from the assumption that firms differ from one another according to their financial robustness, captured by the ratio of the equity base or net worth to the capital stock (equity ratio for short). This framework can be characterized as an optimizing IS-LM model in which the moments of the distribution of the equity ratio – i.e. the cross-sectional mean and variance – determine the average or economywide external finance premium which is a shifter of the IS schedule. In the macroeconomic equilibrium, therefore, the interest rate and output turn out to be functions of these moments.

In order to determine the dynamics of the moments, we have to build and calibrate a *multi-agent model* of the firms’ equity ratio. From the synthetic data obtained through simulations we estimate a linear dynamic system which describes the evolution over time of the *mean and the variance of the distribution* of the equity ratio.

In the "long run" – i.e. when the system settles in the steady state – the distribution reaches a configuration which is summarized by the steady state cross sectional mean and variance of the equity ratio. In other words, we extract from the simulated data an ergodic process such that the actual distribution of the equity ratio converges over time to a long run stable distribution.

The long run moments determine the long run external finance premium, which in turn determines the equilibrium interest rate and output gap.

We put the model to work exploring the impact of macroeconomic shocks, such as a *financial shock* – i.e. a sudden exogenous increase of the probability of bankruptcy – or a *monetary shock*, i.e. an increase of money supply.

Following a negative financial shock the IS curve shifts down along the LM curve on impact. This is the first round effect. The increase of the probability of bankruptcy then activates a second round effect which takes the form of a *financial amplification* mechanism. The decrease of the average equity ratio and the increase of the variance make the external finance premium increase, pushing the IS curve further down. *Heterogeneity contributes to amplification* because the increase in dispersion contributes to the increase of the external finance premium.
As to the monetary shock, on impact the LM curve shifts down along the IS curve, reducing the interest rate. The first round effect on the interest rate activates a financial amplification mechanism. The increase of the average equity ratio and the decrease of the variance make the external finance premium decrease. This second round effect pushes the IS curve up along the new LM schedule. Heterogeneity contributes to amplification because the decrease in dispersion contributes to the decrease of the external finance premium.

We have performed some Montecarlo experiments to evaluate the robustness of our results. While the results concerning a financial shock are robust, the results concerning the macroeconomic effects of the monetary shock are indeed not very robust. Our goal in the present paper, however, is essentially methodological: We want to show how to restore the macroeconomic intuition and clarify the interpretation of the transmission mechanism of shocks in a multi-agent setting. The combined exploitation of the linear dynamic system obtained from the simulation and of the optimizing IS-LM framework allows to cope with heterogeneity in the simplest way at a purely macroeconomic level.

The paper is organized as follows. In section 2 we develop the macroeconomic framework. First of all we describe the behaviour of financially constrained firms and apply the stochastic aggregation procedure to the investment ratio (see subsection 2.1). In subsection 2.2 we analyze the behaviour of households. The macroeconomic equilibrium of the resulting optimizing IS-LM framework is derived in subsection 2.3.

In section 3 we develop the agent based model. In subsection 3.1 we define the law of motion of the equity ratio. Subsection 3.2 is devoted to the dynamical system which describes the evolution over time of the mean and the variance of the distribution of the equity ratio. We examine the long run configuration of the distribution and of the interest rate and the output gap in subsection 3.3.

After a brief comparison with a representative agent framework (section 4) we put the model to test exploring the macroeconomic consequences of a financial shock (section 5) and of a monetary shock (section 6). Section 7 presents the results of our Montecarlo experiments. Finally, section 8 recapitulates the modelling strategy and concludes. Technical details and cumbersome computations concerning the probability of bankruptcy, the household optimization problem, the simulation code and the law of motion of the individual equity ratio are confined to the appendix.
2 The macroeconomic model

2.1 Firms

Financial conditions. Firms are heterogeneous with respect to their financial robustness captured by the equity ratio \(a_{it} = A_{it}/K_{it}\) \((i = 1, 2, ..., z)\) where \(A_{it}\) is the firm’s equity base or net worth and \(K_{it}\) is the firm’s capital stock. In the following we will keep the aggregate price level constant and normalize it to unity (in other words all the variables are in real terms).\(^3\)

The equity ratio \(a_{it}\) falls in the interval \((\alpha, 1)\) where \(\alpha\) is the bankruptcy threshold and 1 is the self-financing threshold.\(^4\) The distribution of the firms’ equity ratio is characterized by the the average or cross-sectional mean of the equity ratio \(E(a_{it}) = a_t\) and by the variance \(E(a_{it} - a_t)^2 = \sigma^2_t\).

Firms cannot raise external finance on the Stock market (due to equity rationing, see Myers and Majluf, 1984; Greenwald et al., 1984) so that they have to rely on bank loans to finance investment. Therefore they run the risk of bankruptcy. Banks extend credit to firms at an interest rate which is uniform across firms and equal to the interest rate on bonds.

Technology. Each firm carries on production by means of a Leontief technology \(Y_{it} = \min(\lambda N_{it}, \nu K_{it})\) where \(Y_{it}, N_{it}\) and \(K_{it}\) represent output, employment and capital, \(\nu\) and \(\lambda\) are positive parameters.

Assuming that labour is always abundant, we can write \(Y_{it} = \nu K_{it}\) and \(N_{it} = \frac{\lambda}{\nu} K_{it}\). \(1/\nu\) is the capital/output ratio; \(\lambda/\nu\) is the the capital/labour ratio or capital intensity.\(^5\)

Market structure. Firms sell goods in a competitive market at a stochastic relative price \(u_{it}\). For the sake of simplicity, we assume that \(u_{it}\) is uniformly distributed on the support \((0, 2)\) so that \(E(u_{it}) = 1\). The rationale for this assumption is the following. Suppose there is a large number of firms (so that firms are price taker). Households allocate their demand for goods to different firms randomly. Let the demand that households allocate to the

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\(^3\)The equity ratio is the reciprocal of leverage. A word of caution is necessary at this point. As will be clear from equation (3.1) \(A_{it}\) are essentially cumulative past retained earnings (cash savings). The notion of equity base or net worth in this paper, therefore, is more restrictive than the usual one because we not account for expected discounted cash flows.

\(^4\)More on this below, under the heading "Bankruptcy".

\(^5\)Since \(\lambda\) and \(\nu\) are constant, output, capital and employment change at the same rate. We will determine the rate of capital accumulation endogenously (see below) and will assume that output and employment change at the same rate of the capital stock.
i-th firm at the end of period $t$ be a fraction $u_{it}$ of total consumption $c_tL_t$ where $c_t$ is consumption per household (to be determined in section 2.2) and $L_t$ is the number of households. $u_{it}$ is a stochastic demand shock specific to the firm (a preference shock) with $E(u_{it}) = 1$. Suppose, moreover, that the firm takes production decisions at the beginning of period $t$. The firm does not know her selling price in advance so that she will equate the marginal cost $MC(Y_{it})$ to the expected relative price $E\left(\frac{P_{it}}{P_t}\right) = 1$. Production will be $Y_{it} = MC^{-1}(1)$. The relative price will change according to the following price adjustment rule:

$$\frac{P_{it}}{P_t} - 1 = \zeta \left[ u_{it}c_tL_t - MC^{-1}(1) \right]$$

This determines the relative price $\frac{P_{it}}{P_t}$ at which the firm sells its predetermined output $Y_{it}$. Therefore the relative price turns out to be an increasing function of the demand disturbance, given the predetermined supply. In our simplified framework we identify the relative price with the stochastic term $u_{it}$. A high realization of $u_{it}$ characterizes a regime of high demand which drives up the relative price. In a regime of low demand, the realization of $u_{it}$ is low and may push the firm out of the market if it is “too low”, i.e. if it generates a loss so big as to deplete equity and make the equity ratio fall below the bankruptcy threshold $\alpha$.

**Profit.** Profit of the i-th firm ($\pi_{it}$) is the difference between revenues ($u_{it}Y_{it}$) and total costs, which consist of production costs ($wN_{it} + rK_{it}$), adjustment costs ($\frac{1}{2}I_{it}^2$) and organizational costs $\theta K_{it}$:

$$\pi_{it} = u_{it}Y_{it} - wN_{it} - r_tK_{it} - \frac{1}{2}I_{it}^2 - \theta K_{it}$$

(2.1)

For simplicity we assume that the real wage $w$ is given and constant. $r_t$ is the real interest rate, $I_{it} = K_{it} - K_{it-1}$ is investment, $K_t = \sum_{i=1}^{z} K_{it}$ is the aggregate capital stock and $\overline{K}_t = K_t/z$ is the average capital stock.

Adjustment costs are quadratic in investment (as usual in investment theory) and decreasing in the average capital stock. We assume a positive externality in the accumulation of capital. The higher the economywide

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6For simplicity we assume that there is no depreciation.
capital stock, the lower adjustment costs will be for the individual firm.

The firm incurs organizational costs to acquire the *soft capital* needed to carry on production. The notion of soft or organizational capital has been put forward, in a different context, by Gertler and Hubbard (1988).

The assumptions on adjustment and organizational costs allow to simplify the firm’s optimization problem and determine a simple interior solution.\(^7\)

**Bankruptcy.** The firm goes bankrupt in \( t \) if \( A_{it-1} \leq \alpha K_{it-1} \) i.e. if net worth reaches a minimum admissible level, which in turn is proportional to total assets. The bankruptcy condition can be rewritten as: \( a_{it-1} \leq \alpha \).

We assume that the probability of bankruptcy is increasing with leverage and therefore decreasing with the equity ratio. For the sake of analytical tractability we assume that the firm adopts the following definition of the probability of bankruptcy:

\[
\Phi_{it} = \left( \frac{1}{a_{it-1}} - 1 \right) \alpha' \tag{2.2}
\]

where \( 0 < \alpha' < 1 \). The firm goes bankrupt with probability one if \( a_{it-1} \leq \alpha := \alpha' / (1 + \alpha') \). Therefore \( \alpha \) represents the *bankruptcy threshold*.

The bankruptcy threshold can be considered a "floor" for the range of admissible equity ratios that the \( i \)-th firm can experience. Symmetrically the "ceiling" is represented by an equity ratio equal to 1. If the equity ratio reaches unity the firm is completely *self-financed* so that she does not need to resort to external finance and therefore does not run the risk of bankruptcy. In other words, when \( a_{it-1} = 1 \) the probability of bankruptcy is 0. Hence \( a_{it-1} \) is defined on the interval \((\alpha, 1)\) and \( \Phi_{it} \) is defined on the interval \((0, 1)\) as shown in figure 2.1.

Definition (2.2) captures the main determinant of the probability of bankruptcy, i.e. a measure of financial robustness, and can be thought of as a "reduced form" formulation. In appendix A we provide a microfoundation of the probability of bankruptcy along the lines of Greenwald and Stiglitz (1993). The adoption of the specific functional form of the probability of bankruptcy that emerges from this microfoundation would have made the model very difficult to manage without adding much in terms of insights. This is the reason why we preferred to adopt the simplified formulation (2.2).

The probability of bankruptcy as defined in (2.2) has an appealing prop-

\(^7\)See below, under the heading "Optimization."
Due to convexity, the marginal probability of bankruptcy $\frac{\alpha_1}{\alpha_0}$ is decreasing with financial robustness. In other words, the stronger the financial condition of the firm - the greater the equity ratio - the lower the decrease of the probability of bankruptcy associated with a further strengthening of the financial condition.

In the following we will characterize a negative financial shock as an exogenous increase of the bankruptcy threshold. A negative financial shock makes the probability of bankruptcy schedule shift to the right, as shown in figure 2.1 where $\alpha$ increases from $\alpha_0$ to $\alpha_1$. It is interesting to note that the convexity of the probability of bankruptcy makes the financial shock hit harder on the relatively fragile firms (those with a low equity ratio) who experience an increase in the probability of bankruptcy higher than the increase affecting robust firms.

Following Greenwald and Stiglitz (1993) we assume that bankruptcy is costly for the borrower and the cost of bankruptcy is an increasing linear function of the scale of activity: $\text{CB}_{it} = \beta K_{it}$ where $\beta > 0$.

**Optimization.** The objective function of the firm $V_{it}$ is the difference between expected profit $E(\pi_{it})$ and bankruptcy cost in case bankruptcy occurs $\text{CB}_{it} \Phi_{it}$:

![Figure 2.1: The probability of bankruptcy](image)
\[ V_{it} = Y_{it} - wN_{it} - (r_{t} + \theta)K_{it} - \frac{1}{2} \frac{I_{it}^2}{K_{t-1}} - \beta K_{it} \left( \frac{1}{a_{it-1}} - 1 \right) \alpha' \quad (2.3) \]

In order to simplify the problem and without loss of generality, we assume that \( \theta = \beta \alpha' \). Since \( Y_{it} = \nu K_{it} \) and \( N_{it} = \frac{\nu}{\lambda} K_{it} \) the problem of the firm boils down to:

\[
\max_{K_{it}} (\gamma - r) K_{it} - \frac{1}{2} \frac{(K_{it} - K_{it-1})^2}{K_{t-1}} - \beta \alpha' \frac{K_{it}}{a_{it-1}}
\]

where \( \gamma \equiv \nu \left( 1 - \frac{w}{\lambda} \right) \) represents earnings before interest – i.e. revenue net of labour costs – per unit of capital. In the following we will refer to \( \gamma \) as the profit rate.\(^9\)

From the FOC we obtain

\[ \tau_{it} = \gamma - (r_{t} + f_{it}) \quad (2.4) \]

The investment ratio \( \tau_{it} \equiv I_{it}/K_{t-1} \) is the difference between the profit rate \( \gamma \) and the interest rate \( r_{t} \) "augmented" by the external finance premium \( \beta \alpha'/a_{it-1} \) (EFP hereafter), i.e. an expected extra-cost due to the risk of bankruptcy. In the present context the EFP is due to the risk of insolvency as perceived by the borrower (borrower’s risk) while in the framework pioneered by Bernanke and Gertler it is traced back to the risk of default as perceived by the lender (lender’s risk) (see Bernanke and Gertler, 1989, 1990; Bernanke, Gertler and Gilchrist, 1999). As in the aforementioned literature, however, also in this context the EFP is increasing with the probability of bankruptcy and therefore decreasing with the equity ratio.\(^10\)

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\(^8\)The specification of the probability of bankruptcy \( (2.2) \) yields an expected cost of bankruptcy \( CB_{it} = (\beta \alpha' / a_{it-1}) - \beta \alpha' K_{it} \) which has a negative part. This negative component of cost plays the role of a component of revenues. By setting \( \theta = \beta \alpha' \) we implicitly assume that this component of marginal revenue is used to pay for the marginal cost of organizational capital.

\(^9\)The control variable in the problem above is the individual capital stock. Due to the Leontief technology, once the stock of capital has been optimally determined, both output and employment are determined because they are proportional to capital.

\(^10\)In principle \( \tau_{it} \) can be negative. In this case the capital stock is shrinking, a situation which we could not rule out – due for instance to a process of “creative destruction” which requires the stripping of obsolete machinery. Of course the capital stock cannot be negative: therefore, in case \( \tau_{it} < 0 \), we impose the following restriction: \( \tau_{it} > -s_{it-1} \) where \( s_{it-1} \equiv K_{it-1}/K_{t-1} \) is the relative size of the firm in \( t-1 \).
As one could expect, the investment ratio is decreasing with input costs $w$, $r_t$ and increasing with the equity ratio.

In the case of the representative agent we get:

$$\tau_t^R = \gamma - (r_t + f_t^R) \quad (2.5)$$

where $\tau_t^R \equiv I_t/K_{t-1}$ and $f_t^R = \beta \alpha'/a_{t-1}$.

In the absence of bankruptcy costs ($\beta = 0$) we obtain the first best investment ratio

$$\tau_t^F = \gamma - r_t \quad (2.6)$$

which is equal to the profit rate after interest payment and depends only on input costs. Of course, in the first best financial robustness has no role to play.

2.1.1 The average investment ratio

Using Taylor’s formula we can approximate the individual investment ratio (around the average equity ratio) as follows:

$$\tau_{it} = \tau_t^R + \frac{\partial \tau_{it}}{\partial a_{it-1}} (a_{it-1} - a_{t-1}) + \frac{1}{2} \frac{\partial^2 \tau_{it}}{\partial a_{it-1}^2} (a_{it-1} - a_{t-1})^2 + \frac{1}{6} \frac{\partial^3 \tau_{it}}{\partial a_{it-1}^3} (a_{it-1} - a_{t-1})^3 + \ldots$$

Taking the expected value of the expression above and recalling that, by definition, $E (a_{it-1} - a_{t-1}) = 0$ one gets:

$$\tau_t = E(\tau_{it}) = \tau_t^R + \frac{1}{2} \frac{\partial^2 \tau_{it}}{\partial a_{it-1}^2} E (a_{it-1} - a_{t-1})^2 + \frac{1}{6} \frac{\partial^3 \tau_{it}}{\partial a_{it-1}^3} E (a_{it-1} - a_{t-1})^3 + \ldots \quad (2.7)$$

where $E (a_{it-1} - a_{t-1})^2 = V_{t-1}$ and $E (a_{it-1} - a_{t-1})^3$ is the third moment around the mean, an indicator of skewness.

From the equation above follows that the average investment ratio – i.e. the investment ratio of the Modified Representative Agent (MRA) – is equal to the investment ratio of the Representative Agent augmented by a weighted sum of all the moments of the distribution of the equity ratio. The procedure to determine the policy function of the MRA in a general setting with heterogeneous agents is thoroughly discussed in Gallegati et al. (2006) where it is labelled the Variant-Representative-Agent, with a somewhat paradoxical
touch.\footnote{The approach has already been used in Agliari \textit{et al.} (2000).}

In the following, in order to keep the analysis as simple as possible, we will cut short the series above at the second term:

\[ \tau_t \approx \tau_t^R + \frac{1}{2} \frac{\partial^2 \tau_{it}}{\partial a_{it-1}^2} V_{t-1} \quad (2.8) \]

where

\[ \frac{\partial^2 \tau_{it}}{\partial a_{it-1}^2} = -\frac{2 \beta \alpha'}{a_{t-1}^3} < 0 \quad (2.9) \]

Summing up, we have taken a second order approximation of the individual investment ratio \((2.4)\) and computed the mean of the approximated investment ratio, getting \((2.8)\). The MRA approach to aggregation in an heterogeneous agents setting is similar to the second order approximation of the policy function proposed by Schmitt Grohé and Uribe (2004) in a RA setting. The procedure they propose, however, does not aim at averaging heterogeneous policy functions but at approximating the non linear policy function of the representative agent.

Recalling \((2.5)\) and \((2.9)\) equation \((2.8)\) can be written as

\[ \tau_t = \gamma - (r_t + f_t) \quad (2.10) \]

where

\[ f_t = \frac{\beta \alpha'}{a_{t-1}} \left(1 + \frac{V_{t-1}}{a_{t-1}^2}\right) \quad (2.11) \]

The investment ratio of the MRA \(\tau_t\) is equal to the difference between the profit rate \(\gamma\) and the interest rate \(r_t\) "augmented" by the \textit{average external finance premium} \(f_t\) which is decreasing with the average equity ratio and increasing with the variance of the equity ratio.\footnote{In other words, the average EFP \(f_t\) can be conceived of as a mark-up \(\mu_{t-1} := \frac{V_{t-1}}{a_{t-1}^2}\) on the EFP of the representative agent \(f_t^R := \frac{\beta \alpha'}{a_{t-1}}\) where the mark-up coincides with the square of the "coefficient of variation" i.e. the ratio of the standard deviation to the mean of the distribution.}

Notice that the MRA investment ratio \(\tau\) is smaller than the RA investment ratio \(\tau_t^R\) which in turn is smaller than the first best \(\tau_t^F\). In other words we have the following hierarchy of investment ratios \(\tau_t^F > \tau_t^R > \tau_t\).
To illustrate this point, in figure 2.2 we represent equation (2.4). The investment ratio of the i-th firm $\tau_{it}$ is an increasing concave function of the individual equity ratio $a_{it-1}$ and tends asymptotically to the first best $\tau^F_t$. Concavity of the investment ratio can be traced back to convexity of the bankruptcy probability function (2.2). In fact $\frac{\partial \tau_{it}}{\partial a_{it-1}} = \beta \left| \frac{d \Phi_{it}}{d a_{it-1}} \right|$. The stronger the financial condition of the firm – i.e. the greater the equity ratio – the smaller the reduction of the probability of bankruptcy associated with an increase of the equity ratio, and therefore the smaller the increase of the investment ratio.

For the sake of discussion, consider the simplest case of a corporate sector consisting of only two firms whose equity ratios are $a_{1t-1}$ and $a_{2t-1}$. Thanks to concavity, by Jensen’s inequality the average investment ratio $\tau_{\text{i}}$ – i.e. the MRA investment ratio – will be smaller than the investment ratio associated with the average equity ratio $\tau_{\text{R}}$ – i.e. the RA investment ratio – which in turn will be smaller than first best. A mean preserving increase in dispersion will bring about a decrease of the MRA investment ratio.
Average investment will be \( \bar{I}_t = \tau_t \bar{K}_{t-1} \) so that in the aggregate \( I_t = \tau_t K_{t-1} \) and \( K_t = (1 + \tau_t) K_{t-1} \). Therefore the MRA investment ratio represents also the rate of growth of the aggregate capital stock. Due to the Leontief technology, employment and output grow at the same rate as the capital stock.

2.2 Households

There are \( L_t \) households that are homogeneous in every respect so that we can safely adopt the representative agent assumption. Each household supplies inelastically one unit of labour. The household has a measure 1 of members. Each member of the household has probability \( x_t \) of being employed. Therefore \( x_t \) coincides with the fraction of household members who are employed and with the employment rate economywide \( x_t = N_t / L_t \). Conversely \( 1 - x_t \) is the probability of being unemployed, the fraction of household members who are unemployed and the unemployment rate economywide.

We assume that all the profits are retained within the firm. In the absence of dividends, the only source of income for the household is the wage rate \( w \) if the member is employed, the unemployment subsidy \( \sigma \) if unemployed, with \( w > \sigma \) (both the real wage and the unemployment subsidy are constant by assumption). We assume that household members pool resources so that there is full insurance in consumption.

The representative household has income \( wx_t + \sigma (1 - x_t) \) and demands goods, financial assets (bonds) and money balances.

The lifetime utility function of the representative household is:

\[
U_t = \sum_{s=0}^{\infty} \xi^s (c_{t+s})^\delta (m_{t+s})^{1-\delta} \tag{2.12}
\]

where \( c_t \) is consumption, \( m_t \) are money balances, \( \xi \) is the discount factor, \( 0 < \delta < 1 \). In the following, for simplicity, we will set \( \delta = 1/2 \).

The household’s budget constraint is:

\[
c_{t+s} + m_{t+s} + b_{t+s} = wx_{t+s} + \sigma (1 - x_{t+s}) + \\
+ m_{t+s-1} + (1 + r_{t+s-1}) b_{t+s-1} \tag{2.13}
\]

\[
s = 0, 1, \ldots \infty
\]

According to the budget constraint, the sum of consumption \( c_t \) and the
demand for money $m_t$ and bonds $b_t$ should be equal to income $wx_t + \sigma (1 - x_t)$ plus interest payments $(1 + r_{t-1}) b_{t-1}$ and money balances $m_{t-1}$ carried over from the previous period.

The problem of the representative household, therefore, consists in maximizing the expected value of (2.12) subject to a sequence of budget constraints (2.13). From the FOCs we obtain the following relation between optimal consumption and money demand:

$$m_t = \frac{1 + r_t}{r_t} c_t$$  \hspace{1cm} (2.14)

We assume that changes in money balances can be implemented only by means of changes in bondholding in the opposite direction through open market transactions:

$$m_t - m_{t-1} = - [b_t - (1 + r_{t-1}) b_{t-1}]$$  \hspace{1cm} (2.15)

Substituting (2.14) and (2.15) into (2.13) we obtain the optimal consumption function and money demand function for the representative household:

$$c_t = wx_t + \sigma (1 - x_t)$$  \hspace{1cm} (2.16)

$$m_t = \frac{1 + r_t}{r_t} [wx_t + \sigma (1 - x_t)]$$  \hspace{1cm} (2.17)

### 2.3 Equilibrium

In this economy there are markets for labor, goods, money and financial assets. Due to real wage rigidity, the labor market can be characterized by underemployment even if both the money and goods markets are in equilibrium.

The goods market is in equilibrium (planned expenditure is equal to actual expenditure) when $C_t + I_t = Y_t$. Aggregate consumption is $C_t = [wx_t + \sigma (1 - x_t)] L_t$. Aggregate investment is $I_t = \tau_t K_{t-1}$. Therefore in equilibrium the following must hold true:

$$[wx_t + \sigma (1 - x_t)] L_t + \tau_t K_{t-1} = Y_t$$  \hspace{1cm} (2.18)

\footnote{See appendix B for details.}
Dividing by total employment \( N_t \), recalling that \( \frac{N_t}{L_t} = x_t \), \( \frac{Y_t}{N_t} = \lambda \), \( \tau_t \frac{K_{t-1}}{N_t} = \frac{\tau_t}{1 + \tau_t \nu} \) we can rewrite (2.18) as

\[
\begin{align*}
  w + \sigma \left( \frac{1}{x_t} - 1 \right) + \frac{\tau_t}{1 + \tau_t \nu} \frac{\lambda}{\nu} &= \lambda \\
 \end{align*}
\]

(2.19)

In the following we will refer to \( x_t \) as the output gap.\(^{14}\) In order to simplify the analysis, we linearize \( 1/x_t \) around full capacity (i.e. \( x = 1 \)) by means of the usual Taylor procedure so that \( \frac{1}{x_t} - 1 \) is approximately equal to \( 1 - x_t \).

Analogously, linearizing \( \frac{\tau_t}{1 + \tau_t} \) around \( \tau_t = 0 \) we get \( \frac{\tau_t}{1 + \tau_t} \approx \tau_t \). Hence (2.19) becomes \( w + \sigma (1 - x_t) + \tau_t \frac{\lambda}{\nu} = \lambda \). Recalling that \( \tau_t = \gamma - r_t - f_t \) in the end we can write:

\[
\begin{align*}
  w + \sigma (1 - x_t) + (\gamma - r_t - f_t) \frac{\lambda}{\nu} &= \lambda \\
\end{align*}
\]

Notice that \( \lambda = w + \gamma \frac{\lambda}{\nu} \). In fact \( \gamma \frac{\lambda}{\nu} \) represents profits per worker, being the product of the profit rate \( \gamma \) times the capital/labour ratio \( \frac{\lambda}{\nu} \). By definition, the sum of labour income per worker – i.e. the wage rate – and profits per worker is equal to income per worker.

Hence equation (2.19) becomes:

\[
\begin{align*}
  x_t &= 1 - \frac{\lambda}{\sigma \nu} (r_t + f_t) \\
\end{align*}
\]

(2.20)

This relation between \( r_t \) and \( x_t \) represents the (optimizing) IS curve of our model. The EFP \( f_t \) is a shifter of the IS curve but the EFP is determined by the moments of the distribution of the equity ratio \( f_t = \frac{\beta \alpha'}{a_{t-1}} \left( 1 + \frac{\nu_{t-1}}{a_{t-1}^2} \right) \).

A reduction of the EFP (due for instance to an increase of the mean or a

\(^{14}\)Thanks to the linearity of technology, the employment rate \( x_t \) can be thought of also as a measure of capacity utilization. In fact, \( N_t = Y_t/\lambda \) and \( L_t = \tilde{Y}_t/\lambda \) where \( \tilde{Y}_t \) is potential output so that \( x_t = \frac{N_t}{L_t} = \frac{Y_t}{\tilde{Y}_t} \). Properly speaking, the output gap is an affine transformation of capacity utilization: \( \left( Y - \tilde{Y}_t \right) / \tilde{Y} = x - 1 \).
reduction of the variance) makes the IS curve shift up.

We now turn to the money market. The demand for money is represented by equation (2.17). Imposing the equilibrium condition \( m_t = \overline{m}_t \) where \( \overline{m}_t \) is exogenous money supply – we get \( x_t = \frac{1}{w - \sigma} \left( \frac{r_t}{1 + r_t} \overline{m}_t - \sigma \right) \).\(^{15}\) Linearizing \( \frac{r_t}{1 + r_t} \) around \( r_t = 0 \) we get \( \frac{r_t}{1 + r_t} \approx r_t \) so that the equation above becomes:

\[
x_t = \frac{1}{w - \sigma} \left( r_t \overline{m}_t - \sigma \right)
\]

This relation between \( r_t \) and \( x_t \) represents the LM curve of our model.

The system (2.20)(2.21) can be solved for the interest rate and the output gap. After some algebra we get

\[
r_t = \Gamma_0 \left[ \sigma \frac{\nu}{\lambda} w - (w - \sigma) f_t \right]
\]

\[
x_t = \frac{\overline{m}_t}{w - \sigma} \Gamma_0 \left[ \sigma \frac{\nu}{\lambda} w - (w - \sigma) f_t \right] - \frac{\sigma}{w - \sigma}
\]

where \( \Gamma_0 = \left( w - \sigma + \sigma \frac{\nu}{\lambda} \overline{m}_t \right)^{-1} \) is positive (since \( w > \sigma \)) and decreasing with per capita money supply.\(^{16}\)

Both \( r_t \) and \( x_t \) in the reduced form are decreasing with the EFP.\(^{17}\)

---

\(^{15}\)In the following we will assume that the rate of growth of money supply will be equal to the rate of growth of population so that per capita money supply will be constant over time.

\(^{16}\)We assume \( \sigma \frac{\nu}{\lambda} w - (w - \sigma) f_t > \frac{\sigma}{\Gamma_0 \overline{m}_t} \) in order to guarantee that both \( r_t \) and \( x_t \) are positive.

\(^{17}\)Notice that the interest rate augmented by the EFP is:

\[
r_t + f_t = \Gamma_0 \sigma \frac{\nu}{\lambda} w + [1 - \Gamma_0 (w - \sigma)] f_t
\]

Since the expression in brackets is positive, it turns out that the augmented interest rate is increasing in \( f_t \). Hence a reduction of the EFP makes the interest rate increase but the augmented interest rate decrease (because the reduction of the EFP is greater in absolute value than the increase of the interest rate).

Recalling that \( \tau_t = \gamma - (r_t + f_t) \) we can conclude that a reduction of the EFP unambiguously boosts capital accumulation.
3 The agent based model

3.1 The individual equity ratio

In this section we build the agent based model of the equity ratio and show how it is nested into the macroeconomic model developed so far. The starting point is the law of motion of the firms’ net worth

\[ A_{it} = A_{it-1} + \pi_{it} \]  

(3.1)

Recalling (2.1) and proceeding as described in appendix C we obtain the law of motion of the individual equity ratio

\[ a_{it} = a_{it-1} \left( \frac{s_{it-1}}{s_{it-1} + \tau_{it}} \right) + u_{it} - w \frac{\nu}{\lambda} - r_t - \frac{1}{2} \frac{\tau_{it}^2}{s_{it-1} + \tau_{it}} \]  

(3.2)

where \( \tau_{it} = \gamma - \left( r_t + \frac{\beta \alpha'}{a_{it-1}} \right) ; s_{it-1} = K_{it-1} \frac{K_{t-1}}{K_{t-1}}. \)

Equation (3.2) is the cornerstone of the agent based model. Given the real wage and technological parameters, the equity ratio of the i-th firm is a function of (i) the investment ratio \( \tau_{it} \) which is determined by the equity ratio, (ii) the relative size \( s_{it-1} \), (iii) a stochastic disturbance \( u_{it} \) and (iv) the interest rate \( r_t \) which is defined as in (2.22). The cross-sectional mean and variance of the equity ratio impact upon the individual law of motion through the interest rate. This is the source of a macroeconomic externality. Firms’ financial conditions at the macroeconomic level, i.e. \( a_{t-1} \), \( V_{t-1} \) determine the EFP which in turn affects the interest rate \( r_t \) and therefore the individual firm’s financial condition.

In a multi-agent setting, there is a large number of laws of motion of the individual equity ratios which can be conceptualized as a multi-dimensional system of non-linear coupled difference stochastic equations.\(^{19}\) Since it is impossible to compute closed form solutions, we have to resort to computer simulations.

We consider a virtual economy consisting of \( z = 1000 \) firms over a time

---

\(^{18}\)As already pointed in section 2.1, in the present paper \( A_{it} \) are cumulative retained earnings (cash savings). Properly speaking, the equity base or net worth is the sum of expected discounted cash flows and retained earnings. In this paper, therefore, the notion of net worth is more restrictive than the usual one.

\(^{19}\)Individual dynamics are coupled via the macroeconomic externality mentioned above.
span of $T = 1000$ periods ("quarters"). We aim at assessing qualitatively the dynamic properties of the economy under scrutiny. Therefore we build the model as sparingly as possible, abstracting from features which would certainly enrich the model but would also increase the complexity of the mechanisms at work. In fact, there are only 7 free parameters in the model which are set as in table 3.1.

The configuration of parameters we have chosen will yield dynamic patterns of the main macroeconomic variables – i.e. the interest rate and the output gap – roughly in line with the empirical evidence.

As shown in section 2.1, for modelling reasons the threshold level of the equity ratio $\alpha$ below which the firm goes bankrupt cannot be exactly zero. The bankruptcy threshold we have chosen, however is close to zero.

Bankruptcy costs too are assumed to be very small, amounting to 0.6% of output.\footnote{In our setting bankruptcy costs amount to 0.2% of the capital stock and the capital/output ratio is 3.} The empirical assessment of bankruptcy costs is controversial because the definition itself is elastic and, once defined, they are measurement sensitive. A survey by White (1989) gave a range of 3-21%. More recently Bris, Nzhu and Welch (2006) suggest a range of 0-20%. Our modelling choice is therefore close to the lower endpoint of available estimates of bankruptcy costs.

The productivity of capital $\nu$ is set to 1/3 to capture the empirical stylized fact according to which the capital/output ratio is close to 3. We do not have strong priors concerning the other parameters. The real wage and the

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bankruptcy threshold</td>
<td>$\alpha = 0.02$</td>
</tr>
<tr>
<td>Bankruptcy cost per unit of capital</td>
<td>$\beta = 0.002$</td>
</tr>
<tr>
<td>Average productivity of capital</td>
<td>$\nu = 1/3$</td>
</tr>
<tr>
<td>Average productivity of labour</td>
<td>$\lambda = 1$</td>
</tr>
<tr>
<td>Wage rate</td>
<td>$w = 0.7$</td>
</tr>
<tr>
<td>Unemployment subsidy</td>
<td>$\sigma = 0.2$</td>
</tr>
<tr>
<td>Money supply</td>
<td>$\bar{m} = 200$</td>
</tr>
</tbody>
</table>

Table 3.1: Parameters setting
productivity of labour \( \lambda \) are set so that, together with the productivity of
capital, they yield a profit rate (before interest) \( \gamma \) equal to 10%.

The unemployment subsidy is substantially lower (less than 30%) than
the wage rate.

The quantity of money per household is set to \( m_t = 200 \).

The code which governs the simulations follows the logical sequence spelled
out in appendix D.

3.2 The cross sectional mean and variance

The simulations described in appendix D generate the time series of the cross-
sectional mean and variance of the equity ratio over 1000 periods. We discard
the transient consisting of the first 100 periods and run an OLS regression
on 900 simulated data to estimate the \( \alpha_{ij} \) coefficients \( (i, j = 0, 1, 2) \) of the
linear system

\[
\begin{align*}
a_t &= \alpha_{10} + \alpha_{11} a_{t-1} + \alpha_{12} V_{t-1} \\
V_t &= \alpha_{20} + \alpha_{21} a_{t-1} + \alpha_{22} V_{t-1}
\end{align*}
\]  

(3.3) (3.4)

We have run several simulations changing the random seed. The esti-
mated coefficients in one of these simulations are

\[
\begin{align*}
\alpha_{10} &= 0.32045; \alpha_{11} = 0.47211; \alpha_{12} = -0.30102 \\
\alpha_{20} &= 0.05469; \alpha_{21} = 0.03137; \alpha_{22} = 0.22436
\end{align*}
\]

They are all statistically significant (at the 5% level). Notice that the autore-
gressive coefficients \( \alpha_{11} \) and \( \alpha_{22} \) are positive and smaller than one (a robust
feature of the simulations) and that lagged variance has a negative impact on
the current mean equity ratio, i.e. \( \alpha_{12} \) is negative. This last feature is typical
for low values of the bankruptcy threshold \( \alpha \). For relatively high values of \( \alpha \)
the lagged variance has a positive impact on the current mean equity ratio,
i.e. \( \alpha_{12} \) turns positive (see section 7 for details).

In order to provide a macroeconomic interpretation of the system (3.3)(3.4),
it is convenient to consider the continuous time approximation:

\[
\begin{align*}
da &= \alpha_{10} + (\alpha_{11} - 1) a + \alpha_{12} V \\
\dot{V} &= \alpha_{20} + \alpha_{21} a + (\alpha_{22} - 1) V
\end{align*}
\]
It is easy to see that the trace of the Jacobian is always negative, thanks to the fact that autoregressive coefficients are smaller than one. Since $\alpha_{12}$ is negative, the determinant is unambiguously positive so that the resulting steady state will be stable.\footnote{This is confirmed by computing the eigenvalues of the system.}

Imposing $da = 0$ we determine the demarcation line $AA$, i.e. the locus of $(a, V)$ pairs such that $a$ is constant. It is easy to see that the line is downward sloping.\footnote{This is due to the fact that $\alpha$ is low so that $\alpha_{12}$ is negative. For relatively high values of $\alpha$ the coefficient $\alpha_{12}$ turns positive and the AA line slopes up.} Points above (below) the line are characterized by a tendency of $a$ to decrease (increase) as shown by the horizontal arrows in figure 3.1.

Imposing $dV = 0$ we determine the demarcation line $VV$, i.e. the locus of $(a, V)$ pairs such that $V$ is constant. The line is upward sloping. Points above (below) the line are characterized by a tendency of $V$ to decrease (increase) (as shown by the vertical arrows).

The steady state is at the intersection of the two curves (point $E_0$). From the estimated linear system we get the coordinates of this point, namely:

$$a_0 = 0.55405; \quad V_0 = 0.09292$$

We have extracted from the simulated data an ergodic process such that the actual distribution of the equity ratio converges over time to a long run stable distribution whose first and second moments are $a_0$ and $V_0$.

Not surprisingly the steady state of the cross sectional mean (and of the variance) almost coincide with the long run average of the same variables which can be computed directly from simulated data, as shown in the upper panels of figure 3.2.

### 3.3 The macroeconomic equilibrium in the steady state

We are now able to compute the EFP in the "long run" i.e when the distribution of the equity ratio has reached her long run equilibrium captured by the steady state cross sectional mean and variance:

$$f_0 \equiv \frac{\beta \alpha'}{a_0} \left( 1 + \frac{V_0}{a_0^2} \right) = 9.4049 \times 10^{-5}$$

This is the crucial datum we have to retrieve from the agent based model and plug into the reduced form (2.22)(2.23) of the macroeconomic model in
Figure 3.1: Phase diagram of (3.3)(3.4)

Figure 3.2: Simulated time series and long run average of $a$, $V$, $r$, $x$. 

21
order to compute the real interest rate and the output gap \textit{in equilibrium} and \textit{in the long run}. They are:

\begin{align*}
x_0 &= 0.94804 \\
r_0 &= 0.003370
\end{align*}

Therefore, in the long run the unemployment rate is approximately 5\% while the annualized interest rate is around 1\%. \footnote{These numbers are just satisfactory for our purposes. They are not too far from the macroeconomic scenario in the USA before the crisis. In this paper, however, we are not aiming at replicating accurately macroeconomic reality. Our aim consists in illustrating a strategy to build a fairly reasonable microfounded macroeconomic model with heterogeneous agents.} The equilibrium values of the output gap and the interest rate are quite close to the long run average of the same variables which can be computed directly from simulated data, as shown in the lower panels of figure 3.2.

In figure 3.3, the equilibrium values of the interest rate and the output gap \((x_0, r_0)\) are the coordinates of the macroeconomic equilibrium \(E_0\). By construction, the macroeconomic equilibrium is anchored to \(f_0\).
4 A comparison with the Representative Agent

Setting $\alpha_{12} = 0$ in (3.3) we get $a_t = \alpha_{10} + \alpha_{11}a_{t-1}$ with $\alpha_{10} = 0.32045; \alpha_{11} = 0.47211$. This is the law of motion of the equity ratio of the representative agent. The steady state of this AR(1) process is the steady state equity ratio of the representative agent

$$a_0^R = 0.60704$$

This is represented by point $R \equiv (a_0^R, 0)$ in figure 3.1. The steady state is stable. Notice that the long run mean of the equity ratio is higher in the representative agent case: $a_0 < a_0^R$. We can easily retrieve the output gap and interest rate in the representative agent case. The EFP in this case, in fact, is

$$f_0^R \equiv \frac{\beta \alpha'}{a_0^R} = 6.5894 \times 10^{-5}$$

so that the output gap and the interest rate are

$$\begin{align*}
    x_0^R &= 0.94844 \\
r_0^R &= 0.003371
\end{align*}$$

(coordinates of point R in figure 3.3).

Summing up: $a_0 < a_0^R; f_0^R < f_0; x_0^R > x_0; r_0^R > r_0$.

Heterogeneity (captured by the cross sectional variance) plays the role of a dampening factor with respect to the accumulation of net worth. Where does this dampening role comes from?

Let’s consider the RA case first. Suppose the initial cross sectional mean is low: $a_{t-1} < a_0^R$. The initial condition, for instance, can be visualized as the abscissa of point B in figure 3.1. The equity ratio increases (the system moves to the right along the x-axis). Hence the EFP goes down, boosting net worth and capital accumulation. The IS curve shifts up along the LM curve so that both the output gap and the interest rate go up. The system will settle in

\begin{footnotesize}
\begin{enumerate}
\item[24] If, for instance, the initial condition of the equity ratio of the representative agent is $a < a_0^R$ (any point of the x-axis between the origin and point R are characterized by this inequality), then $a$ will increase and converge to $a_0^R$.
\item[25] In order to simplify the graphical representation we do not show the first best case, which is characterized by $\tau^F = \gamma - r_t$. The first best would be captured by an IS schedule characterized by $f = 0$. This schedule would be time invariant and located above the IS schedule in the representative agent case.
\end{enumerate}
\end{footnotesize}
the steady state and macroeconomic equilibrium points characterized by the letter $R$ in figures 3.1 and 3.3.

Let’s consider now the Heterogeneous Agents (HAs) case. Suppose the initial condition is point B in figure 3.1. The system moves along the dashed trajectory starting from B in figure 3.1. In this case therefore the mean equity ratio increases but also the variance goes up. The reduction of the EFP due to the increase of the cross-sectional mean is somehow offset by the increase of the variance. Hence net worth and capital accumulation are somehow attenuated. In the end (i.e. in the steady state) the cross sectional equity ratio go up less than in the RA case. The system will settle in the steady state and macroeconomic equilibrium points characterized by the letter $R$ in figures 3.1 and 3.3.

5 The effects of a financial shock

In this section and in the following one we put the M&ABM to the test exploring the consequences of a financial and a monetary shock respectively.

A negative financial shock translates into an exogenous increase of the bankruptcy threshold.\textsuperscript{26} The benchmark bankruptcy threshold (as in table 3.1) is $\alpha_0 = 0.02$. In this section we will set the new threshold at $\alpha_1 = 0.05$, the other parameters in the table being equal. Estimating the $\alpha_{ij}$ coefficients of the system (3.3)(3.4) on the new artificial dataset generated by one of the simulations we get

$$
\begin{align*}
\alpha_{10} &= 0.32727; ~ \alpha_{11} = 0.44562; ~ \alpha_{12} = -0.26421 \\
\alpha_{20} &= 0.06264; ~ \alpha_{21} = 0.02380; ~ \alpha_{22} = 0.20623 
\end{align*}
$$

The financial shock makes the $AA$ curve shift down and the $VV$ curve shift up. Moreover the $AA$ curve become steeper while the $VV$ curve becomes flatter. Figure 5.1 depicts the situation in the proximity of the steady state.

If the shock is permanent, the economy moves from the old steady state $E_0$ to the new one $E_1$ whose coordinates are

$$
\begin{align*}
\alpha_1 &= 0.54494; ~ V_1 = 0.09525 \\
\text{with } \alpha_1 < \alpha_0 \text{ and } V_1 > V_0. \text{ Since the mean equity ratio is smaller and the}
\end{align*}
$$

\textsuperscript{26}See the discussion under the heading "Bankruptcy" in section 2.1.
variance is greater than in the old steady state, in the new steady state the equilibrium EFP

\[ f_1 \equiv \frac{\beta \alpha'_1}{a_1} \left( 1 + \frac{V_1}{a_1} \right) = 2.424 \times 10^{-4} \]

will be higher than in the old one (in fact \( f_0 = \frac{\beta \alpha'_0}{a_1} \left( 1 + \frac{V_0}{a_1} \right) = 9.4049 \times 10^{-5} \)). The real interest rate and the output gap can be computed from the reduced form (2.22)(2.23) using the value for \( f_1 \) above. They are:

\[ x_1 = 0.94589 \]
\[ r_1 = 0.003364 \]

The increase in \( \alpha \) makes both the output gap and the interest rate decrease.

The total effect of the shock can be decomposed into first round and second round effects. The first round effect is the macroeconomic impact of

The parameter \( \alpha'_0 \) is equal to \( \frac{\alpha_0}{1-\alpha_0} = \frac{0.02}{0.98} = 0.0204 \) while \( \alpha'_1 \) is equal to \( \frac{\alpha_1}{1-\alpha_1} = \frac{0.05}{0.95} = 0.0526 \).
Figure 5.2: Effects of an increase in $\alpha$ on the IS-LM apparatus. Solid lines: $\alpha_0 = 0.02$, dashed lines $\alpha_1 = 0.05$.

the shock measured at the initial distribution of the equity ratio (i.e. given the pre-shock steady state distribution). The second round or indirect effect measures the macroeconomic repercussion of the changes in the distribution triggered by the shock. Heterogeneity contributes to the second round effect through changes in the variance of the distribution.

The first round effect of the financial shock is the increase of the probability of bankruptcy for each firm, given the equity ratio of the firm in the original steady state distribution. The EFP increases on impact by

$$\Delta f_{1st} = (\alpha'_1 - \alpha'_0) \left[ \frac{\beta}{\alpha_0} \left( 1 + \frac{V_0}{\alpha_0^2} \right) \right] = \frac{\Delta \alpha'}{\alpha'_0} f_0$$

making the IS curve shift down along the LM curve (from $IS(\alpha = 0.02)$ to $IS(f'_0)$ in figure 5.2).

Due to this initial shift also the interest rate and the output change on impact. From the reduced form of the IS-LM model we get

$$\Delta r_{1st} = -\Gamma_0 \left( w - \sigma \right) \Delta f_{1st}$$
$$\Delta x_{1st} = \frac{\bar{m}}{w - \sigma} \Gamma_0 \Delta r_{1st} = -\bar{m} \Gamma_0 \Delta f_{1st}$$

The first round effect triggers a dynamic downward adjustment of the
As a consequence the cross sectional mean of the equity ratio goes down (pushing the EFP up) and the variance increases (as shown in figure 5.1) pushing the EFP further up. The second round increase of the EFP is

$$\Delta f_{2nd} = \beta \alpha'_1 \left[ \left( \frac{1}{a_1} - \frac{1}{a_0} \right) + \left( \frac{V_1}{a_1^2} - \frac{V_0}{a_0^2} \right) \right]$$

The second round effect, in turn, can be decomposed into two terms. The first one depends only on the adjustment of the cross sectional mean

$$\beta \alpha'_1 \left( \frac{1}{a_1} - \frac{1}{a_0} \right)$$

while the second one depends on the variance

$$\beta \alpha'_1 \left( \frac{V_1}{a_1^2} - \frac{V_0}{a_0^2} \right).$$

The latter measures the contribution of heterogeneity to the second round effect.

The new increase of the EFP will push the IS curve further down along the LM curve. Due to this second shift also the interest rate and the output change. From the reduced form of the IS-LM model we get

$$\Delta r_{2nd} = -\Gamma_0 (w - \sigma) \Delta f_{2nd}$$

$$\Delta x_{2nd} = \frac{\bar{m}}{w - \sigma} \Gamma_0 \Delta r_{2nd} = -\bar{m} \Gamma_0 \Delta f_{2nd}$$

At the end of the transition from the old to the new steady state the equilibrium moves from $E_0$ to $E_1$ as shown in figure 5.2.

The initial shock (first round increase of the EFP) is amplified through the financial accelerator mechanism (second round increase of the EFP). Heterogeneity contributes to the magnification of the initial shock. Due to the non-linear specification of the probability of bankruptcy the financial shock hit the fragile firms (those with a low equity ratio) harder. They will experience an increase in the probability of bankruptcy higher than the increase affecting robust firms. Hence the equity ratio of the fragile firms will fall more than the equity ratio of the robust firms, increasing the variance.

In (5.1) we report the magnitude of the effects computed using the numerical values of the steady state moments of the distribution and the parameters set in table 3.1.
Most of the effect of the shock must be traced back to the first round impact. The second round effect on the EFP is two orders of magnitude smaller than the first round effect. This is due essentially to the negligible size of the bankruptcy cost which we set at 0.6% of output. Due to this assumption, the impact of changes in the distribution on the EFP is limited. The contribution of heterogeneity on EFP (4.22 × 10^{-6}), however, is more than half of the second round effect.

The second round effect on the interest rate is one order of magnitude smaller than the first round effect. The same applies to the effect on the output gap.

6 The effects of a monetary policy shock

Let us now assess the effects of a monetary shock.

Suppose money supply increases from $\bar{m}_0 = 200$ to $\bar{m}_1 = 350$, the other parameters of table 3.1 being equal. We run a number of new simulations and estimate the $\alpha_{ij}$ coefficients of the system (3.3)/(3.4) on the new artificial dataset. In one of the simulations we get the following:

\[
\begin{align*}
\alpha_{10} &= 0.33195; \quad \alpha_{11} = 0.46721; \quad \alpha_{12} = -0.38036 \\
\alpha_{20} &= 0.05583; \quad \alpha_{21} = 0.02517; \quad \alpha_{22} = 0.24602
\end{align*}
\]

Following the shock the AA line shifts up while the VV line shifts down (slightly). Both lines become flatter. Figure 6.1 magnifies a small portion of the $(a, V)$ plane in the proximity of the steady state.

The coordinates of the new steady state $E_1$ are

\[
a_1 = 0.55691; \quad V_1 = 0.09263
\]

with $a_1 > a_0$ and $V_1 < V_0$. The EFP in the new steady state will therefore
Figure 6.1: Effects of a positive monetary policy shock on the demarcation lines. Solid lines: $\bar{m}_0 = 200$; dashed lines: $\bar{m}_1 = 350$.

be lower than in the old steady state:

$$f_1 = \frac{\beta \alpha'}{a_1} \left( 1 + \frac{V_1}{a_1^2} \right) = 9.3276 \times 10^{-5}$$

The real interest rate and the output gap can be computed from the reduced form (2.22)(2.23) using the value for $f_1$ above. They are:

$$x_1 = 0.96926$$
$$r_1 = 0.0019561$$

with $x_1 > x_0$ and $r_1 < r_0$.

The first round effect of the monetary shock is the decrease of the interest rate and the increase in output given the original steady state distribution of the equity ratio due to the shift of the LM curve down along the original IS curve as shown in figure 6.2.

The interest rate and the output gap change on impact by
Figure 6.2: Effects of a positive monetary policy shock on the IS-LM apparatus. Solid lines: $\bar{m}_0 = 200$; dashed lines: $\bar{m}_1 = 350$.

$$
\Delta r_{1st} = (\Gamma_1 - \Gamma_0) \left[ \frac{\sigma}{\chi} w - (w - \sigma) f_0 \right]
$$

$$
\Delta x_{1st} = (\bar{m}_1 \Gamma_1 - \bar{m}_0 \Gamma_0) \frac{\sigma^2 w - (w - \sigma) f_0}{w - \sigma}
$$

where $\Gamma_i = (w - \sigma + \sigma \chi \bar{m}_i)^{-1}; i = 0, 1$. Notice that $\Gamma_1 - \Gamma_0 < 0$ while $\bar{m}_1 \Gamma_1 - \bar{m}_0 \Gamma_0 > 0$.

The first round effect triggers a dynamic upward adjustment of the equity ratio for each and every firm. As a consequence the cross sectional mean goes up and the variance decreases (as shown in figure 6.1) pushing down the EFP. The reduction of the EFP is due to the change in the distribution and therefore occurs only in the second round.

$$
\Delta f = \beta \alpha' \left[ \left( \frac{1}{a_1} - \frac{1}{a_0} \right) + \left( \frac{V_1}{a_1^3} - \frac{V_0}{a_0^3} \right) \right] < 0
$$

The contribution of heterogeneity is $\beta \alpha' \left( \frac{V_1}{a_1^3} - \frac{V_0}{a_0^3} \right)$. 

30
The decrease of the EFP pushes the IS curve up along the new LM curve. Due to this second shift also the interest rate and output change as follows

\[ \Delta r_{2nd} = -\Gamma_1 (w - \sigma) \Delta f \]

\[ \Delta x_{2nd} = -\frac{\bar{m}_1 \Gamma_1}{w - \sigma} \Delta f \]

Notice that the second round effect on the interest rate is positive and offsets in part the first round negative effect. In fact in the second round the IS curve shifts up along the positively sloped LM curve (see the zoom on the area of interest in figure 6.2). For the same reason, the second round effect on the output gap is positive, magnifying the first round effect.\(^{28}\)

The initial shock (first round increase of the output gap due to first round decrease of the interest rate) is amplified through the financial accelerator mechanism (second round decrease of the EFP). Heterogeneity contributes to the magnification of the initial shock because the reduction of dispersion will add to the decrease of the EFP.

In (6.1) we report the magnitude of the effects using the numerical values of the steady state moments of the distribution and the parameters set in table 3.1

\[
\begin{array}{|c|c|}
\hline
1st round effect & 2nd round effect \\
\hline
\Delta r_{1st} = -1.41 \times 10^{-3} & \Delta r_{2nd} = 1.6 \times 10^{-8} \\
\Delta x_{1st} = 2.12 \times 10^{-2} & \Delta x_{2nd} = 1.13 \times 10^{-5} \\
\hline
\end{array}
\]  

(6.1)

Also in this case most of the effect of the shock must be traced back to the first round impact. The second round effect on the interest rate is five orders of magnitude smaller than the absolute value of the first round effect. The second round effect on the output gap is three orders of magnitude smaller than the absolute value of the first round effect. The contribution of heterogeneity \((-1.61 \times 10^{-7})\) is a quarter of the effect on the EFP.\(^{28}\)

\(^{28}\) This result is reminiscent of a similar outcome in Bernanke and Blinder’s CC-LM model (Bernanke and Blinder, 1988). In their framework a monetary shock makes the LM curve shift down and the CC curve shift up. There is therefore an amplification of the shock on output.
The financial amplification is truly small, almost negligible. This result is prima facie surprising in the light of the magnitude of the monetary shock. Money supply has almost doubled but the financial accelerator has contributed relatively little to the increase of output. This is due to the limited impact of changes in distribution on the EFP, which in turn can be traced back at least in part to the limited size of bankruptcy costs.

7 Montecarlo analysis

We have run Montecarlo simulations focusing on the parameters $\alpha$ and $\bar{m}$ in order to check the robustness and sensitivity of the results discussed in the previous sections to the specific configurations of the simulation procedure.

First of all, we have simulated the model for six regularly spaced values of the bankruptcy threshold $\alpha$ in the interval $[0.02, 0.17]$ (the other parameters values are listed in table 3.1). For each value of $\alpha$ we have run 20 simulations with different random seeds. Each simulation generates artificial data for the cross sectional mean and variance of the equity ratio. We have estimated the coefficients $\alpha_{ij} (i, j = 0, 1, 2)$ of the dynamic system (3.3) (3.4) by means of linear regression on the data generated by each simulation. Hence we have got 20 estimates $\hat{\alpha}_{ij}^s, s = 1, 2, ..., 20$ for each coefficient. Finally, we have taken the mean (across simulations) of the 20 estimates $\alpha_{ij} = \frac{1}{20} \sum_{s=1}^{20} \hat{\alpha}_{ij}^s$.

For each of the six $\alpha_{ij}$ coefficients we have plot the mean estimate as a function of $\alpha$ in figure 7.1. Visual inspection leads to conclude that $\alpha_{12}, \alpha_{20}$ and $\alpha_{21}$ are increasing with $\alpha$ while $\alpha_{11}$ and $\alpha_{22}$ are decreasing with $\alpha$. The coefficient $\alpha_{10}$ has no clear tendency. It is worth noting that the coefficient $\alpha_{12}$ turns from negative to positive with an increase in $\alpha$. The changes in the coefficients due to the increase of $\alpha$ impact upon the slope and the intercept of the AA and VV lines.\textsuperscript{29}

The linear regression of the mean estimate of each coefficient on $\alpha$ returns the $\beta$ parameters defined as follows:

$$\alpha_{ij} = \beta_1 + \beta_2 \alpha \quad i, j = 0, 1, 2$$

which are shown in table 7.1. The results of the regression confirm the

\textsuperscript{29}When $\alpha$ becomes greater than 0.07, the AA line becomes upward sloping. The resulting steady state may turn into a saddle point.
conclusion drawn from visual inspection. Coefficients which are increasing (decreasing) with $\alpha$ are characterized by a positive (negative) $\beta_2$, statistically significant (at the 5% level). In this case we replace the coefficient with a linear relationship. For example $\alpha_{12} = -0.3891 + 5.6192\alpha$. When $\beta_2$ is statistically not significant we approximate the coefficient with $\beta_1$. For example $\alpha_{10} = 0.3227$.

Plugging the linear relationships estimated above into the equations that define the steady state $(a_s, V_s)$ of system (3.3) (3.4) we get the plots of the first row of figure 7.2. Since $a_s$ is decreasing and $V_s$ is increasing with $\alpha$ the external finance premium is unambiguously increasing with $\alpha$. The interest rate and output gap are therefore decreasing with $\alpha$ as shown by the second row and the first panel in the third row of figure 7.2.

The last panel of the figure shows the AA and VV lines generated by the mean estimated coefficients with $\alpha_0 = 0.02$ and $\alpha_1 = 0.05$. The financial shock makes the AA line shift down and the VV curve shift up (albeit only marginally) so that $a_s$ goes down and $V_s$ goes up. The values of the coefficients and of the steady state values are reported in table 7.2. These

In figure 7.3 we have shown the shift of the IS curve as a consequence of the financial shock. These results are consistent with the conclusions we have drawn in section 5 on the basis of a run of simulations only.

We have simulated the model also for regularly spaced values of the money supply $\hat{m}$ in the interval $[200, 450]$ (the other parameters values are in table 3.1). For each value of $\hat{m}$ we have run 20 simulations with different random
\[ a_t = \alpha_{10} + \alpha_{11}a_{t-1} + \alpha_{12}V_{t-1} \]

<table>
<thead>
<tr>
<th>( \alpha_{1j} = \beta_1 + \beta_2\alpha; \ j = 0, 1, 2 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{10} )</td>
<td>0.3227*</td>
<td>-0.035</td>
</tr>
<tr>
<td>( \alpha_{11} )</td>
<td>0.5162*</td>
<td>-1.7686*</td>
</tr>
<tr>
<td>( \alpha_{12} )</td>
<td>-0.3891*</td>
<td>5.6192*</td>
</tr>
</tbody>
</table>

\[ V_t = \alpha_{20} + \alpha_{21}a_{t-1} + \alpha_{22}V_{t-1} \]

<table>
<thead>
<tr>
<th>( \alpha_{2j} = \beta_1 + \beta_2\alpha; \ j = 0, 1, 2 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{20} )</td>
<td>0.0569*</td>
<td>0.0685*</td>
</tr>
<tr>
<td>( \alpha_{21} )</td>
<td>0.0133*</td>
<td>0.3025*</td>
</tr>
<tr>
<td>( \alpha_{22} )</td>
<td>0.2561*</td>
<td>-0.9489*</td>
</tr>
</tbody>
</table>

Table 7.1: Effects of changes in \( \alpha \) on the coefficients of the dynamical system (3.3) (3.4). Linear regressions on artificial data generated by the Montecarlo simulations.

| \( \alpha = 0.02 \) | \( \alpha = 0.05 \) |
|---|---|---|---|
| \( \alpha_{10} \) | 0.3227 | \( \alpha_{20} \) | 0.0583 |
| \( \alpha_{11} \) | 0.4808 | \( \alpha_{21} \) | 0.0194 |
| \( \alpha_{12} \) | -0.2767 | \( \alpha_{22} \) | 0.2371 |
| \( a \) | 0.5730 | \( a \) | 0.5459 |
| \( r \) | 0.003397 | \( r \) | 0.003396 |

Table 7.2: Steady state
Figure 7.2: Steady state values of $a$, $V$, $f$, $r$, $x$ as functions of $\alpha$

Figure 7.3: Shift of the IS curve as a consequence of the increase of $\alpha$
seeds, we have got 20 estimates for each coefficient and we have taken the mean (across simulations) of the 20 estimates.

In figure 7.4 we plot the mean estimate of each $\alpha_{i,j}$ coefficient as a function of $\bar{m}$.

Only $\alpha_{10}$ and $\alpha_{12}$ show a clear tendency to increase and decrease respectively with $\bar{m}$. In all the other cases, the coefficients fluctuate around their mean. We interpret this behaviour as a symptom of an unstable relationship between money supply and the structure of the dynamic system.

The linear regression of the mean estimate of each coefficient on $\bar{m}$ returns the $\beta$ parameters defined as follows:

$$\alpha_{ij} = \beta_1 + \beta_2 \bar{m} \quad i, j = 0, 1, 2$$

which are shown in the following table. The results of the regression confirm the conclusion we can draw by visual inspection.

In fact the $\beta_2$ coefficients are not statistically significant in most of the cases. Only $\alpha_{10}$ and $\alpha_{12}$ are related in a significant but weak way to $\bar{m}$, namely $\alpha_{10} = 0.3212 + 0.00002\bar{m}$ and $\alpha_{12} = -0.2882 - 0.00013\bar{m}$. In all the other cases $\beta_2$ is not statistically significant so that we approximate the coefficient with $\beta_1$ assuming away the inherent volatility of the coefficient revealed by figure 7.4.

Since the parameters that change with money supply affect only the AA line, by construction the VV line is unaffected by monetary shocks.
\[ a_t = \alpha_{10} + \alpha_{11} a_{t-1} + \alpha_{12} V_{t-1} \]

<table>
<thead>
<tr>
<th></th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{1j} = \beta_1 + \beta_2 \bar{m} ); ( j = 0, 1, 2 )</td>
<td>( \alpha_{10} )</td>
<td>0.3212*</td>
</tr>
<tr>
<td></td>
<td>( \alpha_{11} )</td>
<td>0.4670*</td>
</tr>
<tr>
<td></td>
<td>( \alpha_{12} )</td>
<td>-0.2882*</td>
</tr>
</tbody>
</table>

\[ V_t = \alpha_{20} + \alpha_{21} a_{t-1} + \alpha_{22} V_{t-1} \]

<table>
<thead>
<tr>
<th></th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{2j} = \beta_1 + \beta_2 \bar{m} ); ( j = 0, 1, 2 )</td>
<td>( \alpha_{20} )</td>
<td>0.0585*</td>
</tr>
<tr>
<td></td>
<td>( \alpha_{21} )</td>
<td>0.0182*</td>
</tr>
<tr>
<td></td>
<td>( \alpha_{22} )</td>
<td>0.2626*</td>
</tr>
</tbody>
</table>

Table 7.3: Effects of changes in \( \bar{m} \) on the coefficients of the dynamical system (3.3) (3.4). Linear regressions on artificial data generated by the Montecarlo simulations.

Plugging these estimates into the equations that define the steady state we get the relationship between \( (a_s, V_s) \) and \( \bar{m} \), plotted in the panels of the first row of figure 7.5. Both \( a_s \) and \( V_s \) are increasing with \( \bar{m} \). In principle therefore the relationship between the EFP and money supply is unclear. It turns out however that the positive relationship between \( a_s \) and \( \bar{m} \) is prevailing so that the EFP is decreasing with \( \bar{m} \). Therefore the interest rate is decreasing and the output gap is increasing with \( \bar{m} \) (see panels in the second row and first panel in the third row of the figure).

The last panel of the figure shows the AA and VV lines generated by the mean estimated coefficients when \( \bar{m} = 200 \) and \( \bar{m} = 350 \). The monetary shock makes the AA line shift up along the upward sloping VV curve so that both \( a_s \) and \( V_s \) go up. In section 6 the AA line shift up and the VV curve shifts down so that \( V_s \) goes down. The values of the coefficients and of the steady state values are reported in table 7.4.

Comparing these results with those of 6, we conclude that the effect of the monetary shock on the cross sectional mean is similar but the effect on the variance is of opposite sign. Since the latter effect is negligible, the effects on the interest rate and the output gap is also broadly consistent with the previous ones.
\begin{table}
\centering
\begin{tabular}{llllllllll}
\hline
 & $\bar{m} = 200$ &  &  & $\bar{m} = 350$ &  &  &  \\
\hline
$\alpha_{10}$ & 0.3252 & $\alpha_{20}$ & 0.0585 & $\alpha_{10}$ & 0.3282 & $\alpha_{20}$ & 0.0585 &  \\
$\alpha_{11}$ & 0.4670 & $\alpha_{21}$ & 0.0182 & $\alpha_{11}$ & 0.4670 & $\alpha_{21}$ & 0.0182 &  \\
$\alpha_{12}$ & -0.3142 & $\alpha_{22}$ & 0.2626 & $\alpha_{12}$ & -0.3337 & $\alpha_{22}$ & 0.2626 &  \\
$a$ & 0.5553 & $V$ & 0.093 & $a$ & 0.5575 & $V$ & 0.0931 &  \\
r & 0.0034 & $x$ & 0.9480 & r & 0.0020 & $x$ & 0.9692 &  \\
\hline
\end{tabular}
\caption{Steady state}
\end{table}

In figure 7.6 we have shown the shift of the IS and LM curves as a consequence of the monetary shock. The second round effect which shifts the IS curve up is negligible; we have to zoom in to appreciate it visually.

8 Conclusions

We illustrate a methodology to resume macroeconomic thinking in a setting characterized by heterogeneous agents building a M&ABM in two interlinked parts.

First of all we develop a macroeconomic model in the following steps:

**Step M1.** We derive a microeconomic behavioural rule for investment. The individual investment ratio is a function of the the interest rate and of the individual equity ratio (which determines the EFP at the micro level).

**Step M2.** We apply a stochastic aggregation procedure (the "modified" representative agent) to the individual investment ratio in order to derive the average investment ratio. The average investment ratio depends non linearly upon the mean and the variance of the distribution of the equity ratio (which determine the average EFP). The distribution of the equity ratio is changing over time and affects investment accordingly.

**Step M3.** We model households’ choice of the optimal consumption plan and desired money balances in a fairly standard conceptual framework.

**Step M4.** Equilibrium on the goods market yields a relationship between the interest rate and the output gap reminiscent of the IS curve. The EFP (and therefore the moments of the distribution of the equity ratio) is a shifter of the IS curve. Equilibrium on the money market yields a rela-
Figure 7.5: Coefficients $\alpha_{1j}$ (first column) and $\alpha_{2j}$ (second column) as functions of $\bar{m}$

Figure 7.6: Shifts of the IS and LM curves as a consequence of the monetary shock
tionship between the interest rate, the output gap and real money balances reminiscent of the LM curve. In the end we obtain a simple optimizing IS-LM model, which can be solved for the equilibrium values of the interest rate and the output gap. In equilibrium $r_t$ and $x_t$ turn out to be functions of the moments of the distribution of the equity ratio $a_{t-1}$ and $V_{t-1}$.

The difference between the traditional microfoundations based on the representative agent and the new ones is the explicit consideration of the moments of the distribution of the equity ratio. By focusing on moments we resume macroeconomic thinking in its purest form, i.e. at a general, non microeconomic, level, in a setting with heterogeneous agents.

So far we have treated the moments of the distribution as pre-determined variables ($a_t$ and $V_t$). In order to endogenize the dynamics of the moments, we have to go back to the micro level. We build an agent based model in the following steps.

**Step A1.** We define the law of motion of the individual equity ratio $a_{it}$ which is a function, among other things, of the interest rate. We plug the equilibrium value of the interest rate $r_t$ derived in step M4 into the individual law of motion. Since in equilibrium the interest rate is a function of the moments of the distribution, the current individual equity ratio turns out to be a function of the lagged cross-sectional mean and variance of the equity ratio.

**Step A2.** We run computer simulations to determine the evolution over time of the individual equity ratios. The time series of the cross sectional moments (mean $a_t$ and variance $V_t$) are computed directly from the individual time series of simulated equity ratios.

**Step A3.** We estimate a linear system of first order difference equations in $a_t, V_t$ by means of OLS regression on the time series of the cross sectional moments. Once we get numerical values for the parameters we can study this system by standard analytical techniques. Given the initial condition $(a_{t-1}, V_{t-1})$ the system generates recursively $(a_t, V_t)$ which determines new external finance premium $f_{t+1}$ and – through the reduced form of the IS-LM model – the new interest rate $r_{t+1}$ which in turn will impact upon $a_{it+1}$ and so on, as shown in figure 8.1.

The steady state of this system $(a_0, V_0)$ are the numerical values of the first and second moments of the "long run" distribution of the equity ratio. Moreover we can easily determine the long run external finance premium $f_0$.

**Step A4.** We plug the long run external finance premium into the equilibrium values of the macroeconomic endogenous variables. Therefore we
Figure 8.1: The structure of the M&ABM
determine the long run output gap and interest rate $x_0$ and $r_0$.

We have assessed the impact of a negative financial shock and of a monetary expansion, showing that the financial amplification effect is captured diagrammatically by shifts of the IS curve due to changes in the moments of the distribution of financial conditions. The financial amplification is small, especially in the case of a monetary shock. This is due, at least in part, to the small size of the bankruptcy cost that we have assumed in the parameter setting.

The benchmark model lends itself to a wide range of possible extensions, such as a different monetary policy setting in a flexprice environment, the explicit consideration of income and wealth inequality among households, the role of fiscal policy and many others. We find the results reached so far encouraging. We hope the methodology we propose can open a path to the development of hybrid models which nest and ABM into a macroeconomic framework in such a way as to allow for a clear conceptual understanding of macroeconomic developments.

A The probability of bankruptcy

In this section we describe a microfoundation for the probability of bankruptcy along the lines of Greenwald and Stiglitz (1993). In the text, for the sake of simplicity, we adopt a simplified "reduced form" specification (equation (2.2)) of the probability of bankruptcy which focuses on only one of the determinants – namely the equity ratio – brought to the fore by the following microfoundation,

Net worth is $A_{it} = A_{i,t-1} + \pi_{it}$ where $\pi_{it} = u_iY_{it} - wN_{it} - rK_{it} - \frac{1}{2} \frac{I_{it}^2}{K_{t-1}}$. We define total cost as $TC_i = wN_i + rK_i + \frac{1}{2} \frac{I_{it}^2}{K_{t-1}}$. Hence $A_{it} = A_{i,t-1} + u_iY_{it} - TC_i$. A firm goes bankrupt if $A_{it} < 0$, i.e. if:

$$u_i < AC_i - \frac{A_{i,t-1}}{Y_{it}} \equiv \bar{u}_i$$

where $AC_i = TC_i/Y_{it}$ is average cost. In words: The firm goes bankrupt if the realization of the random shock is smaller than a threshold $\bar{u}_i$ which in turn depends on equity, output, and the average cost. By assumption, the shock is a uniformly distributed random variable $u_i$ with support $(0,2)$, so
that the probability of bankruptcy is:

$$\Pr(u_i < \bar{u}_i) = \frac{\bar{u}_i}{2} = \frac{1}{2} \left( AC_i - \frac{A_{i-1}}{Y_{it}} \right)$$  \hspace{1cm} (A.1)$$

Let’s assume, as in the text of the paper, that the cost of bankruptcy is $CB_i = \beta K_i$.

Plugging $Y_{it} = \nu K_{it}$ and $N_{it} = \frac{\nu}{\lambda} K_{it}$ into (A.1) and rearranging, the probability of bankruptcy turns out to be:

$$\Pr(u_i < \bar{u}_i) = \frac{1}{2} \left\{ \frac{w}{\lambda} + \frac{r}{\nu} + \frac{1}{2} \left( \frac{K_{it} - K_{it-1}}{K_{it}} \right)^2 - \frac{a_{it-1} K_{it-1}}{\nu K_{it}} \right\}$$  \hspace{1cm} (A.2)$$

The probability of bankruptcy depends on a large number of parameters and endogenous variables, the equity ratio being only one of them. Adopting the specification (A.2) would have made the analysis very messy. In order to simplify the argument, we adopt the approximation (2.2) of the text.

### B Household’s optimization

The problem of the representative household is

$$\max_{c_t, m_t, b_t} U_t = \sum_{s=0}^{\infty} \xi^s (c_{t+s})^\delta (m_{t+s})^{1-\delta}$$

s.t. $c_t + m_t + b_t = wx_t + \sigma (1 - x_t) + m_{t-1} + (1 + r_{t-1}) b_{t-1}$

(B.1)

From which we obtain the following Lagrangian:

$$\mathcal{L} = \sum_{s=0}^{\infty} \xi^s \left\{ (c_{t+s})^\delta (m_{t+s})^{1-\delta} + \lambda_{t+s} [wx_{t+s} + \sigma (1 - x_{t+s}) + m_{t+s-1} + (1 + r_{t+s-1}) b_{t+s-1} - c_{t+s} - m_{t+s} - b_{t+s}] \right\}$$

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \delta (c_t)^{\delta-1} (m_t)^{1-\delta} = \lambda_t$$  \hspace{1cm} (B.2)$$
\[ \frac{\partial L}{\partial m_t} = (1 - \delta) (c_t)^{\delta} (m_t)^{-\delta} - \lambda_t + \xi \lambda_{t+1} = 0 \]  
\quad \text{(B.3)}

\[ \frac{\partial L}{\partial b_t} = -\lambda_t + \xi \lambda_{t+1} (1 + r_t) = 0 \]  
\quad \text{(B.4)}

Solving (B.2) (B.3) (B.4) for \( c_t, m_t \) we obtain the following relation between optimal consumption and money demand:

\[ m_t = \frac{1 - \delta}{\delta} \frac{1 + r_t}{r_t} c_t \]

assuming, for the sake of simplicity, \( \delta = 1/2 \) we get equation (2.14) in the text.

\section{Law of motion of the equity ratio}

Assuming that there are no dividends, the is defined as \( A_{it} = A_{it-1} + \pi_{it} \).

Recalling (2.1) we get:

\[ A_{it} = A_{it-1} + u_{it} Y_{it} - w N_{it} - r_{it} K_{it} - \frac{1}{2} \frac{f_{it}^2}{K_{t-1}} \quad i = 1, 2, ..., z \]

Dividing by \( K_{it} \) we obtain the law of motion of the equity ratio:

\[ a_{it} = a_{it-1} \frac{K_{it-1}}{K_{it}} + u_{it} \nu - w \nu \frac{1}{\lambda} r_{it} - \frac{1}{2} \frac{f_{it}^2}{K_{it-1}^2 K_{t-1}} \]

Recall now that \( g_{it} = \frac{I_{it}}{K_{it}} = \frac{\tau_{it}}{s_{it-1}} \) where \( s_{it-1} = \frac{K_{it-1}}{K_{t-1}} \). Therefore

\[ \frac{K_{it-1}}{K_{it}} = \frac{1}{1 + g_{it}} = \frac{s_{it-1}}{s_{it-1} + \tau_{it}} \]

Moreover

\[ \frac{f_{it}^2}{K_{it}^2 K_{t-1}^2 K_{it-1} K_{t-1}} = \frac{\tau_{it}^2}{s_{it-1} + \tau_{it}} \]

Plugging these expressions into (??) we obtain:

\[ a_{it} = a_{it-1} \left( \frac{s_{it-1}}{s_{it-1} + \tau_{it}} \right) + u_{it} \nu - w \frac{\nu}{\lambda} r_{it} - \frac{1}{2} \frac{\tau_{it}^2}{s_{it-1} + \tau_{it}} \]

which is (3.2).
D Coding the ABM

In this appendix, we detail the logical sequence of the code which governs the simulations of the ABM. The parameter setting is defined in table 3.1.

At the beginning of the time horizon considered, i.e. in quarter t=1, the initial conditions are chosen as follows:

- the equity ratio $a_{i1}$ of the i-th firm, $i = 1,\ldots,1000$, is drawn from a uniform distribution over the $(\alpha,1)$ support. Therefore the cross-sectional mean and variance of the initial equity ratios can be computed as $a_1 = \text{mean}(a_{i1}); V_1 = \text{variance}(a_{i1});$

- the initial capital stock $K_{i1}$ is drawn from a uniform distribution. Therefore, the initial relative size can be computed as $s_{i1} = K_{i1}/\bar{K}_1$ where $\bar{K}_1 = \text{mean}(K_{i1});$

From $t = 2$ on:

- the external finance premium can be computed as $f_2 = \frac{\beta \alpha'}{a_1} \left( 1 + \frac{V_1}{a_1^2} \right)$.

- the interest rate can be computed from (2.22). For example in $t = 2$ we have $r_2 = \Gamma_0 \left[ \sigma \frac{\nu}{\lambda} w - (w - \sigma) f_2 \right]$ where $\Gamma_0 = \left( w - \sigma + \sigma \frac{\nu}{\lambda} \right)^{-1}$

- the output gap can be computed from (2.23). For example in $t = 2$ we have $x_2 = \Gamma_0 \frac{\bar{m}}{w - \sigma} \left[ \sigma \frac{\nu}{\lambda} w - (w - \sigma) f_2 \right] - \frac{\sigma}{w - \sigma}$

- the individual investment ratio can be computed from (2.4). For example in $t = 2$ $\tau_{i2} = \gamma - \left( r_2 + \frac{\beta \alpha'}{a_{i1}} \right)$
• plugging these data into (3.2) we can track the evolution over time of the individual equity ratio. For instance, in period 2 we get:

\[ a_{i2} = a_{i1} \left( \frac{s_{i1}}{s_{i1} + \tau_{i2}} \right) + u_{i2} \nu - w \frac{\nu}{\lambda} - r_2 - \frac{1}{2} \frac{\tau_{i2}^2}{s_{i1} + \tau_{i2}} \]

where \( s_{i1}, \tau_{i2}, r_2 \) have already been determined as above and \( u_{i2} \) is an idiosyncratic shock drawn from a uniform distribution over the \((0, 2)\) support;

• the individual capital stock can be determined according to the following law: \( K_{it} = K_{it-1} + \tau_{it} \bar{K}_{t-1} \) where \( \bar{K}_{t-1} = mean (K_{it-1}) \) because, by definition, \( I_{it} = \tau_{it} \bar{K}_{t-1} \). For instance

\[ K_{i2} = K_{i1} + \tau_{i2} \bar{K}_1 \]

so that the relative size will be \( s_{i2} = \frac{K_{i2}}{\bar{K}_2} \).

• the cross-sectional mean and variance of the equity ratios can be computed – for instance \( a_2 = mean (a_{i2}) ; V_2 = variance (a_{i2}) \) – and the sequence can be iterated.

The i-th firm goes \textit{bankrupt} and exits if the equity ratio hits the bankruptcy threshold. In this case, the exiting firm is replaced by a new firm whose equity ratio is drawn from a uniform distribution over the \((0, 2)\) support.

Symmetrically, if the equity ratio reaches unity, the firm is completely \textit{self-financed}, so that it does not need to resort to external finance to carry on production and therefore does not run the risk of bankruptcy. To keep the analysis as simple as possible, we will imagine that when the equity ratio hits the ceiling, the firm will be restarted with an equity ratio drawn from a uniform distribution over the \((\alpha, 1)\) support.

In a sense, therefore, the i-th firm is a \textit{dynasty}: every time the firm goes bankrupt or becomes completely self-financed, the firm will be restarted with a new stochastic equity ratio. This device allows to keep the equity ratio within the admissible \((\alpha, 1)\) range.

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